The Convex Geometry of Blind Deconvolution

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July 12, 2019
Joint work Felix Krahmer (TUM),
Funded by the DFG in the context of SPP 1798 CoSIP
Blind deconvolution in imaging

- **Blind deconvolution ubiquitous in many applications:**
  - Imaging: $\mathbf{x}$ signal, $\mathbf{y}$ blur

![Image of blurred and deblurred images]

- **(Circular) convolution of $\mathbf{w}, \mathbf{x} \in \mathbb{C}^L$:**
  
  $$(\mathbf{w} \ast \mathbf{x})_k := \sum_{\ell=1}^{L} w_{k} x_{(\ell-k)} \mod L.$$
Blind deconvolution in wireless communications

- **Task:** deliver message $m \in \mathbb{C}^N$ via unknown channel.
- **Proposed approach:** introduce redundancy before transmission.

- **Linear encoding:**
  
  $x = Cm$ with $C \in \mathbb{C}^{L \times N}$
  
  the signal $x$ is transmitted

- **Channel model:**
  
  only most direct paths are active
  
  $w = Bh$, where $B \in \mathbb{C}^{L \times K}$

- **Received signal:** $e$ noise
  
  $y = wx + e \in \mathbb{C}^L$

- Introduced by Ahmed, Recht, Romberg (IEEE IT ’14)

**Goal:** recover $m$ from $y$
Lifting

• Observation: \( w \ast x = Bh \ast Cm \) is bilinear in \( h \) and \( m \)
  \[ \Rightarrow \] There is a unique linear map \( A : \mathbb{C}^{K \times N} \rightarrow \mathbb{C}^{L} \) such that
  \[ Bh \ast Cm = A(hm^*) \]

for arbitrary \( h \) and \( m \)
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- Thus, the rank 1 matrix \( X_0 = hm^* \) satisfies
  \[ y = A(X_0) + e \]

- Finding \( X_0 \) is a **low rank matrix recovery problem**
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  \[ y = A(X_0) + e \]
- Finding \( X_0 \) is a **low rank matrix recovery problem**
- Ideally find
  \[ \arg\min \ \text{rank} \ X \ \text{subject to} \ ||A(X) - y||_2 \leq \eta \]
- Such problems are NP-hard in general
  \[ \Rightarrow \] try convex relaxation
A convex approach

SDP relaxation (Ahmed, Recht, Romberg ’14)

Solve the semidefinite program (SDP)

\[ \tilde{X} = \text{argmin} \| X \|_* \text{ subject to } \| \mathcal{A}(X) - y \|_2 \leq \eta. \]  

(SDP)

The **nuclear norm** \( \| X \|_* := \sum_{j=1}^{\text{rank}(X)} \sigma_j(X) \) is the sum of all singular values.
A convex approach

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Model assumptions:

- \( y = Bh^* \bar{C}m + e \)
- **Adversarial noise:** \( \|e\|_2 \leq \eta \)
- \( C \in \mathbb{C}^{L \times N} \) has i.i.d. standard Gaussian entries
- \( B \in \mathbb{C}^{L \times K} \) satisfies \( B^*B = \text{Id} \) and is such that \( FB \) (for \( F \) the DFT) has rows of equal norm.
Recovery guarantees

**Theorem (Ahmed, Recht, Romberg ’14)**

Assume

\[
\frac{L}{\log^3 L} \geq C \left( K + N \mu_h^2 \right).
\]

Then with high probability every minimizer \(\tilde{X}\) of (SDP) satisfies

\[
\|\tilde{X} - h m^*\|_F \preceq \sqrt{K + N} \eta.
\]

- \(\mu_h\) coherence parameter (typically small)
Recovery guarantees

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- **Consequences:**
  - **No noise**, i.e., \( \eta = 0 \):  
    \( \rightarrow \) Exact recovery with a near optimal-amount of measurements
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- **Consequences:**
  - **No noise**, i.e., \( \eta = 0 \):
    \( \rightarrow \) Exact recovery with a near optimal-amount of measurements
  - **Noisy scenario**, i.e., \( \eta > 0 \):
    \( \rightarrow \) dimension factor \( \sqrt{K + N} \) appears in the noise
    Does not explain empirical success of (SDP)
Noise robustness in low-rank matrix recovery

- Gaussian measurement matrices (implies RIP) ✓
- phase retrieval ✓
- *blind deconvolution* (this presentation) ？
- matrix completion ？
- Robust PCA ？
- many more... ？

Despite the popularity of convex relaxations for low-rank matrix recovery in the literature, their noise robustness is not well-understood.
Noise robustness in low-rank matrix recovery

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Despite the popularity of convex relaxations for low-rank matrix recovery in the literature, their *noise robustness is not well-understood*. 
What is the problem?

- **Proof technique for these models:**
  - *Idea:* Show existence of (approximate) dual certificate w.h.p.
  - *Golfing scheme* originally developed by D. Gross.
What is the problem?

- **Proof technique for these models:**
  - **Idea:** Show existence of (approximate) dual certificate w.h.p.
  - *Golfing scheme* originally developed by D. Gross.

  - Works well in the noiseless case, where $X_0$ is expected to be the minimizer
  - **Problem:** In noisy models we do not know the minimizer
Are the dimension factors necessary?

**Recall:** We are interested in the scenario $L \ll KN$ and we optimize

$$\tilde{X} = \text{argmin} \| X \|_* \quad \text{subject to} \quad \| A(X) - y \|_2 \leq \eta.$$  

(SDP)

Theorem (Krahmer, DS '19)

There exists an admissible $B$ such that:

With high probability there is $\tau_0 > 0$ such that for all $\tau \leq \tau_0$ there exists an adversarial noise vector $e \in C$ with $\| e \|_2 \leq \tau$ that admits an alternative solution $\tilde{X}$ with the following properties.

• $\tilde{X}$ is feasible, i.e., $\| A(\tilde{X}) - y \|_2 = \tau$

• $\tilde{X}$ is preferred to $h^\star$ by (SDP) i.e., $\| \tilde{X} \|_* \leq \| h^\star \|_*$, but

• $\tilde{X}$ is far from the true solution in Frobenius norm, i.e., $\| \tilde{X} - h^\star \|_F \geq \tau C^3 \sqrt{KNL}$.  

(Dominik Stöger (TUM) | AIP 2019 Grenoble | July 12, 2019 9)
Are the dimension factors necessary?

Recall: We are interested in the scenario \( L \ll KN \) and we optimize
\[
\tilde{\mathbf{X}} = \arg\min \| \mathbf{X} \|_* \quad \text{subject to} \quad \| \mathbf{A}(\mathbf{X}) - \mathbf{y} \|_2 \leq \eta.
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There exists an admissible \( \mathbf{B} \) such that:
With high probability there is \( \tau_0 > 0 \) such that for all \( \tau \leq \tau_0 \) there exists an adversarial noise vector \( \mathbf{e} \in \mathbb{C}^L \) with \( \| \mathbf{e} \|_2 \leq \tau \) that admits an alternative solution \( \tilde{\mathbf{X}} \) with the following properties.

- \( \tilde{\mathbf{X}} \) is feasible, i.e., \( \| \mathbf{A}(\tilde{\mathbf{X}}) - \mathbf{y} \|_2 = \tau \)
- \( \tilde{\mathbf{X}} \) is preferred to \( \mathbf{h}m^* \) by (SDP) i.e., \( \| \tilde{\mathbf{X}} \|_* \leq \| \mathbf{h}m^* \|_* \), but
- \( \tilde{\mathbf{X}} \) is far from the true solution in Frobenius norm, i.e.,
\[
\| \tilde{\mathbf{X}} - \mathbf{h}m^* \|_F \geq \frac{\tau}{\mathcal{C}_3} \sqrt{\frac{KN}{L}}.
\]
What does this mean?

- Assume $K = N$ and $L \approx CK$ up to log-factors

\[ \Rightarrow \| \tilde{X} - hm^* \|_F \gtrsim \tau \sqrt{\frac{KN}{L}} \approx \tau \sqrt{K + N}. \]

up to log-factors

\[ \rightarrow \text{The factor } \sqrt{K + N} \text{ is not a pure proof artifact.} \]

- **Caution:** $\tilde{X}$ might not be the minimizer of (SDP)!

- Analogous result can be shown for matrix completion.
Ideas of the analysis I

\[ x_0 + \ker A \]

- Crucial geometric object: Descent cone for \( x_0 \in \mathbb{C}^{K \times N} \)

\[ \mathcal{K}_\star(x_0) = \left\{ \mathbf{z} \in \mathbb{C}^{K \times N} : \| x_0 + \varepsilon \mathbf{z} \|_\star \leq \| x_0 \|_\star \text{ for some small } \varepsilon > 0 \right\} \]
Ideas of the analysis II

• **Minimum conic singular value:**

\[
\lambda_{\text{min}}(\mathcal{A}, \mathcal{K}_*(X_0)) := \min_{Z \in \mathcal{K}_*(X_0)} \frac{\|\mathcal{A}(Z)\|_2}{\|Z\|_F}
\]

• **Noiseless scenario, i.e., \( \eta = 0 \):**

Exact recovery \( \iff \lambda_{\text{min}}(\mathcal{A}, \mathcal{K}_*(X_0)) > 0 \)

• **Noisy scenario:** Conic singular value controls stability [Chandrasekaran et al. '12]:

\[
\|\tilde{X} - X_0\|_F \leq \frac{2\eta}{\lambda_{\text{min}}(\mathcal{A}, \mathcal{K}_*(X_0))}
\]

(As \( \mathcal{A} \) is Gaussian, \( \lambda_{\text{min}}(\mathcal{A}, \mathcal{K}_*(hm^*)) \approx 1 \) w.h.p., whenever \( L \gtrsim K + N \))
Ideas of the analysis III

Lemma (Krahmer, DS ’19)

There exists $B \in \mathbb{C}^{L \times K}$ satisfying $B^* B = \text{Id}_K$ and $\mu_{\text{max}}^2 = 1$, whose corresponding measurement operator $\mathcal{A}$ satisfies the following:

Let $m \in \mathbb{C}^N \setminus \{0\}$ and let $h \in \mathbb{C}^K \setminus \{0\}$ be incoherent. Then with high probability it holds that

$$
\lambda_{\text{min}}(\mathcal{A}, \mathcal{K}_*(hm^*)) \leq C_3 \sqrt{\frac{L}{KN}}.
$$

• Lemma can be used to prove the previous theorem.
• (Analogous result holds for matrix completion.)
All hope is lost???
Recovery for high noise levels

**Theorem (Krahmer, DS ’19)**

Let $\alpha > 0$. Assume that

$$L \geq C_1 \frac{\mu^2}{\alpha^2} (K + N) \log^2 L.$$  

Then with high probability the following statement holds for all $h \in S^{K-1}$ with $\mu_h \leq \mu$, all $m \in S^{N-1}$, all $\tau > 0$, and all $e \in \mathbb{C}^L$ with $\|e\|_2 \leq \tau$:

Any minimizer $\tilde{X}$ of (SDP) satisfies

$$\|\tilde{X} - hm^*\|_F \leq \frac{C_3 \mu^{2/3} \log^{2/3} L}{\alpha^{2/3}} \max\{\tau; \alpha\}.$$  

→ **Near-optimal recovery guarantees** for high noise-levels.
Proof sketch I

- Descent cone **local** approximation to descent set near $hm^*$.
- **Geometric Intuition:** Close to $\ker A$, the descent set is not pointy.

- Consider the partition $\mathcal{K}_*(hm^*) = \mathcal{K}_1 \cup \mathcal{K}_2$, where
  - $\mathcal{K}_1$ contains all elements in $\mathcal{K}_*(hm^*)$, which are near-orthogonal to $hm^*$
  - $\mathcal{K}_2 := \mathcal{K}_*(hm^*) \setminus \mathcal{K}_1$
Proof sketch II

**Geometric intuition:** No large error can occur in directions belonging to $\mathcal{K}_1$ due to the curved nature of the nuclear norm ball.

- $\lambda_{\min}(\mathcal{A}, \mathcal{K}_2)$ can be bounded from below using *Mendelson’s small-ball method*.
- $\rightarrow$ No large error can occur in these directions.

Combining these two ideas yields the result.

S. Mendelson
Outlook and open questions

• What can we say about the actual minimizer in the scenario of small noise?
• Stability of matrix completion?
Thank you for your attention!