Blind Demixing and Deconvolution at Near-Optimal Rate

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A problem in Wireless Communication

- $r$ different devices
- each device wants to deliver a message $m_i \in \mathbb{C}^N$
- **Channel model:**
  Only few active paths $w_i = Bh_i$, where $B \in \mathbb{C}^{L \times K}$
- **Linear encoding:**
  $x_i = C_i m_i$ with $C_i \in \mathbb{C}^{L \times N}$
  Device $i$ transmits $x_i$
- **Received signal:**

\[
y = \sum_{i=1}^{r} w_i \ast x_i \in \mathbb{C}^L
\]

**Goal:** recover all $m_i$ from $y$
Assumptions on $B_i$ and $C_i$

- Assume $w_i$ is concentrated on the first few entries (most direct paths)
- $B$: First $K$ columns of the $L \times L$ identity
  $\Rightarrow$ extends $h_i$ by zeros
- (More general models for $B$ are possible.)
- Choice of $C_i$ arbitrary $\Rightarrow$ randomize
- Choose $C_i$ to have i.i.d. standard complex normal entries, i.e.,
  $$(C_i)_{jk} \sim \mathcal{CN}(0, 1)$$
Lifting

- \( w_i \ast x_i = Bh_i \ast C_i m_i \) bilinear in \( h_i \) and \( m_i \)
- \( \Rightarrow \) There is a unique linear map \( A_i : \mathbb{C}^{K \times N} \rightarrow \mathbb{C}^L \) such that
  \( Bh_i \ast C_i m_i = A_i (h_i m_i^*) \) for arbitrary \( h_i \) and \( m_i \)

\[
y = \sum_{i=1}^{r} A_i (h_i m_i^*) = A (X^0) ,
\]

where

\[
X^0 = (h_1 m_1^*, \ldots, h_r m_r^*)
\]

- **Low rank matrix recovery problem**
- Corresponding combinatorial problem NP-hard in general
  \( \rightarrow \) convex relaxation
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A convex approach for recovery

[Ling, Strohmer 2017]

- Solve the semi-definite program

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{r} \| Y_i \|_* \\
\text{subject to} & \quad \sum_{i=1}^{r} A_i (Y_i) = y. \\
\end{align*}
\]  

(SDP)

- \( \| \cdot \|_* \): nuclear norm, i.e., the sum of the singular values

- Recovery is guaranteed with high probability, if

\[
L \geq C r^2 \left( K + \mu_h^2 N \right) \log^3 L \log r
\]

- \( \mu_h \) coherence parameter, ranges between \( 1 \leq \mu_h^2 \leq K \)

- (Near-)optimal dependence on \( K, N \), suboptimal dependence on \( r \).

- Previously established for \( r = 1 \) in [Ahmed, Recht, Romberg 2015]
Main result

Theorem (Jung, Krahmer, S., 2017)

Let $\omega \geq 1$. Assume that

$$L \geq C_\omega r \left( K \log K + N \mu_h^2 \right) \log^3 L,$$

where $C_\omega$ is a universal constant only depending on $\omega$. Then with probability $1 - O(L^{-\omega})$ the recovery program (SDP) is successful, i.e., $X^0$ is its unique minimizer.

- (Near) optimal dependence on $K$, $N$, and $r$
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Proof overview

Our proof follows the same strategy as [Ling, Strohmer 2016]. It consists of the following two main steps:

- Establishing sufficient conditions for recovery
  ⇒ approximate dual certificate
- Constructing the dual certificate ("Golfing Scheme")
Proof sketch

Proof overview

Our proof follows the same strategy as [Ling, Strohmer 2016]. It consists of the following two main steps:

- Establishing sufficient conditions for recovery \( \Rightarrow \) approximate dual certificate
- Constructing the dual certificate ("Golfing Scheme")

The following subspace is important for both steps of the proof:

\[
\mathcal{T} = \left\{ \left( u_1 m_1^* + h_1 v_1^*, \ldots, u_r m_r^* + h_r v_r^* \right) : u_1, \ldots, u_r \in \mathbb{C}^K, v_1, \ldots, v_r \in \mathbb{C}^N \right\}
\]

\( \mathcal{T}_i \) is defined by

\[
\mathcal{T}_i = \left\{ um_i^* + h_i v^* : u \in \mathbb{C}^K, v \in \mathbb{C}^N \right\}.
\]
Proof sketch

Local Isometry Property

- Crucial ingredient for the proof:

**Definition**
We say that $A$ fulfills the $\delta$-local isometry property, if

$$
(1 - \delta) \sum_{i=1}^{r} \|X_i\|_F^2 \leq \left\| \sum_{i=1}^{r} A_i(X_i) \right\|_{\ell_2}^2 \leq (1 + \delta) \sum_{i=1}^{r} \|X_i\|_F^2
$$

for all $X = (X_1, \ldots, X_r) \in \mathcal{T}$. 
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\]

for all \( X = (X_1, \ldots, X_r) \in T \).

- Our goal: Show that \( A \) fulfills the local isometry property, if \( L \) scales linearly with \( r \)
**Proof sketch**

**Local isometry property**

- Define $\hat{T} = \{ X = (X_1, \ldots, X_r) \in T : \sum_{i=1}^{r} \| X_i \|_F^2 = 1 \}$
- The $\delta$-local isometry property is equivalent to

$$\delta \geq \sup_{X \in \hat{T}} \left| \sum_{i=1}^{r} A_i (X_i) \|_2^2 - \sum_{i=1}^{r} \| X_i \|_F^2 \right|$$

$$= \sup_{X \in \hat{T}} \left| \sum_{i=1}^{r} A_i (X_i) \|_2^2 - \mathbb{E} \left[ \sum_{i=1}^{r} A_i (X_i) \|_2^2 \right] \right|$$

$$= \sup_{X \in \hat{T}} \left| \| V_X \text{vec}([C_1, \ldots, C_r]) \|_2^2 - \mathbb{E} \left[ \| V_X \text{vec}([C_1, \ldots, C_r]) \|_2^2 \right] \right|,$$

where for $X = (u_1 m_1^* + h_1 v_1^*, \ldots, u_r m_r^* + h_r v_r^*) \in T$

$$V_X \left( \text{vec}([C_1, \ldots, C_r]) \right) = \sum_{i=1}^{r} A_i (X_i)$$
**Suprema of Chaos Processes**

**Theorem (Krahmer, Mendelson, Rauhut 2014)**

Let $\mathcal{X}$ be a symmetric set of matrices, i.e., $\mathcal{X} = -\mathcal{X}$, and let $\xi$ be a random vector whose entries $\xi_i$ are independent and have distribution $\mathcal{CN}(0, 1)$. Then, for $t > 0$,

$$P\left(\sup_{A \in \mathcal{X}} \|A\xi\|_{\ell_2}^2 - \mathbb{E}\|A\xi\|_{\ell_2}^2 \geq c_1 E + t\right) \leq 2 \exp\left(-c_2 \min\left(\frac{t^2}{V^2}, \frac{t}{U}\right)\right)$$

where, setting $D(\mathcal{X}) = \int_0^{+\infty} \sqrt{\log \mathcal{N}(\mathcal{X}, \|\cdot\|_{2\rightarrow2}, t)} dt$ the quantities $E$, $V$, and $U$ are defined as

$$E = D(\mathcal{X})(D(\mathcal{X}) + d_F(\mathcal{X}))$$

$$V = d_{2\rightarrow2}(\mathcal{X})(D(\mathcal{X}) + d_F(\mathcal{X}))$$

$$U = d_{2\rightarrow2}^2(\mathcal{X}).$$
The next steps

- Apply Theorem for $\mathcal{X} = \{ V_X : X \in T; \sum_{i=1}^r \| X_i \|_F^2 = 1 \}$.
- Bound covering numbers

$\implies$ $\delta$-local isometry property holds with high probability