Sequents-of-Relations Calculi: Proof Theory for Semiprojective Logics

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November 5 2012

Introduction

Semiprojective Logics

Sequents-of-Relations Calculi

Hypersequents of Relations

Introduction - 1

Fuzzy Logics model reasoning under uncertainty, assigning to each event a value in the real unit interval $[0, 1]$ instead of just 0 and 1.

So, in general, a (propositional) *fuzzy logic* is a logic with signature $\langle \wedge, \vee, \&, \to, 0, 1\rangle$ where each connective is interpreted as a function over $[0, 1]$.

Design choices, essentially, are made as to which properties each connective satisfies, e.g.

- in Gödel logic $x \& y = x \wedge y = \min(x, y)$
- in Nilpotent Minimum logic
$$x \& y = \begin{cases} \min(x, y) & \text{if } x + y > 1 \\ 0 & \text{otherwise} \end{cases}$$

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Proof systems (or just *calculi*) are sets of rules and axioms which, through a truth-preserving processe, allow us to produce true formulas from the axioms.

Calculi for fuzzy logics are the key to applications since they model the way men deal with Mathematics.

In particular, *analytic* calculi (i.e. calculi with rules which read bottom up decompose each formula in simpler formulas) are the main candidates for *inference engines* in fuzzy knowledge representation in Artificial Intelligence.

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Our aim was that to isolate certain features of fuzzy logics which allow us to introduce, for the encompassed logics, analytic calculi in a systematic way (i.e., with an homogeneous procedure)

What we will do:

1. introduce this class of fuzzy logics;
2. present the kind of calculi we will associate to it and the algorithm we have designed to introduce them
3. give a list of the results accomplished in my Master's Thesis, focusing on the most prominent one

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Semiprojective Logics - 1

Definition ([3])

A fuzzy logic $\mathcal{L}$ is said to be *semiprojective* with respect to the first order classical theory $\mathbf{T}$, called the semantic theory of $\mathcal{L}$, if:

1. the constant symbols of the language of $\mathbf{T}$ are the constant symbols of $\mathcal{L}$ and the only function symbols are the unary function symbols $u_1, \ldots, u_m$ corresponding to some of the unary connectives of $\mathcal{L}$
2. the set
$$\Pi_1^{\mathbf{T}} = \{\varphi \mid \mathbf{T} \models \varphi \text{ and } \varphi \text{ is universal}\}$$
is decidable
3. there is a simple formula (i.e., quantifier-free, only conjunctions and disjunctions) Des$(x)$of $\mathbf{T}$ such that for every formula φ of $\mathcal{L}$
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$$\mathbf{T} \models \mathrm{Des}(\varphi) \iff \mathcal{L} \models \varphi$$

Semiprojective Logics - 2

4. for every $\square \in \{\circ, u_p(\circ), u_p(u_q) \mid \circ$ connective of $\mathcal{L}\}$ the truth function of $\square$ has the form

$$\widetilde{\square}(x_1, \ldots, x_n) = \begin{cases} t_1 & \text{if } P^1_\square(s_1, \ldots, s_h) \\ \vdots & \vdots \\ t_k & \text{if } P^k_\square(s_1, \ldots, s_h) \end{cases}$$

where t_i and s_i are either constants, variables in $\{x_1, \ldots, x_n\}$ or terms of the form $u_p(x)$ or $u_p(c)$, where $x \in \{x_1, \ldots, x_n\}$ and c is a constant, and $P^1_\square, \ldots, P^k_\square$ are *simple* formulas satisfying

Totality: $\mathbf{T} \models \forall x_1 \cdots \forall x_n \bigvee_{1 \leq i \leq m} P^i_\square$

Functionality: for all $i \neq j, i, j \in \{1, \ldots, m\}$,

$$\mathbf{T} \models \forall x_1 \cdots \forall x_n \neg(P^i_\square \wedge P^j_\square)$$

Examples of Semiprojective Logics - 1

Gödel logics with an involutive negation $\sim$

$$x \wedge y = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } y < x \end{cases} \qquad \sim(x \wedge y) = \begin{cases} \sim x & \text{if } x \leq y \\ \sim y & \text{if } y < x \end{cases}$$

$$x \vee y = \begin{cases} y & \text{if } x \leq y \\ x & \text{if } y < x \end{cases} \qquad \sim(x \vee y) = \begin{cases} \sim y & \text{if } x \leq y \\ \sim x & \text{if } y < x \end{cases}$$

$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } y < x \end{cases} \qquad \sim(x \to y) = \begin{cases} 0 & \text{if } x \leq y \\ \sim y & \text{if } y < x \end{cases}$$

$$\sim(\sim x) = x$$

This logic is semiprojective with respect to the semantic theory $\mathbf{T}_{\leq,<,\sim}$ axiomatizing the bounded total orders with an involutive and order-reversing unary operation $\sim$

Nilpotent Minimum logic

$$x\&y = \begin{cases} 0 & \text{if } x \leq \sim y \\ x & \text{if } \sim y < x \wedge x \leq y \\ y & \text{if } \sim y < x \wedge y < x \end{cases} \qquad \sim(x\&y) = \begin{cases} 1 & \text{if } x \leq \sim y \\ \sim x & \text{if } \sim y < x \wedge x \leq y \\ \sim y & \text{if } \sim y < x \wedge y < x \end{cases}$$

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This logic is semiprojective with respect $\mathbf{T}_{\leq,<,\sim}$

Projective Logics

Semiprojective logics with function-free semantic theories, introduced in [1].

1. Gödel logics

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$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } y < x \end{cases}$$

It is projective with respect to the theory $\mathbf{T}_{\leq,<}$ ($\mathbf{T}_{\leq,<,\sim}$ without the operation $\sim$)

2. Every finite-valued logic

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Sequents-of-Relations Calculi

The main objects of the calculi we will introduce are *sequents of relations*.

Given a semiprojective logic $\mathcal{L}$ with semantic theory $\mathbf{T}$, a sequent of relations is a multiset of the form

$$R_{i_1}(\varphi^1_1, \ldots, \varphi^1_{r_1}) \mid \ldots \mid R_{i_k}(\varphi^k_1, \ldots, \varphi^k_{r_k})$$

where, for $1 \leq j \leq k$, R_{i_j} is a predicate symbol with arity r_j of the semantic theory $\mathbf{T}$, and all φ^i_j are formulas of the logic $\mathcal{L}$.

Each sequent of relations must be interpreted as a disjunction of the components, e.g. the meaning of the object above is "one of the $R_{i_j}(\varphi^j_1, \ldots, \varphi^j_{r_j})$ holds".

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The Logical Rules - 1

In general, we will have a logical rule for each $\Box \in \{\circ, u_p(\circ), u_p(u_q) \mid \circ \text{ connective of } \mathcal{L}\}$, for each predicate symbol R of the semantic theory of $\mathcal{L}$ and for each position p' of the predicate symbol.

So each rule has a tag $(\Box; R; p')$.

Each logical rule $(\Box; R; p')$ is obtained from the particular format of the truth function of $\Box$ in a systematic way.

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The Logical Rules - 2

Take, for example, the truth function of the connective $\to$ in Gödel logic with an involutive negation

$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } y < x \end{cases}$$

and suppose we want to write the rule $(\to, <, \ell)$ (ℓ stands for *left*)

So, the formula we want to produce is $A \to B < C$

According to the truth function of the connective, $x \to y < z$ holds if

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So, the formula we want to produce is $A \to B < C$

According to the truth function of the connective, $x \to y < z$ holds if

when $x \leq y$ then $1 < z$ and
when $y < x$ then $y < z$

Therefore, the premises of the rule are

$$x \leq y \Rightarrow 1 < z \text{ AND } y < x \Rightarrow y < z$$

Removing $\Rightarrow$ and substituting $\vee$ with $|$

$$y < x \mid 1 < z \text{ AND } x \leq y \mid y < z$$

Hence, the rule $(\to, <, \ell)$ is

$$\frac{B < A \mid 1 < C \qquad A \leq B \mid B < C}{A \to B < C} \, (\to, <, \ell)$$

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Axioms and Structural Rules

Axioms

If $\mathbf{T} \models \forall \vec{x} \bigvee_{1 \leq j \leq n} B_j$, where each B_j is an atomic formula and $\vec{x}$ is the vector of free variables in $\bigvee_{1 \leq j \leq n} B_j$, and σ is a substitution of the variables in $\vec{x}$ with formulas of $\mathcal{L}$, then

$$\sigma(B_1) \mid \ldots \mid \sigma(B_n)$$

is an axiom of $\mathbf{R}\mathcal{L}_{\mathbf{T}}$.

Structural rules

$$\frac{\mathcal{H}}{A \mid \mathcal{H}}\,(\mathrm{wk}) \qquad \frac{A \mid A \mid \mathcal{H}}{A \mid \mathcal{H}}\,(\mathrm{cn})$$

where A is an arbitrary atomic relation between formulas of the logic and $\mathcal{H}$ is ana arbitrary (possibly empty) *side sequent*.

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An Example of a Derivation - 1

Suppose we want to know if the formula $p \wedge (p \to q) \leq q$ is true in Godel logic with an involutive negation

What we do is decomposing the formula with the rules read bottom up until only atomic sequents of relations are left; then we check the validity of each of them as formulas of the semantic theory: if they are all true then $p \wedge (p \to q) \leq q$ is also true.

This technique is called *proof-searching* and it is crucial for many applications: our calculi are specifically designed for this task.

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An Example of a Derivation - 2

We begin with

$$\dfrac{p \wedge (p \to q) \leq q}{p \to q < p \mid p \leq q \qquad p \leq p \to q \mid p \to q \leq q}\ (\wedge; \leq; \ell)$$

Then, we decompose the bottom-left formula

$$\dfrac{\underline{p \to q < p} \mid p \leq q}{\underline{q < p \mid 1 < p} \mid p \leq q \qquad \underline{p \leq q \mid q < p} \mid p \leq q}\ (\to; <; \ell)$$

and we obtain two atomic sequents, so here we are done.

We decompose the bottom-right formula

$$\dfrac{p \leq p \to q \mid \underline{p \to q \leq q}}{p \leq p \to q \mid \underline{q < p \mid 1 \leq q} \qquad p \leq p \to q \mid \underline{p \leq q \mid q \leq q}}\ (\to; \leq; \ell)$$

and we see that the sequents we obtain are not atomic.

An Example of a Derivation - 2

We begin with

$$\frac{p \wedge (p \to q) \leq q}{p \to q < p \mid p \leq q \qquad p \leq p \to q \mid p \to q \leq q}\ (\wedge; \leq; \ell)$$

Then, we decompose the bottom-left formula

$$\frac{\underline{p \to q < p} \mid p \leq q}{\underline{q < p \mid 1 < p} \mid p \leq q \qquad \underline{p \leq q \mid q < p} \mid p \leq q}\ (\to; <; \ell)$$

and we obtain two atomic sequents, so here we are done.

We decompose the bottom-right formula

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An Example of a Derivation - 3

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As it is easily seen, each remaining sequent is valid in the semantic theory $\mathbf{T}_{\leq,<,\sim}$, since it is the first order theory axiomatizing bounded **total** orders. So, the formula $p \wedge (p \to q) \leq q$ is true.

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Properties of Sequents-of-Relations Calculi

1. Analiticity (i.e. every subformula of the conclusion of a rule is a subformula of the premises of the same rule)
2. Cut-free (i.e. no cut rule among the rules)
3. *Sound and invertible* rules (this implies that each calculus has the same expressive power of the associated semiprojective logic)
4. The rules respect the subformula property *up to* the symbols $u_1, \ldots, u_m$
5. If it is possible to recognize the axioms of $\mathbf{R}\mathcal{L}$ in polynomial time, then the set

$$\{\varphi \in FORM(\mathcal{L}) \mid \mathcal{L} \models \varphi\}$$

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My Master's Thesis

In my work I have tackled the following problems:

1. providing a polynomial time algorithm for the recognition of the axioms of the sequents-of-relations calculi for logics semiprojective with respect to the semantic theory $\mathbf{T}_{\leq,<,\sim}$
2. extending a cut-elimination result already available for Gödel logics to Gödel logics with an involutive negation
3. proving that *ordinal sum construction* and the *rotation construction* preserve the semiprojectivity of the logics involved
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The main drawback of semiprojective logics is that each one has a locally finite equivalent algebraic semantics.

This implies that fundamental fuzzy logics like Łukasiewicz logic, Product logic and BL logic are not semiprojective, since their equivalent algebraic semantics is not locally finite.

Since we want to preserve the *semiprojective approach* (i.e. focusing on semantic theories which describe the truth functions of the connectives), we see from the truth function of the connective & of Łukasiewicz logic

$$x\&y = \begin{cases} x+y-1 & \text{if } 0 < x+y-1 \\ 0 & \text{if } x+y-1 \leq 0 \end{cases}$$

that we are forced to include binary functions in the language of semantic theories.

Sequents-of-Relations Calculi: Proof Theory for Semiprojective Logics

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In order to do so, we change the object of our calculi.

Definition

An *hypersequent of relations* of the first order classical theory $\mathbf{T}$, semantic theory of the logic $\mathcal{L}$, is an object of the form

$$R_{i_1}(\Gamma^1_1, \ldots, \Gamma^1_{r_{i_1}}) \mid \ldots \mid R_{i_s}(\Gamma^s_1, \ldots, \Gamma^s_{r_{i_s}})$$

where for all $1 \leq j \leq k$: $i_j \in \{1, \ldots, n\}$, R_{i_ℓ} is a relation of $\mathbf{T}$, r_ℓ is the arity of R_{i_ℓ}, and all Γ^i_j are *multisets* of formulas of $\mathcal{L}$.

Since a multiset of formulas is a set of formulas $\{\varphi_1, \ldots, \varphi_n\}$ in which repetitions are allowed, we can consider semantic theories with a *single binary function* which will be interpreted, in our calculi, as the comma between formulas in our multisets.

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Who is the comma?

The problem is that we may need more than one binary function symbol, such as in the case of Product logic, where

$$x\&y = x \cdot y \text{ (product of reals)} \qquad x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{if } y < x \end{cases}$$

Question: which one shall be interpreted as the comma?

Answer: anyone, since it is true that

$$y/x \lhd z \iff y \lhd z \cdot x$$

where $\lhd \in \{\leq, <\}$, so every one can be replaced by the other one.

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Hyperprojective Logics - 1

So, informally, we shall say that a logic is *hyperprojective* if it is semiprojective with respect to a semantic theory $\mathbf{T}$ with binary function symbols $f, g_1, \ldots, g_k$ such that $\mathbf{T}$ satisfies conditions allowing us to substitute each occurrence of the undesired binary functions $g_1, \ldots, g_k$ with terms involving only the designated binary function f, which shall be interpreted in the calculus as the comma.

As it is expected, for hyperprojective logics is available a procedure for introducing analytic calculi based on hypersequents of relations which is similar to the one we have for semiprojective logics (the main differences are the extra steps made for eliminating the functions $g_1, \ldots, g_k$ and the introduction of multisets of formulas instead of just formulas).

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Hyperprojective Logics - 2

These calculi enjoy all the previously mentioned properties of sequents-of-relations calculi, in particular the one which says that the logic is Co-NP provided there is a polynomial time algorithm for the recognition of the axioms of the calculus, and they also generalize calculi already existing in literature but designed for specific logics (see [2] and [4]).

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Open Problems

1. Formal investigation of the expressive power of sequents-of-relations calculi: a sequents-of-relations calculus is associated to every semiprojective logic but is the contrary true? If not, what can not be formalized within this framework?
2. Find interesting examples of hyperprojective logics, e.g. can MTL logic be captured by this notion? If so, how?

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Thank you for your attention!

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