Learning and Sparse Control of Multiagent Systems

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“Psychohistory is the mathematical study of the reactions of human conglomerates in response to economic and social stimuli.”
– Isaac Asimov, Foundation

“One believes things because one has been conditioned to believe them.”
– Aldous Huxley, Brave New World

“History merely repeats itself. It has all been done before. Nothing under the sun is truly new.”
– Bible, Ecclesiastes 1:9

Abstract. In the past decade there has been a large scope of studies on mathematical models of social dynamics. Self-organization, i.e., the autonomous formation of patterns, has been so far the main driving concept. Usually first or second order models are considered with given predetermined nonlocal interaction potentials, tuned to reproduce, at least qualitatively, certain global patterns (such as flocks of birds, milling school of fish or line formations in pedestrian flows etc.). However, it is common experience that self-organization of a society does not always spontaneously occur. In the first part of this survey paper we address the question of whether it is possible to externally and parsimoniously influence the dynamics, to promote the formation of certain desired patterns. In particular we address the issue of finding the sparsest control strategy for finite agent models in order to lead the dynamics optimally towards a given outcome. In the second part of the paper we show the rigorous limit process connecting finite dimensional sparse optimal control problems with ODE constraints to an infinite dimensional sparse mean-field optimal control problem with a constraint given by a PDE of Vlasov-type, governing the dynamics of the probability distribution of the agent population. Moreover, often in practice we do not dispose of a precise knowledge of the governing dynamics. In the last part of this paper we present a variational and optimal transport framework leading to an algorithmic solution to the problem of learning the interaction potentials from the observation of the dynamics of a multiagent system.

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1. Introduction

In this survey manuscript for the 7ECM Proceedings we concisely present the scope of results from the recent papers [9, 13, 7, 42, 39, 10, 8], constituting the current core output of the ERC-Starting Grant Project “High-Dimensional Sparse Optimal Control” (HDSPCONTR)\(^1\).

We address the conditional and unconditional sparse controllability of self-organizing multi-agent models, in particular for systems of alignment/consensus promoting interactions and attraction-repulsion interactions [9, 13, 7]. We explore how sparse optimal control problems of multi-agent systems of finite number of agents can be properly approached by mean-field approximations [42, 39, 10]. For the applicability of such models in real-life contexts, we also investigate the learn-ability of multi-agent systems from data acquired from observations of their evolutions [8].

2. Self-organizing multi-agent systems

Self-organization in social interactions is a fascinating mechanism, which inspired the mathematical modeling of multi-agent interactions towards formation of coherent global behaviors, with applications in the study of biological, social, and economical phenomena. Recently there has been a vigorous development of literature in applied mathematics and physics describing collective behavior of multi-agent systems [29, 31, 32, 44, 50, 51, 77], towards modeling phenomena in biology, such as cell aggregation and motility [11, 52, 53, 67], coordinated animal motion [5, 16, 19, 26, 23, 24, 32, 60, 63, 64, 71, 76, 81], coordinated human [27, 34, 72] and synthetic agent behavior and interactions, such as cooperative robots [20, 57, 65, 74]. As it is very hard to be exhaustive in accounting all the developments of this very fast growing field, we refer to [14, 15, 17, 78] for recent surveys.

Two main mechanisms are considered in such models to drive the dynamics. The first, which takes inspiration, e.g., from physics laws of motion, is based on binary forces encoding observed “first principles” of biological, social, or economical interactions. Most of these models start from particle-like systems, borrowing a leaf from Newtonian physics, by including fundamental “social interaction” forces within classical systems of 2nd order equations. In this review paper we consider mainly large particle/agent systems of form (below we consider also first order

\(^1\)A wider scope of results can be found at https://www-m15.ma.tum.de/Allgemeines/PublicationsEN.
models as well):

\[
\begin{aligned}
\dot{x}_i &= v_i, \\
\dot{v}_i &= (S + H \ast \mu_N)(x_i, v_i), \\
&\quad i = 1, \ldots, N, \quad t \in [0, T],
\end{aligned}
\]

(1)

where \(\mu_N = \frac{1}{N} \sum_{j=1}^{N} \delta(x_i, v_i)\),

where the “social forces” are encoded in \(S\) and \(H\), modeling, e.g., repulsion-attraction, alignment, self-propulsion/friction etc.

Of course, possible noise/uncertainty can be considered by adding stochastic terms. For large we mean that the number \(N\) of agents is very large.

The second mechanism, which we do not address in detail here, is based on evolutive games, where the dynamics is driven by the simultaneous optimization of costs by the players, perhaps subjected to selection, from game theoretic models of evolution [48] to mean-field games, introduced in [56] and independently under the name Nash Certainty Equivalence (NCE) in [49], later greatly popularized, e.g., within consensus problems, for instance in [61, 62].

The common point of view of these branches of mathematical modeling of multi-agent systems is that the dynamics is based on the free interaction of the agents or decentralized control. The wished phenomenon to be described is their self-organization in terms of the formation of complex macroscopic patterns.

One fundamental goal of these studies is in fact to reveal the possible relationship between the simple binary forces acting at individual level, being the “first principles” of social interaction or the game rules, and the potential emergence of a global behavior in the form of specific patterns. For patterns we do not necessarily mean only steady states, as one considers in the study of the formation of crystalline structures in material sciences, but rather structured evolutions, such as the formation of flocks or swarms in animal motion. Let us recall a few examples of such multi-agent systems and their conditional pattern formation features.

2.1. Alignment models as proposed by Cucker and Smale. For \(S \equiv 0\) and \(H(x, v) = a(||x||)(-v)\), where here \(||\cdot||\) is the \(L^2\)-Euclidean norm on \(\mathbb{R}^d\), the general system (1) specifies the Cucker-Smale model of alignment [32, 33], see also the generalizations in [45],

\[
\begin{aligned}
\dot{x}_i &= v_i, \\
\dot{v}_i &= \frac{1}{N} \sum_{j=1}^{N} a(||x_i - x_j||)(v_j - v_i), \quad i = 1, \ldots, N.
\end{aligned}
\]

(2)

A typical choice for the rate of communication function is \(a(r) = \frac{1}{(1+r)^\beta}\), where \(\beta \in [0, +\infty]\). In matrix notation we can rewrite the Cucker-Smale system as

\[
\begin{aligned}
\dot{x} &= v \\
\dot{v} &= -L^a(x)v
\end{aligned}
\]

\footnote{In (1) the convolution of the kernel \(H\) with the probability measure of the agent group \(\mu\) is defined by \(H \ast \mu(x, v) = \int_{\mathbb{R}^d} H(x-y, v-w)d\mu(y, w)\). For \(\mu = \mu_N\) one has \(H \ast \mu_N(x, v) = \frac{1}{N} \sum_{i=1}^{N} H(x-x_i, v-v_i)\). We write the system already in a measure theoretical setting, because below we consider also limits for \(N \to \infty\).}
where the matrix $L^a(x)$ is the Laplacian of the $N \times N$ adjacency matrix $(a(||x_j - x_i||)/N)_{i,j=1}^N$, and depends on $x$. For a concise description of the behavior of this system, let us mention first of all that it conserves the mean-velocity:

$$\frac{d}{dt} \bar{v}(t) = \frac{1}{N} \sum_{i=1}^N \dot{v}_i(t) = \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N \frac{v_j - v_i}{1 + ||x_j - x_i||^2} \equiv 0.$$ 

In a some sense, it is a diffusion process over the graph of the positions of the agents and the diffused information is precisely the mean-velocity. At the beginning the agents have different velocities and are “unaware” of the global mean-velocity, but with the time they “learn” through the diffusion of information the mean-velocity of the system and they tend to assume it, successfully if consensus on alignment is reached. As we explain below, consensus towards assuming the mean-velocity does not always occur and may depend on the initial conditions.

### 2.2. Repulsion-attraction models proposed by Cucker and Dong.

For $S(x,v) = -bv$ and $H(x,v) = a(||x||^2)(-x) - f(||x||^2)(-x)$, the general system (1) specifies the Cucker-Dong model [30] of repulsion-attraction

$$\begin{cases}
\dot{x}_i = v_i, \\
\dot{v}_i = -b_i v_i + \sum_{j=1}^N (a - f)(||x_i - x_j||^2)(x_j - x_i), \quad i = 1, \ldots, N,
\end{cases}$$

Typical choices for the rate of attraction and repulsion are $a(r) = \frac{1}{(1+r^2)^\beta}$ and $f(r) = \frac{1}{r^\delta}$ respectively, and $t \to b_i(t)$ is a nonnegative function for all $i = 1, \ldots, N$.

As explained below sufficient conditions for self-organization are given in terms of the quantities

$$E(x,v) := \sum_{i=1}^N ||v_i(t)||^2 + \frac{1}{2} \sum_{i<j}^N \left( \int_0^{||x_i - x_j||^2} a(r)dr + \int_{||x_i - x_j||^2}^{+\infty} f(r)dr \right),$$

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where $\beta \leq \frac{1}{2}$ and alignment occurs everytime, on the right $\beta > \frac{1}{2}$ and alignment occurs only for certain initial data.
which is the total energy of the system and the energy threshold

\[ \theta = \frac{N - 1}{2} \int_{0}^{+\infty} a(r)dr. \]  

(5)

In the following it will be convenient to write \( E(t) = E(x(t), v(t)) \).

2.3. Repulsion-attraction and self-propulsion models proposed by D’Orsogna et al.

For \( S(x, v) = (\alpha - \beta \|v\|^2)v \) and \( H(x, v) = -\nabla U(\|x\|) \frac{x}{\|x\|} \), the general system (1) specifies the D’Orsogna-Bertozzi et al. model [19, 16] used to explain, e.g., the formation of certain patterns in fish schools:

\[
\begin{align*}
\dot{x}_i &= v_i, \\
\dot{v}_i &= (\alpha - \beta \|v_i\|^2)v_i - \frac{1}{N} \sum_{j=1}^{N} \nabla U(\|x_i - x_j\|) \frac{x_i - x_j}{\|x_i - x_j\|}, \quad i = 1, \ldots, N,
\end{align*}
\]

A typical example of attraction and repulsion potential is \( U(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R} \). For \( \frac{C_A}{\ell_A} \left( \frac{\ell}{\ell_A} \right)^d < 1 \) the dynamics often converges to crystalline structures compactly translating altogether. If \( \frac{C_A}{\ell_A} \left( \frac{\ell}{\ell_A} \right)^d \geq 1 \) mill patterns arise.

Figure 3. Mills in nature and in our simulations. J. A. Carrillo, M. Fornasier, G. Toscani, and F. Vecil, *Particle, kinetic, hydrodynamic models of swarming*, Birkhäuser 2010 [17].
3. Sparse controllability of multi-agent systems

The mathematical property for a system to form patterns is actually its persistent compactness. There are certainly several mechanisms of promotion of compactness to yield eventually self-organization. In the recent paper [59], for instance, the authors name the heterophilia, i.e., the tendency to bond more with those who are “different” rather than those who are similar, as a positive mechanism in consensus models to reach accord. But also in homophilious societies influenced by more local interactions, global self-organization towards consensus can be expected as soon as enough initial coherence is given. At this point, and perhaps reminiscently of biblical stories from the Genesis, one could enthusiastically argue “Let us give them good rules and they will find their way!” Unfortunately, this is not true, at all. In fact, in homophilious regimes there are plenty of situations where patterns will not spontaneously form. Below we mathematically demonstrate with a few simple examples the incompleteness of the self-organization paradigm, and we propose to amend it by allowing possible parsimonious external interventions. The human society calls them government.

3.1. Sparse stablization and optimal control of alignment models. A relatively simple, but also very instructive mathematical description of such a situation is given by the alignment promoting model (2) by Cucker and Smale already mentioned above, where consensus emergence is shown to be conditional to initial conditions of coherence whenever the system is predominantly homophilious. Let us recall the mathematical model more precisely. Consider now the symmetric bilinear form

\[ B(u,v) = \frac{1}{2N^2} \sum_{i,j} (u_i - u_j, v_i - v_j) = \frac{1}{N} \sum_{i=1}^{N} (u_i, v_i) - \langle \bar{u}, \bar{v} \rangle, \]

and the quantities

\[ X(t) = B(x(t), x(t)), \quad V(t) = B(v(t), v(t)), \]

representing the spread in space and velocity of the system (2). The following result provides sufficient conditions to ensure that consensus towards mean-velocity is reached.

**Theorem 3.1** (Ha-Ha-Kim [45]). Let \((x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N\) be such that \(X_0 = B(x_0, x_0)\) and \(V_0 = B(v_0, v_0)\) satisfy

\[ \gamma(X_0) := \sqrt{N} \int_{\sqrt{2r}/N}^{\infty} a(\sqrt{2r}) dr > \sqrt{V_0}. \] (6)

Then the solution of the Cucker-Smale system with initial data \((x_0, v_0)\) tends to consensus, i.e., \(\lim_{t \to \infty} V(t) = 0.\)

The inequality (6) defines a region in the space \((X, V)\) of initial conditions, which will lead to consensus. We call this set consensus region. If the rate of
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communication function $a$ is integrable, i.e., far distant agents are influencing very weakly the dynamics, i.e., the system has a homophilious regime, then such a region is essentially bounded, and actually not all initial conditions will realize self-organization, as the following example shows.

**Example 3.2** (Non-consensus events – Caponigro-Fornasier-Piccoli-Trélat [13]).
Consider $N = 2$, $d = 1$, $\beta = 1$, and $x(t) = x_1(t) - x_2(t)$, $v(t) = v_1(t) - v_2(t)$ relative position and velocity of two agents on the line. Then

\[
\begin{aligned}
\dot{x} &= v \\
\dot{v} &= -\frac{v}{1 + x^2}
\end{aligned}
\]

with initial conditions $x(0) = x_0$ and $v(0) = v_0 > 0$. By direct integration

\[ v(t) = -\arctan x(t) + \arctan x_0 + v_0. \]

Hence, if $\arctan x_0 + v_0 > \pi/2 + \varepsilon$ we have

\[ v(t) > \pi/2 + \varepsilon - \arctan x(t) > \varepsilon, \quad \forall t \in \mathbb{R}_+, \]

and no consensus towards mean-velocity can be reached.

Besides this mathematical model of failure of self-organization, it is also common experience that coherence in a homophilious society can be lost, leading sometimes to dramatic consequences, questioning strongly the role and the effectiveness of governments, which may be called to restore social stability.

First of all, it is very important to mention and to stress it very much that decentralized rules of control are in general doomed to fail, as we clarified in [9]. In fact, although we allow agents to self-steer towards consensus according to additional decentralized rules computed with local information, their action results in general in a minor modification of the initial homophilious model with no improvement in terms of promoting unconditional consensus formation! Hence, blindly insisting and believing on decentralized control is certainly fascinating, but rather wishful, as it does not secure pattern formation.
For this reason, in the work [13] we explored the conditions, under which alignment promoting systems such as (2) can be stabilized by means of centralized feedback controls. Let us consider then any system of the type

\[ \begin{align*}
\dot{x} &= v \\
\dot{v} &= -L^a(x)v + u,
\end{align*} \tag{7} \]

where the main state of the group of \( N \) agents is given by the \( N \)-tuple \( x = (x_1, \ldots, x_N) \). Similarly for the consensus parameters \( v = (v_1, \ldots, v_N) \). The vector \( u = (u_1, \ldots, u_N) \) represents the external controls, such that \( \sum_{i=1}^N \|u_i(t)\| \leq M \).

The constant \( M > 0 \) represents the budget of strength of the control force that we can apply on the system and it is of course bounded, likely \( M \) is supposed to be very small.

The space of main states and the space of consensus parameters is \((\mathbb{R}^d)^N\) for both; the matrix \( L^a(x) \) is again the Laplacian of the \( N \times N \) adjacency matrix \((a(||x_j - x_i||)/N)_{i,j=1}^N\), where \( a \in C^1([0, +\infty)) \) is an arbitrary nonincreasing, positive, and bounded function. The aim is then to find admissible controls steering the system to the consensus region starting from any possible initial condition.

\[ \begin{align*}
&\text{Proposition 3.3 (Total Control – Caponigro-Fornasier-Piccoli-Trélat [13]). For any initial datum } (x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \text{ and } M > 0 \text{ there exist } T > 0 \text{ and } u : [0, T] \to (\mathbb{R}^d)^N, \text{ with } \sum_{i=1}^N \|u_i(t)\| \leq M \text{ for every } t \in [0, T] \text{ such that the associated solution reaches the consensus region in time } T > 0. \\
\text{Proof. Let us mention that the Laplacian } L^a(x) \text{ is a nonnegative operator, not only with respect to the Euclidean product, but also with respect to the bilinear form } B(\cdot, \cdot), \text{ hence } B(L^a(x)v, v) \geq 0. \text{ Consider a solution of the system with initial data } (x_0, v_0) \text{ associated with a feedback control } u = -\alpha(v - \bar{v}), \text{ with } 0 < \alpha \leq M/(N\sqrt{B(v_0, v_0)}). \text{ In particular it holds } \sum_{i=1}^N \|u_i\| \leq M \text{ and the control is admissible. Then}
\end{align*} \]

\[ \begin{align*}
\frac{d}{dt}V(t) &= \frac{d}{dt}B(v(t), v(t)) \\
&= -2B(L^a(x)v(t), v(t)) + 2B(u(t), v(t)) \\
&\leq 2B(u(t), v(t)) = -2\alpha B(v - \bar{v}, v - \bar{v}) = -2\alpha V(t).
\end{align*} \]
Therefore $V(t) \leq e^{-2\alpha t}V(0)$ and $V(t)$ tends to 0 exponentially fast as $t \to \infty$. In particular $X(t)$ keeps bounded and the trajectory reaches the consensus region in finite time.

This result, although very simple, is somehow remarkable. Not only it shows that we can steer to consensus the system from any initial condition, but that the strength of the control $M > 0$ can be arbitrarily small!

However, this result has perhaps only theoretical validity, because the chosen stabilizing control $u = -\alpha(v - \bar{v})$ needs to act instantaneously on all the agents and needs to be informed of the status of the entire system. For that, we may call it total control. We wonder then, whether we can stabilize the system by means of more parsimonious interventions, which are more realistically modeling actual governments. From the instructive proof above, we learn that a good strategy to steer the system to consensus is actually the minimization of $B(u(t), v(t))$ with respect to $u$, for all $t$. However, we may want to impose additional sparsity constraints on the control, which will enforce that most of its component will be zero, modeling its parsimonious interaction with the system. This leads us very quickly into the difficult combinatorial problem of the selection of the best few control components to be activated. How can we solve it?

The problem resembles very much the one in information theory of finding the best possible sparse representation of data in form of vector coefficients with respect to an adapted dictionary [58, Chapter 1] for the sake of their compression. We shall borrow these concepts for investigating best policies in stabilization and control of dynamical systems modeling multi-agent interactions. Beside stabilization strategies in collective behavior already considered in the recent literature, see e.g. [70, 75], the conceptually closest work to our approach is perhaps the seminal paper [57], where externally driven “virtual leaders” are inserted in a collective motion dynamics in order to enforce a certain behavior. Nevertheless our modeling still differs significantly from this mentioned literature, because we allow us directly, externally, and instantaneously to control the individuals of the group, with no need of introducing predetermined virtual leaders, and we shall specifically seek for the most economical (sparsest) control for leading the group towards a certain global behavior. The relationship between control choices and result will be usually highly nonlinear, especially for several known dynamical systems, modeling social dynamics. Were this relationship more simply linear instead, then a rather well-established theory would predict how many degrees of freedom are minimally necessary to achieve the expected outcome. Moreover, depending on certain spectral properties of the linear model, the theory allows also for efficient algorithms to compute the relevant degrees of freedom, relaxing the associated combinatorial problem. This theory is known in mathematical signal processing and information theory under the name of compressed sensing, see the seminal work [12] and [36], see also the review chapter [41]. The major contribution of these papers was to realize that one can combine the power of convex optimization, in particular $\ell_1$-norm minimization, and spectral properties of random linear models in order to achieve optimal results on the ability of $\ell_1$-norm minimization of recovering robustly linearly constrained sparsest solutions. Borrowing a leaf from compressed sensing, we
model sparse stabilization and control strategies by penalizing the class of vector valued controls \( u = (u_1, \ldots, u_N) \in (\mathbb{R}^d)^N \) by means of a mixed \( \ell_1^N - \ell_2 \)-norm, i.e.,
\[
\sum_{i=1}^N \|u_i\|. 
\]
This mixed norm has been used for instance in [40] as a joint sparsity constraint and it has the effect of optimally sparsifying multivariate vectors in compressed sensing problems [38]. The use of (scalar) \( \ell_1 \)-norms to penalize controls dates back to the 60’s with the models of linear fuel consumption [25]. More recent work in dynamical systems [79] resumes again \( \ell_1 \)-minimization emphasizing its sparsifying power. We also refer to the recent and very appropriate paper [69] on sparse stabilization of linear dynamical systems. Also in optimal control with partial differential equation constraints it became rather popular to use \( \ell_1 \)-norms to enforce sparsity of controls, for instance in the modeling of optimal placing of actuators or sensors [18, 21, 22, 47, 73, 80]. In the light of the sparsifying power of \( \ell_1 \)-minimization we showed the following result.

**Theorem 3.4** (Greedy Control – Caponigro-Fornasier-Piccoli-Trélat [13]). For every initial condition \( (x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \) and \( M > 0 \) there exist \( T > 0 \) and a sparse piecewise constant control \( u: [0, T] \to (\mathbb{R}^d)^N \), with \( \sum_{i=1}^N \|u_i(t)\| \leq M \) for every \( t \in [0, T] \) such that the associated AC solution reaches the consensus region at the time \( T \). More precisely, we can choose adaptively the control law explicitly as one of the solutions of the variational problem
\[
\min B(v, u) + \frac{\gamma(x)}{N} \sum_{i=1}^N \|u_i\| \quad \text{subject to} \quad \sum_{i=1}^N \|u_i\| \leq M, \tag{8}
\]
where \( \gamma(x) = \sqrt{N} \int_0^{\sqrt{N}B(x, x)} a(\sqrt{2}r)dr \), the threshold in Theorem 3.1.

This choice of the control makes \( V(t) = B(v(t), v(t)) \) vanishing in finite time, in particular there exists \( T \) such that \( B(v(t), v(t)) \leq \gamma(x)^2, t \geq T. \)

Notice that the variational principle (8) is balancing the minimization of \( B(u, v) \), which we mentioned above as relevant to promote convergence to consensus, and the \( \ell_1 \)-norm term \( \sum_{i=1}^N \|u_i\| \) expected to promote sparsity.

Define \( U(x, v) \) the sets of solution controls to (8). For \( (x, v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \) and \( u(x, v) \in U(x, v) \) there exist nonnegative real numbers \( \alpha_i \)'s such that
\[
u_i(x, v) = \begin{cases} 0 & \text{if } v_{\perp i} = 0, \\ -\alpha_i \frac{v_{\perp i}}{\|v_{\perp i}\|} & \text{if } v_{\perp i} \neq 0, \end{cases}
\]
where \( 0 \leq \sum_{i=1}^N \alpha_i \leq M \) and \( v_{\perp} = v - \bar{v} \). The space \( (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \) can be partitioned in the disjoint union of subsets \( \mathcal{C}_i, \ldots, \mathcal{C}_4 \) defined by
\[
\mathcal{C}_1 = \{(x, v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \mid \max_{1 \leq i \leq N} \|v_{\perp i}\| < \gamma(B(x, x)) \}. 
\]
Figure 4. Shepherd dog strategy, controlling the most “stubborn”.

\[ C_2 = \{ (x,v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \mid \max_{1 \leq i \leq N} \|v_{\perp i}\| = \gamma(B(x,x)) \}, \]

\[ C_3 = \{ (x,v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \mid \max_{1 \leq i \leq N} \|v_{\perp i}\| > \gamma(B(x,x)) \text{ and there exists a unique } i \in \{1, \ldots, N\} \text{ such that } \|v_{\perp i}\| > \|v_{\perp j}\| \text{ for every } j \neq i \}, \]

\[ C_4 = \{ (x,v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \mid \max_{1 \leq i \leq N} \|v_{\perp i}\| > \gamma(B(x,x)) \text{ and there exist } k \geq 2 \text{ and } i_1, \ldots, i_k \in \{1, \ldots, N\} \text{ such that } \|v_{\perp i_1}\| = \cdots = \|v_{\perp i_k}\| \text{ and } \|v_{\perp i_1}\| > \|v_{\perp j}\| \text{ for every } j \neq \{i_1, \ldots, i_k\} \}. \]

The subsets \( C_1 \) and \( C_3 \) are open, and the \( L^{2Nd}((C_1 \cup C_3)^e) = 0 \). For \( (x,v) \in C_1 \cup C_3 \), the set \( U(x,v) \) is single valued: \( U|_{C_1} = \{0\} \) and \( U|_{C_3} = \{(0, \ldots, 0, -Mv_{\perp i}/\|v_{\perp i}\|, 0, \ldots, 0)\} \) for some unique \( i \in \{1, \ldots, N\} \). If \( (x,v) \in C_2 \cup C_4 \) then a control in \( U(x,v) \) may chosen not be sparse.

However, there is one choice of control, which is always solution of (8): If \( \|v_{\perp i}\| \leq \gamma(x) \) for every \( i = 1, \ldots, N \), then the consensus region is actually reached by (6), and the only solution in \( U(x,v) \) is actually

\[ u_1 = \cdots = u_N = 0. \]

In fact no control is needed in this case. Otherwise there exists some index \( i \in \{1, \ldots, N\} \) such that

\[ \|v_{\perp i}\| > \gamma(x) \quad \text{and} \quad \|v_{\perp j}\| \geq \|v_{\perp j}\| \quad \text{for every } j = 1, \ldots, N, \]

and we can choose in \( U(x,v) \) the control law

\[ u_i = -M \frac{v_{\perp i}}{\|v_{\perp i}\|}, \quad \text{and} \quad u_j = 0, \quad \text{for every } j \neq i. \]

Hence, figuratively, this control acts on the most “stubborn”. We may call this control the “shepherd dog strategy”. We reiterate that this choice of the control makes \( V(t) = B(v(t), v(t)) \) vanishing in finite time, hence there exists \( T \) such that \( B(v(t), v(t)) \leq \gamma(x)^2, t \geq T \).
Figure 5. For $|v| \leq \gamma$ the minimal solution $u \in [-M, M]$ is zero.

Figure 6. For $|v| > \gamma$ the minimal solution $u \in [-M, M]$ is for $|u| = M$.

The geometrical interpretation of the solution of (8) is given by the graphics in Figure 5 and Figure 6 above representing the scalar situation. This result is truly remarkable, because it holds again independently of the initial conditions and of the strength $M > 0$ of the control. Moreover, it indicates the controllability of consensus systems simply by acting on one agent at each time and for nontrivial intervals of time.

Surprisingly enough, the shepherd dog strategy is optimal for consensus problems with respect to any other control strategy, which spreads control over multiple agents.

**Theorem 3.5** (Optimality of the “shepherd dog strategy” – Caponigro-Fornasier-Piccoli-Trélat [13]). Consider generic control $u$ of components

$$
u_i(x, v) = \begin{cases} 0 & \text{if } v_{\perp i} = 0, \\ -\alpha_i \frac{v_{\perp i}}{\|v_{\perp i}\|} & \text{if } v_{\perp i} \neq 0, \end{cases}$$

where $\alpha_i \geq 0$ such that $\sum_{i=1}^{N} \alpha_i \leq M$. The 1-sparse control (the shepherd dog strategy) is the minimizer of $\frac{d}{dt} V(t)$ among all the control of the previous form.

This mathematical result actually suggests a remarkable general real-life principle: if a mediator has to help an assembly to reach consensus, then the mediator
should better convince the current most dissenting member of the assembly rather than addressing the entire assembly to average on a common point of view.

Nevertheless, the shepherd dog strategy is only instantaneously optimal, in a some sense instantaneously guided by the greedy principle (8). If we were allowed to reliably simulate future developments of the system, it would be better to consider instead a sparse open loop optimal control. The problem is then to minimize, for a given $\gamma > 0$

$$J(u) = \int_0^T \frac{1}{N} \sum_{i=1}^{N} \left( \left( \frac{1}{N} \sum_{j=1}^{N} v_j(t) \right)^2 + \gamma \|u_i(t)\| \right) dt,$$

s.t. $\sum_{i=1}^{N} \|u_i(t)\| \le M, \quad t \in [0, T],$

where the state is a trajectory of the controlled system

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j - x_i\|)(v_j - v_i) + u_i, \end{cases}$$

with initial conditions

$$(x(0), v(0)) = (x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N.$$

By an analysis of the associated Pontryagin Minimum Principle\(^4\) we could derive the following result, which states the optimality of sparse controls. We report it without technical details.

**Theorem 3.6** (Essential sparsity of optimal controls – Caponigro-Fornasier-Piccoli-Trélat [13]). For “generic” data $(x_0, v_0)$ in $(\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, for every $M > 0$, and for every $\gamma > 0$ the optimal control problem has an optimal solution. The optimal control $u$ is “generically” instantaneously a vector with at most one nonzero coordinate.

### 3.2. Sparse stabilization of systems driven by attraction and avoidance forces.

In the previous section we showed that, if, on the one side, the homophilious character of a society plays against its compactness, on the other side, it may plays at its advantage if we allow for sparse controls. Let us however stress that such results have more far-reaching potential, as they can address also situations, which do not match the structure (7), such as the Cucker and Dong model of cohesion and avoidance (3), where the system has actually the form

$$\begin{cases} \dot{x} = v, \\ \dot{v} = (L^f(x) - L^a(x))x + u, \end{cases}$$

\(^4\)The Pontryagin Minimum Principle (PMP) can be viewed as yet another application of the Lagrange multiplier theorem in Banach spaces to characterize solutions in terms of first order optimality conditions.
where $L^a(x)$ and $L^f(x)$ are graph-Laplacians associated to competing avoidance and cohesion forces respectively. Similar models considering attraction, repulsion and other effects, such as alignment or self-drive, appear in the recent literature and they seem effectively describing realistic situations in nature of conditional pattern formation, see, e.g., some of the most related contributions [16, 19, 29, 37]. For the system without control, i.e., $u \equiv 0$, under certain conditions on the attraction and repulsion forces $a$ and $f$, fulfilled by the examples mentioned in Section 2.2, if the total energy of the system (9) is below the energy threshold $\theta$, then such systems are known to converge autonomously to the stable configuration of keeping confined and collision avoiding in space, uniformly in time.

**Theorem 3.7** (Cucker-Dong [30, Theorem 2.1]). If $E(0) < \theta$, then there exist finite constants $c_0 > 0, C_0 > 0$ such that

$$c_0 \leq \|x_i(t) - x_j(t)\| \leq C_0, \quad \text{for all } t \geq 0.$$ 

As for the model (2) we could construct non-consensus events if one violates the sufficient condition (6), also for the model (3) and in violation of the threshold $E(0) < \theta$, one can exhibit non-cohesion events.

**Example 3.8** (Non-cohesion events – Cucker-Dong [30]). Consider $N = 2, d = 1, \beta > 1, f \equiv 0, b_i \equiv 0$, and $x(t) = x_1(t) - x_2(t)$, $v(t) = v_1(t) - v_2(t)$ relative position and velocity of two agents on the line. Then

$$\begin{cases}
\dot{x} = v \\
\dot{v} = -\frac{x}{(1 + x^2)^{\beta}}.
\end{cases} \quad \text{(10)}$$

If we are given the initial conditions $x(0) = x_0 > 0$ and $v(0) = v_0 > 0$, and

$$v(0)^2 \geq \frac{1}{(\beta - 1)(1 + x(0)^2)^{\beta - 1}},$$

then $x(t) \to \infty$ for $t \to \infty$. In fact, by direct integration in (10) one obtains

$$v(t)^2 = \frac{1}{(\beta - 1)(1 + x(t)^2)^{\beta - 1}} + \left[\frac{v(0)^2 - 1}{(\beta - 1)(1 + x(0)^2)^{\beta - 1}}\right]_{0}^{\Psi_0},$$

and it follows that $v(t) > 0$ for all $t$. This implies that $t \to x(t)$ is increasing. Had this function an upper bound $x_*$, then we would have $\dot{x}(t) = v(t) \geq \frac{1}{(\beta - 1)(1 + x(t)^2)^{\beta - 1}} + \Psi_0$ and $x(t) \to \infty$ for $t \to \infty$, hence a contradiction, and $t \to x(t)$ is unbounded.

Again, one can pose the question of whether, given $E(0) > \theta$ a sparse control can bring in finite time $T$ the energy under the threshold $E(T) < \theta$. We showed in [7] that in the latter situation of lost self-organization, one can nevertheless steer the system (9) to return to stable energy levels by sparse feedback controls.
Definition 3.9. Let \((x, v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N\) and \(0 \leq \epsilon \leq \frac{M}{E(0)}\). We define the sparse feedback control 
\[ u(x, v) = [u_1(x, v), \ldots, u_N(x, v)]^T \in (\mathbb{R}^d)^N \]
associated to \((x, v)\) as
\[ u_i(x, v) = \begin{cases} 
-\epsilon E(x, v) \frac{v_i}{\|v_i\|} & \text{if } i = \hat{i}(x, v), \\
0 & \text{if } i \neq \hat{i}(x, v),
\end{cases} \]
where \(\hat{i}(x, v)\) is the minimal index such that
\[ \|v_{\hat{i}(x, v)}\| = \max_{j=1,\ldots,N} \|v_j\| . \]

We summarize the result as follows.

Theorem 3.10 (Bongini-Fornasier [7]). There exist a constant \(\eta > 1\) and a finite time \(T > 0\) such that, if
\[ \theta < E(0) < \eta \theta, \]
then the energy of controlled system (9) with the sparse feedback control of Definition 3.9 assumes energy value \(E(T) < \theta\). Hence, for \(t \geq T\) the uncontrolled system keeps cohesive and collision avoiding.

Notice that, according to this result and differently from the sparsely controlled Cucker and Smale system, the initial conditions need to fulfil a second threshold (11) for (9) to be sparsely controlled by a sparse feedback control as in Definition 3.9. While we do not dispose presently of results, which state the sparse uncontrollability of the system if \(E(0) > \eta \theta\), one can easily provide counterexamples showing that in general no sparse control as in Definition 3.9 (for any \(M > 0\!\!\!\)!) is able to stabilize the system if \(E(0) > \eta \theta\). This suggests that sparse controllability is in general conditional to the choice of the initial conditions; perhaps this is good news for those who may fear the next coming of a Brave New World :-)

4. Mean-field sparse optimal control

"The Three Theorems of Psychohistorical Quantitivity: The population under scrutiny is oblivious to the existence of the science of Psychohistory. The time periods dealt with are in the region of 3 generations. The population must be in the billions (± 75 billions) for a statistical probability to have a psychohistorical validity."

– Isaac Asimov, Foundation

While in some cases, for instance in the simple models mentioned above, it is possible to describe rather precisely the mechanism of pattern formation, for most of the very complex models the analytical description or the numerical simulation of the asymptotic behavior of a large system of particles can become an impossible task. A classical way to approach the global description of the system is then to focus on its mean behavior. In the classical \textit{mean-field} theory one studies the evolution of a large number of small individuals \textit{freely interacting} with each other,
by simplifying the effect of all the other individuals on any given individual by a single averaged effect. This results in considering the evolution of the particle density distribution in the state variables, leading to so-called mean-field partial differential equations of Vlasov- or Boltzmann-type [66]. We refer to [14] and the references therein for a recent survey on some of the most relevant mathematical aspects on this approach to swarming models. In particular, for our system (1) (for \( S \equiv 0 \)) the corresponding mean-field equations are

\[
\partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [ (H * \mu) \mu ],
\]

where \( \mu(t) \) is the particle probability distribution, i.e., \((x(t), v(t)) \sim \mu(t)\). In addition, to deal with the sparse optimal control of a very large number of agents, where the curse of dimensionality destroys any chance of computational solvability, in [42, 39, 10] mean-field sparse optimal control models have been introduced and analyzed in a rather general setting, which includes both (7) and (9) as possible starting particle formulations. Nevertheless, the proper definition of a limit dynamics when an external control is added to the system and it is supposed to have some sparsity surprisingly remains a difficult task. In fact, the most immediate and perhaps natural approach would be to assign as well to the finite dimensional control \( u \) an atomic vector valued time-dependent measure

\[
\nu_N(t) = \sum_{i=1}^{N} u_i \delta_{(x_i(t), v_i(t))},
\]

and consider a proper limit \( \nu \) for \( N \to +\infty \), leading to the controlled PDE

\[
\partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [ (H * \mu + \nu) \mu ],
\]

(12)

where now \( \nu \) represents an external force field. The sequence \( (\nu_N)_N \) is to be made of minimizers of certain cost functionals over measures, which may allow for the necessary compactness to derive the limit \( \nu_N \to \nu \). For the optimal control problems based on the sparse optimal control of a very large number of agents, where the curse of dimensionality destroys any chance of computational solvability, in [42, 39, 10] mean-field sparse optimal control models have been introduced and analyzed in a rather general setting, which includes both (7) and (9) as possible starting particle formulations. Nevertheless, the proper definition of a limit dynamics when an external control is added to the system and it is supposed to have some sparsity surprisingly remains a difficult task. In fact, the most immediate and perhaps natural approach would be to assign as well to the finite dimensional control \( u \) an atomic vector valued time-dependent measure

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\]

and consider a proper limit \( \nu \) for \( N \to +\infty \), leading to the controlled PDE

\[
\partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [ (H * \mu + \nu) \mu ],
\]

(12)

where now \( \nu \) represents an external force field. The sequence \( (\nu_N)_N \) is to be made of minimizers of certain cost functionals over measures, which may allow for the necessary compactness to derive the limit \( \nu_N \to \nu \). For the optimal control problems based on the \( \ell_1 \)-norm penalization of the controls considered for instance in [13, Section 5], such a limit procedure cannot in general prevent \( \nu \) to be singular with respect to \( \mu \). This means that the interaction of \( \nu \) with \( \mu \) may have effects only at the boundary of the support of \( \mu \). Even if we could consider in (12) an absolutely continuous control \( \nu = f \mu \), we would end up with an equation of the type

\[
\partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [ (H * \mu + f) \mu ],
\]

(13)

where now \( f \) is a force field, which is just an \( L^1 \)-function with respect to the measure \( \mu \). Unfortunately, stability and uniqueness of solutions for equations of the type (13) is established only for fields \( f \) with at least some regularity.

The term “curse of dimensionality” was first introduced by Richard E. Bellman precisely to describe the increasing complexity of high-dimensional optimal control problems.

Here to simplify we assume that \((x_i(t), v_i(t)) \neq (x_j(t), v_j(t))\) a.e. for \( i \neq j \).
4.1. A smooth relaxation of the mean-field sparse optimal control. At this point it seems that our quest for a proper definition of a mean-field sparse optimal control gets to a dead-end, unless we allow for some modeling compromise. A first approach which provides well-posedness actually starts from the equation (13), by assuming the vector valued function \( f(t, x, v) \) being in a proper compact set of a function space of Carathéodory functions, locally Lipschitz continuous functions in \( (x, v) \). More precisely:

**Definition 4.1.** For a horizon time \( T > 0 \), and an exponent \( 1 \leq q < +\infty \) we fix a control bound function \( \ell \in L^q(0, T) \). The class of admissible control functions \( \mathcal{F}_\ell([0, T]) \) is so defined: \( f \in \mathcal{F}_\ell([0, T]) \) if and only if

1. \( f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^d \) is a Carathéodory function,
2. \( f(t, \cdot) \in W^{1, \infty}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^d) \) for almost every \( t \in [0, T] \), and
3. \( |f(t, 0)| + \text{Lip}(f(t, \cdot), \mathbb{R}^d) \leq \ell(t) \) for almost every \( t \in [0, T] \).

Then by proceeding back in a sort of reverse engineering, we reformulate the finite dimensional model, leading to systems of the type

\[
\begin{aligned}
\dot{x}_i &= v_i, \\
\dot{v}_i &= H * \mu_N(x_i, v_i) + f(t, x_i, v_i), \quad i = 1, \ldots, N, \quad t \in [0, T],
\end{aligned}
\]

where now \( f \) is a feedback control. We consider additionally the following assumptions:

- **(H)** Let \( H : \mathbb{R}^{2d} \to \mathbb{R}^d \) be a locally Lipschitz function such that
  \[ |H(z)| \leq C(1 + |z|), \quad \text{for all } z \in \mathbb{R}^{2d}; \]

- **(L)** Let \( L : \mathbb{R}^{2d} \times \mathcal{P}_1(\mathbb{R}^{2d}) \to \mathbb{R}_+ \) be a continuous function in the state variables \( (x, v) \) and such that if \( (\mu_j)_{j \in \mathbb{N}} \subset \mathcal{P}_1(\mathbb{R}^{2d}) \) is a sequence converging narrowly to \( \mu \) in \( \mathcal{P}_1(\mathbb{R}^{2d}) \), then \( L(x, v, \mu_j) \to L(x, v, \mu) \) uniformly with respect to \( (x, v) \) on compact sets of \( \mathbb{R}^{2d} \);

- **(Ψ)** Let \( \psi : \mathbb{R}^d \to [0, +\infty) \) be a nonnegative convex function satisfying the following assumption: there exist \( C \geq 0 \) and \( 1 \leq q < +\infty \) such that
  \[ \text{Lip}(\psi, B(0, R)) \leq CR^{q-1} \]
  for all \( R > 0 \). Notice that \( \psi(\cdot) = |\cdot| \) is an admissible choice (for \( q = 1 \)).

**Theorem 4.2** (Fornasier-Solombrino [42]). Assume that we are given maps \( H, L, \) and \( \psi \) as in assumptions (H), (L), and (Ψ). For \( N \in \mathbb{N} \) and an initial datum \( ((x_0^N)_1, \ldots, (x_0^N)_N, (v_0^N)_1, \ldots, (v_0^N)_N) \in B(0, R_0) \subset (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \), for \( R_0 > 0 \) independent of \( N \), we consider

\[
\min_{f \in \mathcal{F}_\ell} \int_0^T \int_{\mathbb{R}^{2d}} [L(x, v, \mu_N(t, x, v)) + \psi(f(t, x, v))] \, d\mu_N(t, x, v) dt,
\]

where \( \mu_N : [0, T] \times \mathbb{R}^{2d} \to [0, 1] \) is a probability measure.
where \( \mu_N(t,x,v) = \frac{1}{N} \sum_{j=1}^{N} \delta(x_j(t),v_j(t))(x,v) \), constrained by being the solution of

\[
\begin{aligned}
\begin{cases}
\dot{x}_i = v_i, \\
\dot{v}_i = (H \ast \mu_N)(x_i,v_i) + f(t,x_i,v_i), 
\end{cases} 
\end{aligned}
\]

with initial datum \((x(0),v(0)) = (x_0^N,v_0^N)\) and, for consistency, we set

\[\mu_0^N = \frac{1}{N} \sum_{i=1}^{N} \delta((x_0^N)_i,(v_0^N)_i)(x,v)\].

For all \( N \in \mathbb{N} \) let us denote the function \( f_N \in \mathcal{F} \) as a solution of the finite dimensional optimal control problem (15)-(16). If there exists a compactly supported \( \mu_0 \in \mathcal{P}_1(\mathbb{R}^d) \) such that \( \lim_{N \to \infty} W_1(\mu_0^N, \mu_0) = 0 \), then there exists a subsequence \((f_{N_k})_{k \in \mathbb{N}}\) and a function \( f_\infty \in \mathcal{F} \) such that \( f_{N_k} \) converges to \( f_\infty \) in a weak sense and \( f_\infty \) is a solution of the infinite dimensional optimal control problem

\[
\min_{f \in \mathcal{F}} \int_0^T \int_{\mathbb{R}^d} [L(x,v,\mu(t,x,v)) + \psi(f(t,x,v))] \, d\mu(t,x,v) dt,
\]

where \( \mu : [0,T] \to \mathcal{P}_1(\mathbb{R}^d) \) is the unique weak solution of

\[
\frac{\partial \mu}{\partial t} + v \cdot \nabla_x \mu = \nabla_v \cdot [(H \ast \mu + f) \mu],
\]

with initial datum \( \mu(0) := \mu_0^0 \) and forcing term \( f \).

Notice that choosing \( \psi(\cdot) = \|\cdot\| \) in (17) would result in a penalization of the \( L_1 \)-norm of the control, i.e., \( \int_0^T \int_{\mathbb{R}^d} |f(t,x,v)| \, d\mu(t,x,v) dt \), and it is allowed so to promote a “small” support of \( f \).

The meaning of the theorem is that, by solving the PDE-constrained optimal control problem, which is not affected anymore by the curse of dimension for \( N \) very large, one can obtain a control \( f_\infty \) which can correspond to a good approximation to the optimal control for the multi-agent dynamical system for \( N \) exagerately large. However, although \( f_\infty \approx f_N \) and this could legitimate the use of \( f_\infty \) as a control for (14) for \( N \) large, it is not yet clear how stable such an approximation of the control would be in terms of quasi-optimality of the control \( f_\infty \) for (14).

This is a relevant open issue, which needs to be investigated first numerically.

The approach above has been recently explored in [42], where a proof of a simultaneous \( \Gamma \)-limit and mean-field limit of the finite dimensional optimal controls for (14) to a corresponding infinite dimensional optimal control for (13) has been established. We also mention the related work [6] where first order conditions are derived for optimal control problems of equations of the type (13) for Lipschitz feedback controls \( f(t,x,v) \) in a stochastic setting. Such conditions result in a coupled system of a forward Vlasov-type equation and a backward Hamilton-Jacobi

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7 Here \( \mathcal{P}_1(\mathbb{R}^n) \) denotes the set of probability measures on \( \mathbb{R}^n \) of finite first moment.

8 Here \( W_1 \) denotes the 1-Wasserstein distance.
equation, similarly to situations encountered in the context of mean-field games [56] or the Nash certainty equivalence [49]. Certainly, this calls for a renewed enthusiasm and hope for a proper definition of mean-field sparse optimal control, until one realizes that actually the problem of characterizing the optimal controls \( f(t, x, v) \) with the purpose of an efficient and manageable numerical computation may not have simplified significantly. In fact it is not a trivial task to obtain a rigorous derivation and the well-posedness of the corresponding first order conditions as in [6] in a fully deterministic setting. A reliable numerical realization is still an open issue.

4.2. Mixed granular-diffuse mean-field sparse optimal control. Inspired by the successful construction of the coupled \( \Gamma \)- and mean-field-limits in [42] and the multiscale approach in [26, 27], to describe a mixed granular-diffuse dynamics of a crowd, we modify here our modeling not starting anymore from (13), but actually from the initial system (1).

The idea is to add to (1), or, better, to elect \( m \) particular individuals, which interact freely with the \( N \) individuals given above. We denote by \((y, w)\) the space-velocity variables of these new individuals. We can consider these \( m \) individuals as “leaders” of the crowd, while the other \( N \) individuals may be called “followers”.

However, the interpretation given here to the leaders is considering them as few “discrete representatives” of the entire crowd. In particular, we shall assume that we have a small amount \( m \) of leaders/representatives that have a great influence on the population, and a large amount \( N \) of followers which have a small influence on the population.

Then, the dynamics one shall define is

\[
\begin{aligned}
\dot{y}_k &= w_k, \\
\dot{w}_k &= H \ast \mu_N(y_k, w_k) + H \ast \mu_m(y_k, w_k) \quad k = 1, \ldots, m, \quad t \in [0, T], \\
\dot{x}_i &= v_i, \\
\dot{v}_i &= H \ast \mu_N(x_i, v_i) + H \ast \mu_m(x_i, v_i) \quad i = 1, \ldots, N, \quad t \in [0, T],
\end{aligned}
\]

where we considered the additional atomic measure

\[
\mu_m(t) = \frac{1}{m} \sum_{k=1}^{m} \delta(y_k(t), w_k(t)),
\]

supported on the trajectories \( t \mapsto (y_k(t), w_k(t)), k = 1, \ldots, m \). From now on, the notations \( \mu_N \) and \( \mu_m \) for the atomic measures representing followers and leaders respectively will be considered fixed. Up to now, the dynamics of the system is similar to a standard multi-agent dynamics for \( N + m \) individuals, with the only difference that the actions of leaders and followers have different weights on a single individuals, \( \frac{1}{m} \) and \( \frac{1}{N} \), respectively. Let us now add controls on the \( m \) leaders. We

\[\text{9One can generalize this model to the one where different kernels for the interaction between a leader and a follower, two leaders, etc. are considered. All the results of this paper easily generalize to this setting.}\]
obtain the system

\[
\begin{aligned}
&\dot{y}_k = w_k, \\
&\dot{w}_k = H * \mu_N(y_k, w_k) + H * \mu_m(y_k, w_k) + u_k, \\
&\dot{v}_i = H * \mu_N(x_i, v_i) + H * \mu_m(x_i, v_i) \\
&\text{for } k = 1, \ldots, m, \quad t \in [0, T], \\
&\text{for } i = 1, \ldots, N, \quad t \in [0, T],
\end{aligned}
\] (20)

where \( u_k : [0, T] \to \mathbb{R}^d \) are measurable controls for \( k = 1, \ldots, m \), and we define the control map \( u : [0, T] \to \mathbb{R}^{md} \) by \( u(t) = (u_1(t), \ldots, u_m(t)) \) for \( t \in [0, T] \). In this setting, it makes sense to choose \( u \in L^1([0,T],U) \) where \( U \) is a fixed nonempty compact subset of \( \mathbb{R}^{d \times m} \) and \( U \subset B(0, U) \) for \( U > 0 \). Finite-dimensional control problems in this setting are of interest, and we will focus again on a specific class of control problems, namely optimal control problems in a finite-time horizon with fixed final time. We design a sparse control \( u \) to drive the whole population of \( m + N \) individuals to a given configuration. We model this situation by solving the following optimization problem

\[
\min_{u \in L^1([0,T],U)} \int_0^T \left\{ L(y(t), w(t), \mu_N(t)) + \frac{1}{m} \sum_{k=1}^m |u_k(t)| \right\} dt,
\] (21)

where \( L(\cdot) \) is a suitable continuous map in its arguments.

As one of our main results in [39], we showed that, given a control strategy \( u \in L^1([0,1],U) \), it is possible to formally define a mean-field limit of (20) when \( N \to \infty \) in the following sense: the population is represented by the vector of positions-velocities \((y, w)\) of the leaders coupled with the compactly supported probability measure \( \mu \in \mathcal{P}_1(\mathbb{R}^{2d}) \) of the followers in the position-velocity space. Then, the mean-field limit will result in a coupled system of an ODE with control for \((y, w)\) and a PDE without control for \( \mu \). More precisely the limit dynamics will be described by

\[
\begin{aligned}
&\dot{y}_k = w_k, \\
&\dot{w}_k = H * (\mu + \mu_m)(y_k, w_k) + u_k, \\
&\partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [(H * (\mu + \mu_m)) \mu],
\end{aligned}
\] (22)

where the weak solutions of the equations have to be interpreted in the Carathéodory sense. The atoms \((y_k, w_k)\) constituting the support of \( \mu_m \) are interpreted as representatives of the entire distribution \( \mu \), which we are indirectly controlling, by acting directly on its representatives. See Figure 7 from [27] for an example of a dynamics similar to the one of (22) for a multiscale pedestrian crowd mixing a granular discrete part and a diffuse part, where a first order model was considered.

Besides the mean-field limit of (20) to (22) for \( N \to +\infty \), we simultaneously proved a \( \Gamma \)-convergence result, implying that the optimal controls \( u_N^* \) of the finite dimensional optimal control problems (20)-(21) converge weakly in \( L^1([0, T], U) \)
Figure 7. A mixed granular-diffuse crowd leaving a room through a door. This figure was kindly provided by the authors of [27]. Copyright ©2011 Society for Industrial and Applied Mathematics. Reprinted with permission. All rights reserved.

for $N \to +\infty$ to optimal controls $u^*$, which are minimal solutions of

$$
\min_{u \in L^1([0,T],U)} \int_0^T \left\{ L(y(t),w(t),\mu(t)) + \frac{1}{m} \sum_{k=1}^m |u_k(t)| \right\} dt.
$$

(23)

**Theorem 4.3** (Fornasier-Piccoli-Rossi [39]). Denote $X := \mathbb{R}^{2d \times m} \times \mathcal{P}(\mathbb{R}^d)$. Let $H$ and $L$ be maps satisfying conditions (H) and (L) respectively. Given an initial datum $(y^0,w^0,\mu^0) \in X$, with $\mu^0$ compactly supported, $\text{supp}(\mu^0) \subset B(0,R), R > 0$, the optimal control problem

$$
\min_{u \in L^1([0,T],U)} \int_0^T \left\{ L(y(t),w(t),\mu(t)) + \frac{1}{m} \sum_{k=1}^m |u_k(t)| \right\} dt,
$$

has solutions, where the triplet $(y,w,\mu)$ defines the unique solution of

$$
\begin{align*}
\dot{y}_k &= w_k, \\
\dot{w}_k &= H \ast (\mu + \mu_m)(y_k, w_k) + u_k, \quad k = 1, \ldots, m, \quad t \in [0,T], \\
\partial_t \mu + v \cdot \nabla_x \mu &= \nabla_v \cdot [(H \ast (\mu + \mu_m)) \mu],
\end{align*}
$$

with initial datum $(y^0,w^0,\mu^0)$ and control $u$, and

$$
\mu_m(t) = \frac{1}{m} \sum_{k=1}^m \delta_{(y_k(t),w_k(t))}.
$$

Moreover, solutions to the problem can be constructed as weak limits $u^*$ of sequences of optimal controls $u^*_{N_k}$ of the finite dimensional problems

$$
\min_{u \in L^1([0,T],U)} \int_0^T \left\{ L(y_N(t), w_N(t), \mu_N(t)) + \frac{1}{m} \sum_{k=1}^m |u_k(t)| \right\} dt,
$$

where $N_k$ is a sequence of finite dimensional problems.
where $\mu_N(t) = \frac{1}{N} \sum_{i=1}^{N} \delta(x_i,N(t),v_i,N(t))$ and $\mu_{m,N}(t) = \frac{1}{m} \sum_{k=1}^{m} \delta(y_k,N(t),w_k,N(t))$ are the time-dependent atomic measures supported on the trajectories defining the solution of the system

$$
\begin{align*}
\dot{y}_k &= w_k, \\
\dot{w}_k &= H \ast \mu_N(y_k,w_k) + H \ast \mu_{m,M}(y_k,w_k) + u_k \quad k = 1,\ldots,m, \quad t \in [0,T], \\
\dot{x}_i &= v_i, \\
\dot{v}_i &= H \ast \mu_N(x_i,v_i) + H \ast \mu_{m,M}(x_i,v_i) \quad i = 1,\ldots,N, \quad t \in [0,T],
\end{align*}
$$

with initial datum $(y^0,w^0,x^0_N,v^0_N)$, control $u$, and $\mu_0 = \frac{1}{N} \sum_{i=1}^{N} \delta(x_0^i,v_0^i)$ is such that $W_1(\mu_0,\mu^0) \to 0$ for $N \to +\infty$.

This is actually an existence result of solutions for the infinite-dimensional optimal control problem (22)-(23). Differently from the one proposed in [42] though, this model retains the controls only on a finite and small group of agents, despite the fact that the entire population can be very large (here modeled by the limit $N \to +\infty$). Hence, the control is sparse by construction, and, by the stratagem of dividing the populations in two groups and allowing only one of them to have growing size, again we are not anymore exposed to the curse of dimensionality when it comes to numerically solving the corresponding optimal control problem. Moreover, in this case the analysis of the first order optimality conditions, i.e., Pontryagin Minimum Principle (PMP) resulted significantly facilitated. In [10] we defined the Pontryagin Minimum Principle (PMP) for (22)-(23) as a natural mean-field limit of the one for (20)-(21). In particular we showed the following commutative graph in terms of relationships between optimizations and first order optimality conditions. This characterizing principle can now be the basis for numerical computations. This formulation has found already truly remarkable applications, e.g., in the modeling of the evacuation of pedestrians from unknown environment guided by few informed agents [1].

### 4.3. Mean-field sparse feedback controls

We conclude this section mentioning relevant and related approaches towards mean-field (sparse) feedback control. In [68] the authors proved that one can drive to consensus any group of agents...
governed by the kinetic Cucker-Smale model, by means of a sparse centralized control strategy, and this, for any initial configuration of the crowd. Here, sparse control means that the action at each time is limited over an arbitrary proportion of the crowd, or, as a variant, of the space of configurations. Somehow this result is a mean-field version of Theorem 3.4 for the particle model. In [4] and [2] the authors proposed a very original alternative approach towards mean-field feedback control. Their idea is to compute a model-predictive control exclusively based on the binary interaction of two agents only. Then this feedback control is inserted as an additional term in the binary interaction rules. Those are eventually plugged in the Boltzmann hierarchy to derive a corresponding Boltzmann-type equations, which, by grazing collision limit, is shown to approximate a controlled mean-field equation. A similar approach towards sparse mean-field control as [68] is derived in [3], where selective controls are again derived following the Boltzmann hierarchy construction.

5. Inferring interaction rules from observations of evolutions

What are the instinctive individual reactions which make a group of animals forming coordinated movements, for instance a flock of migrating birds or a school of fish? Which biochemical interactions between cells produce the formation of complex structures, like tissues and organs? What are the mechanisms, which induce certain significant changes in a large amount of players in the financial market? For the analysis, but even more crucially for the reliable and realistic numerical simulation of such phenomena, one presupposes a complete understanding and determination of the governing interactions. Unfortunately, except for physical situations where the calibration of the model can be done by measuring the governing forces rather precisely, for some relevant macroscopic models in physics and most of the models in biology and social sciences the governing interactions are far from being precisely determined. In fact, very often in these studies the governing potential energies are just predetermined ad hoc to be able to reproduce, at least approximately or qualitatively, some of the macroscopic effects of the observed dynamics, such as the formation of certain patterns (we introduced above a few examples), but there has been relatively little effort in the applied mathematics literature towards matching data from real-life cases. In the paper [8] we have been concerned with the “mathematization” of the problem of learning or inferring interaction rules from observations of evolutions. The framework we considered is the one of deterministic first order models. In fact, besides the above mentioned very relevant second order models, very popular to describe collective motion, also first order systems have often been used to model multiagent interactions, e.g., in opinion formation [54, 46], vehicular traffic flow [43], pedestrian motion [28], and synchronisation of chemical and biological oscillators in neuroscience [55], to mention a few.
Consider the dynamics
\[ \dot{x}_i(t) = \frac{1}{N} \sum_{j \neq i} a(|x_i - x_j|)(x_j - x_i), \quad i = 1, \ldots, N. \]
with \( a \in X = \{ b : \mathbb{R}_+ \to \mathbb{R} \mid b \in L_\infty(\mathbb{R}_+) \cap W^{1,\infty}_{\text{loc}}(\mathbb{R}_+) \} \). Can we “learn” the interaction function \( a \) from observations of the dynamics?

5.1. A simple least-square formulation of the learning problem. One possible formulation of the problem would be to recast the learning of the interaction function \( a \) into an optimal control, perhaps also a mean-field optimal control problem. The trouble with that formulation is that the resulting variational problem would be highly nonconvex and no optimal solution would be easily accessed. For this fundamental reason we chose for an approximation to \( a \) to seek for a minimizer of the following convex discrete error functional
\[ E_N(\hat{a}) = \frac{1}{T} \int_0^T \left\| \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N (\hat{a}(|x_i(t) - x_j(t)|)(x_i(t) - x_j(t)) - \dot{x}_i(t)) \right\|^2 dt, \]
among all functions \( \hat{a} \in X \). In particular, given a finite dimensional space \( V \subset X \), we consider the minimizer:
\[ \hat{a}_{N,V} = \arg \min_{\hat{a} \in V} E_N(\hat{a}). \] (24)
The fundamental question is

(Q) For which choice of the approximating spaces \( V \in \Lambda \) (we assume here that \( \Lambda \) is a countable family of invading subspaces of \( X \)) does \( \hat{a}_{N,V} \to a \) for \( N \to \infty \) and \( V \uparrow X \) and in which topology should this convergence hold?

As \( \hat{a}_{N,V} \) are minimizing solutions to (24), it is natural to seek for a proper \( \Gamma \)-convergence result [35], i.e., the identification of a \( \Gamma \)-limit for \( E_N \), for \( N \to \infty \), whose unique (!) minimizer is precisely \( a \). Let us first address the construction of such a \( \Gamma \)-limit and then the uniqueness of its minimizer.

5.2. A mean-field \( \Gamma \)-limit and the well-posedness of the learning problem. For initial conditions drawn at random according to a probability distribution \( \mu_0 \), the empirical measure \( \mu_N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i(t)} \) weakly converges for \( N \to \infty \) to the probability measure valued trajectory \( t \to \mu(t) \) satisfying weakly the equation
\[ \partial_t \mu(t) = -\nabla \cdot ((H[a] * \mu(t)) \mu(t)), \quad \mu(0) = \mu_0. \]
where \( H[a](x) = -a(|x|)x \), for \( x \in \mathbb{R}^d \). By means of this mean-field solution, we define the candidate \( \Gamma \)-limit functional by
\[ \mathcal{E}(\tilde{a}) = \frac{1}{T} \int_0^T \int_{\mathbb{R}^d} \left| \left( H[\tilde{a}] - H[a] \right) * \mu(t) \right|^2 d\mu(t)(x)dt, \]
Let us now investigate the uniqueness of the minimizer. By Jensen inequality
\[ E(\hat{a}) \leq \frac{1}{T} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\hat{a}(x-y)| - a(|x-y|)^2 |x-y|^2 d\mu(t)(x)d\mu(t)(y) dt \]
\[ = \frac{1}{T} \int_0^T \int_{\mathbb{R}_+} |\hat{a}(s) - a(s)|^2 s^2 d\varphi(t)(s) dt \]  
(25)
where \( \varphi(t) = (|x-y|\#\mu_x(t) \otimes \mu_y(t)) \). We define the probability measure
\[ \hat{\rho} := \frac{1}{T} \int_0^T \varphi(t) dt. \]  
(26)
Finally we define the weighted measure
\[ \rho(A) := \int_A s^2 d\hat{\rho}(s), \]
Then one can reformulate (25) in a very compact form as follows
\[ E(\hat{a}) \leq \int_{\mathbb{R}_+} |\hat{a}(s) - a(s)|^2 d\rho(s) = \|\hat{a} - a\|^2_{L_2(\mathbb{R}_+, \rho)}. \]  
(27)
To establish coercivity of the learning problem, i.e., the uniqueness of the minimizer of \( E \), it is essential to assume that there exists \( c_T > 0 \) such that also the following additional lower bound holds
\[ c_T \|\hat{a} - a\|^2_{L_2(\mathbb{R}_+, \rho)} \leq E(\hat{a}), \]  
(28)
for all relevant \( \hat{a} \in X \cap L_2(\mathbb{R}_+, \rho) \). This crucial assumption eventually determines also the natural space \( X \cap L_2(\mathbb{R}_+, \rho) \) for the solutions.

5.3. Approximation spaces and the main result. We now define the proper space for the \( \Gamma \)-convergence result to hold and a proper concept of approximating spaces. For \( M > 0 \) and an interval \( K = [0, 2R] \) define the set
\[ X_{M,K} = \{ b \in W^{1}_\infty(K) : \|b\|_{L_\infty(K)} + \|b'\|_{L_\infty(K)} \leq M \}. \]
Additionally for every \( N \in \mathbb{N} \), let \( V_N \) be a closed subset of \( X_{M,K} \) w.r.t. the uniform convergence on \( K \) forming a sequence of spaces with the following uniform approximation property: for all \( b \in X_{M,K} \) there exists a sequence \( (b_N)_{N \in \mathbb{N}} \) converging uniformly to \( b \) on \( K \) and such that \( b_N \in V_N \) for every \( N \in \mathbb{N} \). Then by a \( \Gamma \)-convergence argument, we showed the following approximation result.

**Theorem 5.1** (Bongini-Fornasier-Hansen-Maggioni [8]). Fix \( M \geq \|a\|_{L_\infty(K)} + \|a'\|_{L_\infty(K)} \) for \( K = [0, 2R] \), for \( R > 0 \) large enough. For every \( N \in \mathbb{N} \), let \( V_N \) be a closed subset of \( X_{M,K} \) w.r.t. the uniform convergence on \( K \) forming a sequence with the uniform approximation property. Then the minimizers
\[ \hat{a}_N \in \arg \min_{\hat{a} \in V_N} \mathcal{E}_N(\hat{a}). \]
converge uniformly for $N \to \infty$ to a continuous function $\hat{a} \in X_{M,K}$ such that $\mathcal{E}(\hat{a}) = 0$. If we additionally assume the coercivity condition (28), then $\hat{a} = a$ in $L_2(\mathbb{R}+, \rho)$. Moreover, in this latter case, if there exist rates $\alpha, \beta > 0$, constants $C_1, C_2 > 0$, and a sequence $(a_N)_{N \in \mathbb{N}}$ of elements $a_N \in V_N$ such that

$$\|a - a_N\|_{L_\infty(K)} \leq C_1 N^{-\alpha},$$

and

$$W_1(\mu_0^N, \mu_0) \leq C_2 N^{-\beta},$$

(again $W_1$ here is the 1-Wasserstein distance) then there exists a constant $C_3 > 0$ such that

$$\|a - \hat{a}_N\|_{L_2(\mathbb{R}+, \rho)}^2 \leq C_3 N^{-\min\{\alpha, \beta\}},$$

for all $N \in \mathbb{N}$. In particular, in this case, it is the entire sequence $(\hat{a}_N)_{N \in \mathbb{N}}$ (and not only subsequences) to converge to $a$ in $L_2(\mathbb{R}+, \rho)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure8.png}
\caption{Different reconstructions of an interaction kernel $a$ for different values of $M$. On the left column: the true kernel in white and its reconstructions for different $M$; the brighter the curve, the larger the $M$. On the right column: the true trajectories of the agents in white, the trajectories associated to the reconstructed potentials with the same color.}
\end{figure}
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