New embedding results for Kondratiev spaces

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Overview

Weighted Sobolev spaces

- $D \subset \mathbb{R}^d$ domain, e.g. with polygonal ($d = 2$) or polyhedral ($d = 3$) structure and singular set S, further $1 < p < \infty$, $m \in \mathbb{N}$ and $a \in \mathbb{R}$
- $\rho : D \longrightarrow [0,1]$... smooth distance to the singular set S of D, in the vicinity of the singular set equal to distance $\mathrm{dist}(x, S)$

Babuska-Kondratiev spaces $\mathcal{K}^m_{a,p}(D)$ on D are defined via the norm:

$$\|u|\mathcal{K}^m_{a,p}(D)\|^p = \sum_{|\alpha|\leq m} \int_D |\rho(x)^{|\alpha|-a} \partial^\alpha u(x)|^p \, dx \, .$$

Extended Bab.-Kondratiev spaces $\mathcal{K}^m_{a,p}(\mathbb{R}^d \setminus S)$ on $\mathbb{R}^d$ are defined by:

$$\|u|\mathcal{K}^m_{a,p}(\mathbb{R}^d \setminus S)\|^p = \sum_{|\alpha|\leq m} \int_{\mathbb{R}^d} |\rho(x)^{|\alpha|-a} \partial^\alpha u(x)|^p \, dx \, .$$

Hansen 2013: There is a linear, bounded Stein extension operator mapping $\mathcal{K}^m_{a,p}(D)$ into $\mathcal{K}^m_{a,p}(\mathbb{R}^d \setminus S)$.

Shift theorem for elliptic operators

Proposition

Let D be some bounded polyhedral domain without cracks in $\mathbb{R}^d$, $d = 2, 3$. Consider the problem

$$-\nabla\big(A(x)\cdot\nabla u(x)\big) = f \quad \text{in} \quad D\,, \qquad u|_{\partial D} = 0\,, \tag{1}$$

where $A = (a_{i,j})_{i,j=1}^d$ is symmetric and

$$a_{i,j} \in \mathcal{K}^m_{0,\infty} = \left\{v : D \longrightarrow \mathbb{C} : \rho^{|\alpha|}\partial^\alpha v \in L_\infty(D)\,, |\alpha| \le m\right\}.$$

Let the bilinear form

$$B(v,w) = \int_D \sum_{i,j} a_{i,j}(x)\partial_i v(x)\partial_j w(x)dx$$

be bounded and coercive in $H^1_0(D)$.

Shift theorem (continued)

Then there exists some $\overline{a} > 0$ such that for any $m \in \mathbb{N}_0$, any $|a| < \overline{a}$ and any $f \in \mathcal{K}^{m-1}_{a-1,2}(D)$ the problem (**??**) admits a uniquely determined solution $u \in \mathcal{K}^{m+1}_{a+1,2}(D)$, and it holds

$$\|u|\mathcal{K}^{m+1}_{a+1,2}(D)\| \leq C\,\|f|\mathcal{K}^{m-1}_{a-1,2}(D)\|$$

for some constant $C > 0$ independent of f.

Shift theorem (continued)

Then there exists some $\overline{a} > 0$ such that for any $m \in \mathbb{N}_0$, any $|a| < \overline{a}$ and any $f \in \mathcal{K}^{m-1}_{a-1,2}(D)$ the problem (**??**) admits a uniquely determined solution $u \in \mathcal{K}^{m+1}_{a+1,2}(D)$, and it holds

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for some constant $C > 0$ independent of f.

Questions:

- Can we embed them into certain suitable function spaces in the class of Besov spaces $B^s_{p,q}(D)$ or Triebel-Lizorkin spaces $F^s_{p,q}(D)$, in particular regarding n-term approximation?
- Can we even find equivalent characterizations or decompositions similar to the ones for function spaces on D?

A first observation

Let's consider the norm of $\mathcal{K}^m_{m,p}(D)$ for $m \in \mathbb{N}$ ...

$$\|u|\mathcal{K}^m_{m,p}(D)\|^p = \sum_{|\alpha|\leq m} \int_D |\rho(x)^{|\alpha|-m}\partial^\alpha u(x)|^p\,dx\,.$$

$$= \|\rho(x)^{-m}u|L_p(D)\|^p + \ldots + \sum_{|\alpha|=m} \|\partial^\alpha u|L_p(D)\|^p$$

$$\sim \|\rho(x)^{-m}u|L_p(D)\|^p + \ldots + \|u|W^m_p(D)\|^p.$$

These two terms play an important role in an equivalent characterization of the so-called

refined localization (Triebel-Lizorkin) spaces $F^{s,\mathrm{rloc}}_{p,q}(\Omega)$.

So let's have a look at them ...

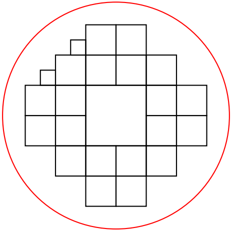
Refined localization spaces - Whitney decomposition

$Q_{j,k}$ open cube, left lower corner $2^{-j}k$, side length $2^{-j}, j \geq 0, k \in \mathbb{Z}^d$

Whitney decomposition of domain Ω ... collection of pairwise disjoint cubes $\{Q_{j,k_\ell}\}_{j\geq 0, \ell=1,\dots,N_j}$ such that

$$\Omega = \bigcup_{j\geq 0} \bigcup_{\ell=1}^{N_j} \overline{Q}_{j,k_\ell}\,, \qquad \mathrm{dist}(2Q_{j,k_\ell}, \partial\Omega) \sim 2^{-j}\,, \quad j \in \mathbb{N}\,,$$

complemented by $\mathrm{dist}(2Q_{0,k_\ell}, \partial\Omega) \geq c > 0$. Example Ω ball:



Refined localization spaces - definition

Let $\{\varphi_{j,\ell}\}$ be a resolution of unity of non-negative C^∞-functions w.r.t. the family $\{Q_{j,k_\ell}\}$, i.e.

$$\sum_{j,\ell} \varphi_{j,\ell}(x) = 1 \text{ for all } x \in \Omega\,, \qquad |\partial^\alpha \varphi_{j,\ell}(x)| \leq c_\alpha 2^{j|\alpha|}\,, \quad \alpha \in \mathbb{N}_0^d\,.$$

Moreover, we require $\operatorname{supp} \varphi_{j,\ell} \subset 2Q_{j,k_\ell}$. Then we define the refined localization spaces $F^{s,\mathrm{rloc}}_{p,q}(\Omega)$ to be the collection of all locally integrable functions f such that (see Triebel 2001, 2008)

$$\|f|F^{s,\mathrm{rloc}}_{p,q}(\Omega)\| = \left(\sum_{j=0}^{\infty}\sum_{\ell=1}^{N_j} \|\varphi_{j,\ell} f|F^s_{p,q}(\mathbb{R}^d)\|^p\right)^{1/p} < \infty\,,$$

where $0 < p < \infty$, $0 < q \leq \infty$ and $s > \sigma_{p,q} = d\max\left(0, \frac{1}{p} - 1, \frac{1}{q} - 1\right)$.

Refined localization spaces - properties

- for $1 < p < \infty$ it holds $F^{0,\mathrm{rloc}}_{p,2}(\Omega) = F^0_{p,2}(\Omega) = L_p(\Omega)$
- always $F^{s,\mathrm{rloc}}_{p,q}(\Omega) \subset F^s_{p,q}(\Omega)$
- smooth functions with compact support $D(\Omega)$ are dense in $F^{s,\mathrm{rloc}}_{p,q}(\Omega)$
- if $F^s_{p,q}(\Omega)$ has trace tr on parts of $\partial\Omega$, then $\mathrm{tr}\, f = 0$ for $f \in F^{s,\mathrm{rloc}}_{p,q}(\Omega)$.

For special kind of domains (smooth, polyhedral, ...) more is known. Let $\Omega = \mathbb{R}^d \setminus \mathbb{R}^\ell = \mathbb{R}^d \setminus \mathbb{R}^\ell \times \{0\}^{d-\ell}$. Then

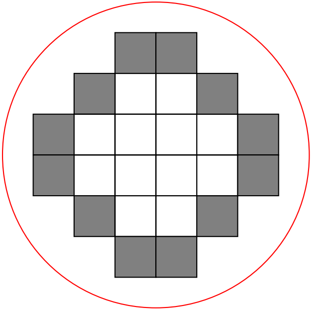
- For $s - (d-\ell)/p \notin \mathbb{N}_0$ it holds

$$F^{s,\mathrm{rloc}}_{p,q}(\mathbb{R}^d \setminus \mathbb{R}^\ell) = \{f \in F^s_{p,q}(\mathbb{R}^d \setminus \mathbb{R}^\ell) : \mathrm{tr}_{\mathbb{R}^\ell} f = 0\}$$

(all existing traces). Furthermore, $F^{s,\mathrm{rloc}}_{p,q}(\mathbb{R}^d \setminus \mathbb{R}^\ell)$ is the completion of $D(\Omega)$ with respect to $F^s_{p,q}(\mathbb{R}^d \setminus \mathbb{R}^\ell)$.

Interior wavelet systems

Today, probably the best understanding of $F^{s,\mathrm{rloc}}_{p,q}(\Omega)$ is by interior wavelet systems, see Triebel 2008. Wavelets of j-th order:



two types of (mother) wavelet functions:

- wavelets with moment conditions and $\mathrm{dist}(\mathrm{supp}\ \Phi^{j,1}_r, \partial\Omega) \lesssim 2^{-j}$
- (boundary/residual) wavelets without moment conditions and $\mathrm{dist}(\mathrm{supp}\ \Phi^{j,2}_r, \partial\Omega) \sim 2^{-j}$, don't overlap too much...

Refined localization spaces have interior wavelet bases

Theorem (Triebel 2008 - Theorem 2.38)

Let $\Omega \subsetneq \mathbb{R}^d$ be a domain and

$$0 < p < \infty, \quad 0 < q \leq \infty, \quad s > \sigma_{p,q}, \quad u \in \mathbb{N}, \quad u > s.$$

Then there exists an interior wavelet basis $\Phi \subset C^u(\Omega)$ for $F^{s,rloc}_{p,q}(\Omega)$, i. e.: An element $f \in L_{\max(1+\varepsilon,p)}$ belongs to $F^{s,rloc}_{p,q}(\Omega)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r(f) 2^{-jd/2} \Phi^j_r \tag{2}$$

*with λ in the sequence space $f^s_{p,q}(\mathbb{Z}^\Omega)$. The representation (**??**) is unique with*

$$\lambda^j_r(f) = 2^{jd/2}(f, \Phi^j_r).$$

...one can develop atomic decompositions for $F^{s,\mathrm{rloc}}_{p,q}(\Omega)$ (Scharf 2014)

Hardy inequality characterization of $F^{s,\mathrm{rloc}}_{p,q}(\Omega)$

$\Omega \subsetneq \mathbb{R}^d$ domain, $d(x) = \mathrm{dist}(x, \partial\Omega)$ and $\delta(x) = \min(d(x), 1)$.

Lemma (Triebel 2001/2008, Scharf 2014)

Let $0 < p < \infty$, $0 < q < \infty$ and $s > \sigma_{p,q}$. Then $f \in F^{s,rloc}_{p,q}(\Omega)$ if, and only if,

$$\|f|F^s_{p,q}(\Omega)\| + \|\delta^{-s}(\cdot)f|L_p(\Omega)\| < \infty.$$

(equivalent norms)

Proof idea:

$\Rightarrow$ homogeneity property and $\mathrm{dist}(x, \partial\Omega) \sim 2^{-j}$ for $x \in \operatorname{supp} \varphi_{j,\ell}$.
$\Leftarrow$ wavelet decomposition into two orthognal wavelet parts, moment conditions and local mean theorem for the first (interior) part, $\mathrm{dist}(x, \partial\Omega) \sim 2^{-j}$ and small overlap for the second (boundary)

Equivalence of $\mathcal{K}^m_{m,p}$ and $F^{s,\mathrm{rloc}}_{p,q}$ (i)

$S \subset \mathbb{R}^d$ singular set with $|S| = 0$ such that $\partial\left(\mathbb{R}^d \setminus S\right) = S$, e.g. S singular set of a polyhedron D. Then $\rho \sim \delta$ and

$$\|u|\mathcal{K}^m_{m,p}(\mathbb{R}^d \setminus S)\| = \sum_{|\alpha|\leq m} \|\rho(x)^{|\alpha|-m}\partial^\alpha u|L_p(\mathbb{R}^d)\|.$$

On the other hand

$$\begin{aligned} \|f|F^{m,\mathrm{rloc}}_{p,2}(\mathbb{R}^d \setminus S)\|^p &\sim \|f|F^m_{p,2}(\mathbb{R}^d \setminus S)\| + \|\delta^{-m}(\cdot)f|L_p(\mathbb{R}^d)\| \\ &\sim \|f|W^m_p(\mathbb{R}^d)\| + \|\delta^{-m}(\cdot)f|L_p(\mathbb{R}^d)\| \\ &\sim \sum_{|\alpha|=m} \|\partial^\alpha u|L_p(\mathbb{R}^d)\| + \|\rho^{-m}(\cdot)f|L_p(\mathbb{R}^d)\|. \end{aligned}$$

Therefore $\mathcal{K}^m_{m,p}(\mathbb{R}^d \setminus S) \hookrightarrow F^{m,\mathrm{rloc}}_{p,2}(\mathbb{R}^d \setminus S)$. In between terms?

Equivalence of $\mathcal{K}^m_{m,p}$ and $F^{m,\mathrm{rloc}}_{p,2}$ (ii)

Theorem

Let S be as before, $m \in \mathbb{N}$ and $1 < p < \infty$. Then it holds

$$\mathcal{K}^m_{m,p}(\mathbb{R}^d \setminus S) = F^{m,rloc}_{p,2}(\mathbb{R}^d \setminus S)\,.$$

Proof.

For $0 < |\alpha| < m$:

$$\begin{aligned} f \in F^{m,\mathrm{rloc}}_{p,2}(\mathbb{R}^d \setminus S) &\Rightarrow f \text{ has an interior wavelet decomposition} \\ &\Rightarrow \partial^\alpha f \text{ has an interior atomic decomposition} \\ &\Rightarrow \partial^\alpha f \in F^{m-|\alpha|,\mathrm{rloc}}_{p,2}(\mathbb{R}^d \setminus S) \\ &\overset{Hardy}{\Rightarrow} \rho(x)^{|\alpha|-m}\partial^\alpha u \in L_p(\mathbb{R}^d). \end{aligned}$$

□

An equivalent characterization of $\mathcal{K}^a_{m,p}$

It is easily seen using Leibniz rule and $|\partial^\alpha \rho| \leq C_\alpha \rho^{1-|\alpha|}$ that

$$u \in \mathcal{K}^m_{a,p}(\mathbb{R}^d \backslash S) \iff \rho^{m-a} u \in \mathcal{K}^m_{m,p}(\mathbb{R}^d \backslash S) \iff \rho^{-a} u \in \mathcal{K}^m_{0,p}(\mathbb{R}^d \backslash S)$$

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Hence

$$u \in \mathcal{K}^m_{a,p}(\mathbb{R}^d \setminus S) \iff \rho^{m-a} u \in F^{m,\mathrm{rloc}}_{p,2}(\mathbb{R}^d \setminus S).$$

!possible extension of Kondratiev spaces to $m \notin \mathbb{N}$ or $p \leq 1$!

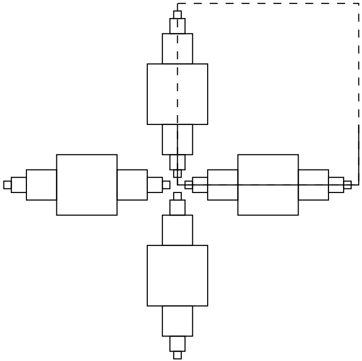
Corollary

For all $a \geq 0$ and $m \in \mathbb{N}$ we have

$$\|u|\mathcal{K}^m_{a,p}(\mathbb{R}^d \setminus S)\| \sim \sum_{|\alpha|=m} \|\partial^\alpha(\rho^{m-a} u)|L_p(\mathbb{R}^d)\| + \|\rho^{-a} u|L_p(\mathbb{R}^d)\|.$$

Idea: Wavelet decompositions for $\mathcal{K}^a_{m,p}$

Using $\mathcal{K}^m_{m,p}(\mathbb{R}^d \setminus S) = F^{m,\mathrm{rloc}}_{p,2}(\mathbb{R}^d \setminus S)$ and the isomorphism ρ^{m-a} we can derive (weighted) wavelet and atomic decompositions for $\mathcal{K}^a_{m,p}(\mathbb{R}^d \setminus S)$.



Support of wavelets of $F^{m,\mathrm{rloc}}_{p,2}(\mathbb{R}^d \setminus S)$ for a square D

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n-term approximation

The (error of the) best n-term approximation is defined as

$$\sigma_n\big(u; L_p(D)\big) = \inf_{\Gamma\subset\Lambda:\#\Gamma\leq n} \inf_{c_\gamma} \left\| u - \sum_{\gamma=(I,\psi)\in\Gamma} c_\gamma \psi_I \Big| L_p(D) \right\|,$$

i.e. as the name suggests we consider the best approximation by linear combinations of the basis functions consisting of at most n terms.

Approximation spaces

We define $\mathcal{A}_q^\alpha(L_p(D))$, $\alpha > 0$, $0 < q \leq \infty$ to consist of all functions $f \in L_p(D)$ such that

$$\|u|\mathcal{A}_q^\alpha(L_p(D))\| = \left(\sum_{n=0}^{\infty}\Big((n+1)^\alpha \sigma_n\big(u; L_p(D)\big)\Big)^\tau \frac{1}{n+1}\right)^{1/q},$$

$0 < q < \infty$, or

$$\|u|\mathcal{A}_\infty^\alpha(L_p(D))\| = \sup_{n\geq 0} n^\alpha \sigma_n\big(u; L_p(D)\big),$$

respectively, are finite. Then a well-known result of DeVore, Jawerth and Popov (1992) may be formulated as

$$\mathcal{A}_\tau^{s/d}(L_p(\mathbb{R}^d)) = B^s_{\tau,\tau}(\mathbb{R}^d), \qquad \frac{1}{\tau} = \frac{s}{d} + \frac{1}{p}.$$

Proposition (Dahlke, Novak, Sickel 2006)

Let D be a bounded domain: If $s > d(\frac{1}{\tau} - \frac{1}{p})$ for $0 < \tau \leq p$, $1 < p < \infty$, then

$$\sigma_n\big(u; L_p(D)\big) \lesssim n^{-s/d} \|u|B^s_{\tau,q}(D)\|\,, \qquad u \in B^s_{\tau,q}(D)\,,$$

independent of the microscopic parameter q.

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independent of the microscopic parameter q.

Proposition (Hansen, Sickel 2011)

For $s > 0$ and $1 < p < \infty$, we have

$$F^s_{\tau_*,\infty}(\mathbb{R}^d) \hookrightarrow \mathcal{A}^{s/d}_\infty(L_p(\mathbb{R}^d))\,, \qquad \frac{1}{\tau_*} = \frac{s}{d} + \frac{1}{p}\,.$$

Moreover, this space $F^s_{\tau_*,\infty}(\mathbb{R}^d)$ is maximal in the sense that if any other Besov or Triebel- Lizorkin space is embedded in $\mathcal{A}^{s/d}_\infty(L_p(\mathbb{R}^d))$, then it is already embedded in $F^s_{\tau_*,\infty}(\mathbb{R}^d)$.

Prototypical result - solutions in approximation spaces

Proposition (Dahlke, DeVore (1997))

Let v be a harmonic function on some Lipschitz domain $D \subset \mathbb{R}^d$. Further assume

$$v \in B^{\lambda}_{p,p}(D)\,, \qquad 1 < p < \infty, \quad \lambda > 0\,.$$

Then it follows

$$v \in B^{s}_{\tau,\tau}(D)\,, \qquad \frac{1}{\tau} = \frac{s}{d} + \frac{1}{p}\,, \quad 0 < s < \lambda \frac{d}{d-1}\,.$$

Under appropriate conditions on the parameters and the smoothness of the right-hand side, this also transfers to solutions of Poisson's problem with homogeneous Dirichlet boundary condition.

Previous Embedding result

Theorem (Hansen 2013)

Let $D \subset \mathbb{R}^d$ some bounded polyhedral domain. Suppose $\min(s, a) > \frac{\delta}{d} m$. *Then there exists some* $0 < \tau_0 \leq p$ *such that*

$$\mathcal{K}^m_{a,p}(D) \cap H^s_p(D) \hookrightarrow B^m_{\tau,\infty}(D)$$

for all $\tau_* < \tau < \tau_0$, *where* $\frac{1}{\tau_*} = \frac{m}{d} + \frac{1}{p}$.

Here $\delta \leq d - 2$ stands for the dimension of the singular set.

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for all $\tau_* < \tau < \tau_0$, *where* $\frac{1}{\tau_*} = \frac{m}{d} + \frac{1}{p}$.

Here $\delta \leq d - 2$ stands for the dimension of the singular set.

A trivial embedding of Kondratiev spaces

In the special case $p > \tau > 1$ it follows by using Hölder inequality

$$\mathcal{K}^m_{a,p}(\mathbb{R}^d \setminus S) \hookrightarrow \mathcal{K}^m_{m,\tau}(\mathbb{R}^d \setminus S)$$

for

$$m - a < (d - \delta)\left(\frac{1}{\tau} - \frac{1}{p}\right).$$

We want to extend this result to $\tau \leq 1$ using $F^{m,\mathrm{rloc}}_{\tau,2}(\mathbb{R}^d \setminus S)$ instead of $\mathcal{K}^m_{m,\tau}(\mathbb{R}^d \setminus S)$.

Embedding result for Kondratiev spaces (i)

Theorem (Hansen, Scharf 2014)

Let $m \in \mathbb{N}$, $a \in \mathbb{R}$, $1 < p < \infty$, *and* $0 < \tau < \infty$. *Moreover, assume either*

$$\tau < p \quad \text{and} \quad m - a < (d - \ell)\Big(\frac{1}{\tau} - \frac{1}{p}\Big)$$

(or $\tau = p$ *and* $a \geq m$*). Furthermore, let* $u \in \mathcal{K}^m_{a,p}(\mathbb{R}^d \setminus \mathbb{R}^\ell)$ *have compact support. Then it holds*

$$\|u|F^{m,rloc}_{\tau,2}(\mathbb{R}^d \setminus \mathbb{R}^\ell)\| \lesssim \|u|\mathcal{K}^m_{a,p}(\mathbb{R}^d \setminus \mathbb{R}^\ell)\| \,.$$

Remark: The conditions are also necessary, counterexamples are known...

Embedding result for Kondratiev spaces (ii)

Theorem (Hansen, Scharf 2014)

Let D be a polytope in $\mathbb{R}^d$ and $\tau < p$. Then we have an embedding

$$\mathcal{K}^m_{a,p}(D) \hookrightarrow F^m_{\tau,2}(D)$$

if, and only if,

$$m - a < (d - \delta)\left(\frac{1}{\tau} - \frac{1}{p}\right),$$

where $\delta \leq d - 2$ is the dimension of the singular set.

- Instead of $F^m_{\tau,2}(D)$ one could use $F^{m,\mathrm{rloc}}_{\tau,2}(\mathbb{R}^d \setminus S)$ restricted to D.
- plug in $\frac{1}{\tau_*} = \frac{m}{d} + \frac{1}{p}$. Then the condition reads $a > \frac{\delta}{d} m$.

Theorem (Hansen, Scharf 2014)

Let D be some bounded polyhedral domain without cracks in $\mathbb{R}^d$, and consider the problem

$$-\nabla\big(A(x)\cdot\nabla u(x)\big) = f \quad \text{in} \quad D\,, \qquad u|_{\partial D} = 0\,.$$

Under the assumptions of the Shift-Theorem, for a right-hand side $f \in \mathcal{K}^{m-1}_{a-1,2}(D)$ the uniquely determined solution $u \in H^1_0(D)$ can be approximated at the rate

$$\sigma_n\big(u; H^1(D)\big) \lesssim n^{-m/d}\|f|\mathcal{K}^{m-1}_{a-1,2}(D)\|\,,$$

where

$$m+1 < \tfrac{d}{\delta}(a+1)\,, \quad a \geq 0\,.$$

Thank you for your attention

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