Mathematics Enters the Picture

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Abstract Can one of the most important Italian Renaissance frescoes reduced in hundreds of thousand fragments by a bombing during the Second World War be re-composed after more than 60 years from its damage? Can we reconstruct the missing parts and can we say something about their original color?
In this short paper we want to exemplify, hopefully effectively by taking advantage of the seduction of art, how mathematics today can be applied in real-life problems which were considered unsolvable only few years ago.

1 Introduction

During last century, perhaps the dominating direction within applied and computational mathematics was oriented to problems of physics and the latter are expected to be of fundamental inspiration for the mathematics of this century also. However, new challenges are now emerging from the engineering world and by social changes motivated, *e.g.*, by our interdependence through technology. Actually, the combination of technological innovations and sophisticated - often interdisciplinary - mathematical methods allow for advances that were not possible by traditional means. As an example of this new trend of applied and computational mathematics, current developments in image processing hardware and the conceptualization of digital images as mathematical objects have led to an explosive growth of the interdisciplinary field of imaging sciences. Mathematics plays here a fundamental role, where applied and computational harmonic analysis (*e.g.*, with the advent of time-frequency and wavelet analysis) [6], singular PDEs, calculus of variations, and geometric measure theory [1] fuse into a new challenging field. The direct interplay between mathematical modelling of images and real-life applications is a continu-

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ous source of new ideas. On the one hand imaging science could exploit successfully classical mathematical tools, on the other hand real-life problems inspired new developments with an impact which often reaches far beyond the original scope. Indeed digital images can serve as a toy-model with a sufficiently rich morphology for sophisticated applications in more complex systems and phenomena.
In this short contribution we would like to exemplify the development of modern mathematical models and methods in imaging, inspired by a real-life problem in art restoration. In particular, we highlight the direct interplay between the application and mathematical advances.
On March 11, 1944, the Eremitani's church in Padua (Italy) was destroyed in an Allied air raid along with the inestimable frescoes by Andrea Mantegna *et al.* contained in the Ovetari Chapel. The importance of these frescoes is reported by the effective words of J. W. Goethe in his *Italienische Reise*: on September 26-27 1786, on his famous Italian journey, Goethe came to Padua and visited the church of the Eremitani. There he saw the frescoes by Mantegna, of the lives of Saint James and Saint Christopher, in the funerary chapel of Antonio degli Ovetari. He stood before them "astounded" at their "scrupulous detail, their imaginative power, their strength, and subtlety". Here he had found one of 'the older painters' who stood behind and inspired the great Masters of the Italian Renaissance: "Thus did art develop after the ages of barbarism"[1].

Fig. 1 Fragments of the frescoes contained in the box 31, tray A2.

In the last 60 years, several attempts have been made to restore the puzzle of the fresco fragments (Fig. 1) by traditional methods, without much success. Most of the difficulties were because the fragments are 'few' (more than 88000 though, with an average surface 6-7 cm^2!) and, eventually, any reconstruction result may appear just

[1] *Tra mistero ed estasi Goethe rimase folgorato*, La Repubblica, August 14, 2006: http://ricerca.repubblica.it/repubblica/archivio/repubblica/2006/08/14/tra-mistero-ed-estasi-goethe-rimase-folgorato.html.

disappointing. However, Cesare Brandi, former Director of the Central Institute for Restoration in Rome in 1947, came to write “... the importance of the Padua cycle was such that [...] also the recovery of a sole square decimeter has an impact that no modesty can hide ...” [3, p. 180]. This sentence clearly turns the problem into a challenging, fascinating, and extraordinary ‘treasure hunt’.

The problem proposed more than 60 years ago and remained unsolved so far has been eventually challenged and overtaken by means of mathematical methods. Of course, mathematics cannot substitute the artistic genius of Mantegna, but the theoretical achievements we reached today, surely extraordinary as well, allow for the solution of problems considered impossible until now.

We contributed to the development of an efficient mathematics-based pattern recognition algorithm to map the original position and orientation of the fragments, based on comparisons with an old gray level image of the fresco prior to the damage. This innovative technique allowed for the partial reconstruction of the frescoes. In Section 2 we review the relevant features of the method we proposed, and a few samples of the results.

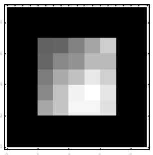
Unfortunately, the surface covered by the colored fragments is only 77 m^2, while the original area was of several hundreds. This means that we can reconstruct only a fraction (less than 8%) of this inestimable artwork. In particular the original color of the blanks is not known. This begs the question of whether it is possible to *estimate mathematically* the original colors of the frescoes by making use of the potential information given by the available fragments and the gray level of the pictures taken before the damage. In Section 3 we review a model recently studied for the recovery of vector valued functions from incomplete data, with applications to the recolorization problem. The model is based on the minimization of a functional which is formed by the discrepancy with respect to the data and additional total variation regularization constraints. We present the numerical solution of the minimization problem, and we show the results of the application of the method on the real-life case of the A. Mantegna’s frescoes.

The goal of this short paper is to provide a popular description (with some mathematics to accommodate the legitimate wish of a bit of cogency) of our work on the Mantegna’s fresco restoration, whereas it is *not* its aim that of presenting it in very detail. For that, we refer the interested reader to [3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15].

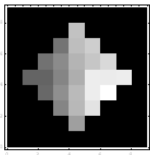
2 Re-puzzling Mantegna

2.1 Digital images and rotations

In order to understand what mathematics has to do with the Mantegna’s art, we need first to point out the relationship between mathematics and digital images. Roughly speaking a digital image is a collection of points (pixels) with different levels of brightness located at nodes of a regular grid. The use of multiple channels allows further to encode color levels. Hence, an image can be represented as a numerical matrix, and, in this form, it can be processed mathematically, see Fig. 2. In partic-



0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	49	50	65	82	103	0	0
0	0	51	68	73	93	93	0	0
0	0	62	85	96	116	104	0	0
0	0	68	97	121	128	115	0	0
0	0	83	98	123	123	112	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0



0	0	0	0	0	0	0	0	0
0	0	0	0	95	0	0	0	0
0	0	0	57	93	101	0	0	0
0	0	56	68	88	97	106	0	0
0	49	49	65	85	116	115	116	0
0	0	51	70	91	116	125	0	0
0	0	0	65	90	111	0	0	0
0	0	0	0	78	0	0	0	0
0	0	0	0	0	0	0	0	0

Fig. 2 Digital images are encoded into numerical matrices. Rotations may produce numerical distortions.

ular, we can compare images, for instance whether they are 'similar', by evaluating the *distance* $|a_{ij} - b_{ij}|$ of the numerical entries (a_{ij}) and (b_{ij}) of the corresponding matrices. Although two images may refer to a photo of the same subject, significant disturbance, due to different photographic techniques, light exposition etc., may occur. In particular, the photos of the frescoes are dated to the 1920 and acquired in black-and-white with the techniques of that time, whereas the photos in color of the fragments were produced in the late-1990s on Kodak film. Therefore the numbers which appears in the corresponding matrices cannot be equal. Moreover, the fragments are rotated with respect to their original orientation, introducing a further fundamental element of uncertainty and complexity. Indeed rotations which are not multiple of 90° on a square grid are affected by *aliasing*. In practice if we rotate a digital image, say, of 45°, the resulting matrix contains entries which are only approximatively close to the original numbers contained in the matrix of the unrotated image, see Fig. 2. Hence, it seems that our 'treasure hunt' is really a challenge which, mathematically speaking, results in the search of a few numbers, only approximatively given, and encoding the fragment image, in the huge matrix of the fresco image, independently of a possible mutual rotation.

2.2 Complexity and computational time

Whatever is the method we choose for evaluating the *distance* of the numerical entries of two digital images, for instance whether it is a suitable norm, eventually we have also to face the problem of the *complexity*, *i.e.*, the number of algebraic operations which are needed in order for a computer to execute the comparison. We may consider, for example, the following *naive* strategy:

- we 'transport' the rotated fragment on each position within the fresco image;
- we rotate the fragment image for a sufficiently large number of rotations according to the resolution;
- for each position and for each rotation we execute the comparison, for instance, by calculating the maximal distance of the entries $\max_{ij}|a_{ij} - b_{ij}|$.

The number of positions within the fresco image equals the dimensions, say, $N \times M$. In particular, each one of the 12 scenes of the Mantegna's frescoes is encoded into a digital image of dimensions $N \times M \approx 3200 \times 2400 \approx 7500000$ pixels. Moreover, if the fragment is represented, for instance, by an image of $n = a \times a = 15 \times 15$ pixels, we are allowed to consider at least $a = 15$ rotations (it makes no sense to consider more rotations since the resolution is limited). Eventually, we need to compute the maximum of the distances, which has a cost of $n\log(n) \approx 1000$ operations. Altogether the search of a fragment on a scene of the frescoes costs $N \times M \times a \times 2a^2 \log(a) \approx 10^{11}$ operations. This number has to be multiplied at least by the number of scenes[2] and further by the number of fragments (c.a. 88000). Therefore, we have to expect a number of operations of order 10^{17}. As of 2008, the fastest PC processors (quad-core) perform over 37 GFLOPS[3], *i.e.*, 37×10^9 floating point operations per second. Hence, the search of the fragments with this method would require at least two years of computational time of the fastest PC processor available today. Clearly this strategy cannot be pursued because in practice a human operator further needs to visually evaluate the result of the computation and several other operations has to be fulfilled for the complete identification of the fragment positions with additional time consumption.

2.3 A clever solution: circular harmonic expansions

As explained above, the request of a fast algorithm excludes the implementation of any comparison *pixel-by-pixel* and suggests that methods based on series expansions can be more efficient. Circular Harmonic decompositions have found a relevant role in pattern matching because of their rotation invariance (self-steerability) properties

[2] Actually the fresco area is much larger and it contains all the decorations of the vault and large portions of destroyed frescoes belonging to the side chapel Dotto

[3] "2007 CPU Charts". Tom's Hardware (2007-07-16). Retrieved on 2008-07-08: http://www.tomshardware.com/reviews/cpu-charts-2007,1644-36.html.

and their effective and successful optical implementations [2]. Compactly supported Circular Harmonics (CH) arise as natural solutions of the Laplace eigenvalues problem on a disk under Dirichlet conditions [16], and they are related to relevant physical problems with rotation invariant symmetries. In fact, since the Laplacian commutes with rotations, CH are also eigenfunctions of any rotation operator.

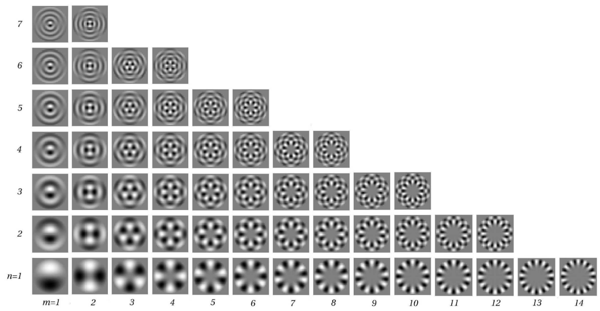


Fig. 3 Real part of a few compactly supported CH, ordered by angular (abscissa) and radial (ordinate) frequencies, depending respectively on the parameters $m \in \mathbb{Z}$ and $n \in \mathbb{N}$.

We denote in the following by $L^p(\Omega)$ the Lebesgue space of p-summable functions on $\Omega \subset \mathbb{R}^d$. Assume $\Omega_a \subset \mathbb{R}^2$ is a disk of radius $a > 0$. The system of Circular Harmonic functions on Ω_a is defined in polar coordinates by

$$e_{m,n,a}(r,\theta) = \frac{c_{m,n}}{a} J_m(j_{m,n} r/a) e^{im\theta}, \quad m \in \mathbb{Z}, \quad n \in \mathbb{N}, \tag{1}$$

where J_m's are Bessel functions of the first kind of order $m \in \mathbb{Z}$, $(j_{m,n})_{n \in \mathbb{N}}$ is the sequence of their positive zeros [17], and $c_{m,n}$ is a normalization constant. We summarize their relevant properties [16]: (i) CH constitute an orthonormal basis for $L^2(\Omega_a)$; (ii)] CH are characterized by special *radial* and *angular* frequencies depending on the parameters n and m respectively, see Fig. 3; (iii) Let R_α be the rotation operator of angle α, *i.e.*, in polar coordinates $R_\alpha f(r,\theta) = f(r, \theta + \alpha)$, for all functions f on Ω_a. Then CH are eigenfunctions of any rotation operator (*self-steerability* property) [2]

$$R_\alpha e_{m,n,a} = e^{im\alpha} \cdot e_{m,n,a}, \tag{2}$$

for all $m \in \mathbb{Z}, n \in \mathbb{N}$. The use of CH allows us to simplify the problem by eliminating an explicit search of the mutual rotation. Indeed, if we (de)compose an image by means of CH, *i.e.*, $f = \sum_{m,n} f_{m,n} e_{m,n,a}$, where $f_{m,n} = \langle f, e_{m,n,a} \rangle_{\ell^2} = \sum_{ij} f_{ij} \overline{e_{m,n,a}}_{ij}$, one can rotate it just by multiplying the moments by unitary eigenvalues of the rotation operator (2):

$$R_\theta f \approx \sum_{m,n} e^{im\theta} f_{m,n} e_{m,n,a}. \tag{3}$$

The approximation symbol "≈" is due to discretization [9]. See Fig. 4 for an example of image (de)composition. Hence, the decision of whether two images f and g

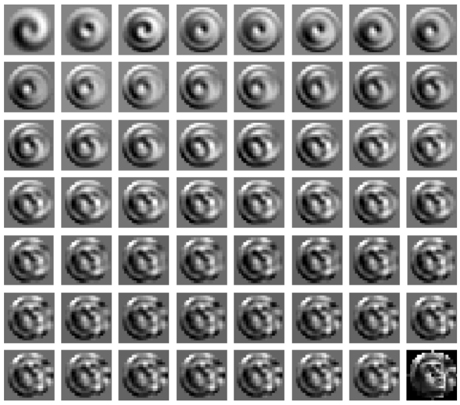


Fig. 4 Example of a (de)composition of an image by means of Circular Harmonics.

are one the rotated of the other stems from checking that the ratios $\frac{g_{m,n}}{f_{m,n}}$ are equal to $e^{im\theta}$ for all $m > 0$. Hence, using $m > 0$, one defines inductively the procedure for an *implicit approximated calculation of optimal angle*: at some $m > 0$, one assumes that a first determination of the optimal angle, say $\alpha_{m-1} \approx \alpha$, is given maybe by means of some calculations on previous coefficients $v_k = \sum_n f_{k,n}\overline{g_{k,n}}$, $k = 1, \ldots, m-1$. Then one computes the next approximation/correction of the optimal angle using the next (independent) complex vector v_m, just rotating back it of α_{m-1}, *i.e.*, multiplying v_m by $e^{-i(m-1)\alpha_{m-1}}$, and setting $\alpha \approx \alpha_m = \arg(e^{-i(m-1)\alpha_{m-1}} v_m)$. An initial approximation from which to start can be deduced from v_1 whenever f and g can be ‘close enough’ up to rotation, see Fig 5.

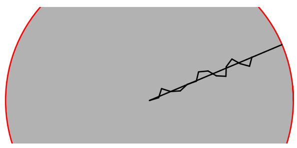


Fig. 5 Iterative angle computation by means of the vectors v_k. The straight line indicates the right angle. The vector v_k are re-aligned and normalized to form the matching coefficient.

The components $f_{m,n}$ of the fragments and $g_{m,n}$ of the frescoes are compared at each positions by ‘transporting’ the fragment via a correlation executed by FFT (fast Fourier transform). Also this operation is very fast and reduce significantly the computational cost. The re-aligned summation of the complex numbers v_k by means of the computed angles α_k, and the normalization of the resulting vector defines a complex number called the *matching coefficient*, see Fig. 5. Its length represents the degree of similarity of the fragment with respect to the underlying fresco image independently of mutual rotation. The highest matching can be found by searching the map of correspondence, Fig. 6. In Fig. 7 we show a sample of the results due

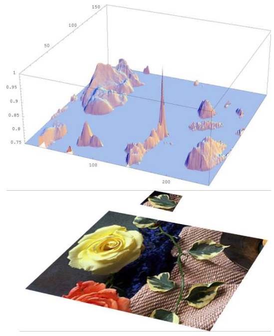


Fig. 6 We associate the length of the matching coefficient to each corresponding position of the fresco. This operation defines the *map of correspondence*. The location with the largest value of the length of the matching coefficient is the most probable location of the fragment.

to the computer-assisted restoration. For more shots and information we refer to the book [5] and to the web-site *www.progettomantegna.it*.

Fig. 7 On the left the scene "St. James Led to Martyrdom" with a few fragments localized by the computer assisted recollocation. On the right, we point out a particular of the scene.

3 Image recolorization

It is evident the sparsity of the re-placed fragments. This is not a failure of the proposed method but it is because the fragments cover only 77 m^2 versus an original surface of several hundreds. A perhaps ungenerous evaluation[4] would argue that, despite the call by Cesare Brandi to the challenging 'treasure hunt'[5], this was not a successful restoration and the placed fragments are just "suspended confetti" (Arturo Carlo Quintavalle, Corriere della Sera, December 11 2006). However, this rushed judgment does not take into account what we could achieve by re-positioning also a few fragments: "... In many cases, also one modest isolated fragment is able to color all the picture where it belongs: in a some sense it diffuses as it developed harmonics ..." (Cesare Brandi [3, p. 180]). Indeed, this is not just a Brandi's poetic hope or a mere imaginative effort, but a concrete possibility: by using the information provided by the few placed color fragments and the gray levels of the photo of the fresco prior to the damage, it is possible to use mathematics again in order to re-color *completely* the frescoes[6]. Note that this re-colorization is the most faithful we can hope to achieve, since for most of the frescoes there is *not* record of any color reproduction, and the colors we use are those diffused from the fragments which have still the original Mantegna's colors.

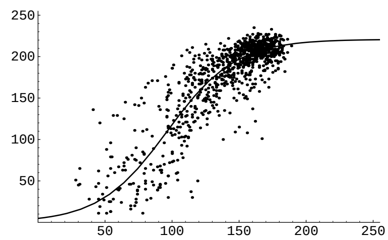


Fig. 8 Estimate of the nonlinear curve L from a distribution of points with coordinates given by the linear combination $\xi_1 r + \xi_2 g + \xi_3 b$ of the (r,g,b) color fragments (abscissa) and by the corresponding underlying gray level of the original photographs dated to 1920 (ordinate).

This innovative mathematical technique gets its inspiration from physics: it is common experience that in an inhomogeneous material heat diffuses anisotropically from heat sources; the mathematical (partial differential) equations that govern this phenomenon are well-known. In turn similar equations can be used to diffuse the color (instead of the heat) from the 'color-sources', which are the placed fragments, keeping into account the inhomogeneity due to the gradients provided by the known gray levels. We describe formally the model as follows. A color image can be modeled as a function $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3_+$, so that, to each "point" $\mathbf{x} \in \Omega$ of the image, one associates the vector $u(\mathbf{x}) = (r(\mathbf{x}), g(\mathbf{x}), b(\mathbf{x})) \in \mathbb{R}^3_+$ of the color represented by the different channels, for instance, red, green, and blue. The gray level of an image can be described as non-linear projection of the colors $\mathscr{L}(r,g,b) := L(\xi_1 r + \xi_2 g + \xi_3 b)$, $(r,g,b) \in \mathbb{R}^3_+$, where $\xi_1, \xi_2, \xi_3 > 0$, $\xi_1 + \xi_2 + \xi_3 = 1$, and $L : \mathbb{R} \to \mathbb{R}$ is a suitable

[4] *Il finto restauro del Mantegna agli Ovetari*, Arturo Carlo Quintavalle, Corriere della Sera, December 11 2006:
http://archiviostorico.corriere.it/2006/dicembre/11/finto_restauro_del_Mantegna_agli_co_9_061211052.shtml

[5] ... the importance of the Padua cycle was such that [...] also the recovery of a sole square decimeter has an impact that no modesty can hide ..." [3, p. 180].

[6] Actually, in the work [11], it has been shown that it is sufficient to have only the 3% of color information randomly distributed, in order to recover with good fidelity the color of a whole image!

non-negative increasing function. For example Fig. 8. describes the typical shape of an L function, which is estimated by fitting a distribution of data from the real color fragments, see Fig. 7. The recolorization is modeled as the minimum (color image)

Fig. 9 The first column illustrates two different data for the recolorization problem. The second column illustrates the corresponding recolorized solution.

solution of the functional

$$F(u) = \mu \int_{\Omega \setminus D} |u(x) - \bar{u}(x)|^2 dx + \int_D |\mathscr{L}(u(x)) - \bar{v}(x)|^2 dx + \int_\Omega \sum_{\ell=1}^{3} |\nabla u^\ell(x)| dx, \quad (4)$$

where we want to reconstruct the vector valued function $u := (u^1, u^2, u^3) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ (for RGB images) from a given observed couple of color/gray level functions $(\bar{u}, \bar{v})$. The observed function $\bar{u}$ is assumed to represent correct information, *e.g.*, the given colors, on $\Omega \setminus D$, and $\bar{v}$ the result of the *nonlinear projection* $\mathscr{L} : \mathbb{R}^3 \to \mathbb{R}$, *e.g.*, the gray level, on D. See [8, 10, 11, 12] for further mathematical details, and Fig. 9 for a sample of the mathematical recolorization.

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References

1. G. Aubert and P. Kornprobst, *Mathematical Problems in Image Processing. Partial Differential Equations and the Calculus of Variation*, Springer, 2002.
2. H. H. Arsenault, Y. N. Hsu, and K. Chalasinska-Macukow, Rotation-invariant pattern recognition, Opt. Eng. 23 (1984) 705-709.
3. C. Brandi, *Il Mantegna Ricostituito* (Italian), in “L’Immagine”, I, 1947, pp.179-180.
4. R. Cazzato, *Un Metodo per la Ricolorazione di Immagini e Altri Strumenti per il Restauro. Il Progetto Mantegna e gli Affreschi nella Chiesa degli Eremitani* (Italian), Laurea thesis, University of Padua, 2007.
5. R. Cazzato, G. Costa, A. Dal Farra, M. Fornasier, D. Toniolo, D. Tosato, and C. Zanuso, Il Progetto Mantegna: storia e risultati (Italian), in “Andrea Mantegna. La Cappella Ovetari a Padova” (Anna Maria Spiazzi, Alberta De Nicolò Salmazo, Domenico Toniolo eds.), Skira, 2006.
6. I. Daubechies, *Ten Lectures on Wavelets*, SIAM, 1992.
7. A. De Nicolò Salmazo, Le “Storie dei santi Giacomo e Cristoforo” nella chiesa degli Eremitani, in Il soggiorno padovano di Andrea Mantegna, Cittadella (Padova) 1993, pp. 31-86.
8. M. Fornasier, Faithful recovery of vector valued functions from incomplete data. Recolorization and art restoration, Lecture Notes in Computer Science, Volume 4485/2007, Proceedings of the First International Conference on Scale Space Methods and Variational Methods in Computer Vision (Sgallari, Fiorella; Murli, Almerico; Paragios, Nikos Eds.), 2007, pp. 116-127.
9. M. Fornasier, Function spaces inclusions and rate of convergence of Riemann-type sums in numerical integration, Numer. Funct. Anal. Opt., Vol. 24, Nos. 1 & 2, 2003, pp. 45-57.
10. M. Fornasier, Nonlinear projection recovery in digital inpainting for color image restoration, J. Math. Imaging Vis. Vol. 24, No. 3, 2006, pp. 359-373.
11. M. Fornasier and R. March, Restoration of color images by vector valued BV functions and variational calculus, SIAM J. Appl. Math., Vol. 68 No. 2, 2007, pp. 437-460.
12. M. Fornasier, R. Ramlau, and G. Teschke, A comparison of joint sparsity and total variation minimization algorithms in a real-life art restoration problem, to appear in Adv. Comput. Math., 2008, 23 pp. DOI:10.1007/s10444-008-9103-6
13. M. Fornasier and D. Toniolo, Fast, robust, and efficient 2D pattern recognition for re-assembling fragmented images, Pattern Recognition, Vol. 38, 2005, pp. 2074-2087.
14. G. Galeazzi and D. Toniolo, I frammenti della Chiesa degli Eremitani: un approccio matematico alla soluzione del problema (Italian), Atti del convegno “Filosofia e Tecnologia del restauro”, gli ‘Emblémata’, Abbazia di Praglia, Oct. 1994, pp. 89-97.
15. G. Galeazzi and D. Toniolo, Il problema della ricostruzione degli affreschi della Chiesa degli Eremitani in Padova. (Italian), Atti del convegno “Il complesso basilicale di San Francesco di Assisi ad un anno dal terremoto”, Assisi 24-26 Sept. 1998.
16. K. B. Wolf, *Integral Transforms in Science and Engineering*, Mathematical Concepts and Methods in Science and Engineering, Vol. 11. New York, London: Plenum Press. XIII, 1979.
17. G. N. Watson, *Theory of Bessel Functions*, Cambridge University Press, 1966.