

Besov regularity of solutions of the p -Laplace equation



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Overview

Introduction and results for the Laplace equation ($p = 2$)

- Introduction to the p -Laplace

- Approximation in Sobolev and Besov spaces

- Known results for the Laplace equation ($p = 2$)

Sobolev and local Hölder regularity of the p -Laplace

- Sobolev regularity of the p -Laplace

- Local Hölder regularity of the p -Laplace equation

Besov regularity of solutions of the p -Laplace equation

- From $B_{p,p}^s(\Omega)$ and $C_{\gamma,loc}^{\ell,\alpha}(\Omega)$ to $B_{\tau,\tau}^\sigma(\Omega)$

- Besov regularity of the p -Laplace

The p -Laplace - Introduction

$\Omega \subset \mathbb{R}^d$ Lipschitz domain, d dimension, $1 < p < \infty$

Inhomogeneous p -Laplace equation:

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Variational (weak) formulation:

$$\int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla v \rangle \, dx = \int_{\Omega} f v \, dx \quad \text{for all } v \in C_0^\infty(\Omega)$$

- has a unique solution $u \in \dot{W}_p^1(\Omega)$ for $f \in W_{p'}^{-1}(\Omega)$,
- has **model character for nonlinear problems**, similar to the Laplace equation ($p = 2$) for linear problems

nice and free introduction: P. Lindqvist. Notes on the p -Laplace equation, 2006.

<http://www.math.ntnu.no/~lqvist/p-laplace.pdf>

Sobolev and Besov spaces

$W_p^s(\Omega)$: Sobolev space of smoothness s and integrability p on Ω

$B_{p,p}^s(\Omega)$: Besov space of smoothness s and integrability p on Ω

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Wavelet representation: $\eta_{l,p} = |l|^{1/2-1/p} \eta_l$ p -normalized wavelets

$$g \in B_{p,p}^s(\mathbb{R}^d) \Leftrightarrow g = P_0(g) + \sum_l \sum_{\eta \in \Psi} \langle g, \eta_{l,p'} \rangle \eta_{l,p}$$

$$\text{and } \left\| P_0(g) \Big|_{L_p(\mathbb{R}^d)} \right\| + \left\| \langle g, \eta_{l,p'} \rangle \Big|_{b_{p,p}^s(\mathbb{R}^d)} \right\| < \infty$$

Here

$$\left\| \langle g, \eta_{l,p'} \rangle \Big|_{b_{p,p}^s(\mathbb{R}^d)} \right\|^p = \sum_l \sum_{\eta \in \Psi} |l|^{-sp/d} |\langle g, \eta_{l,p'} \rangle|^p$$

more smoothness \Leftrightarrow more decay of the wavelet coefficients

Trivial embedding: $B_{p,p}^{s+\varepsilon}(\Omega) \hookrightarrow W_p^s(\Omega) \hookrightarrow B_{p,p}^s(\Omega)$

Linear and Adaptive approximation by wavelets (i)

How to approximate $f \in B_{p,p}^s(\Omega)$, Ω bounded, by wavelet basis?

Linear approximation f_k of f (order k : $\sim 2^{kd}$ terms):

$$f_k = P_0(g) + \sum_{|I| \geq 2^{-k}} \sum_{\eta \in \Psi} \langle g, \eta_{I,p'} \rangle \eta_{I,p}$$

It holds

$$f \in B_{p,p}^s(\Omega) \text{ (or } W_p^s(\Omega)) \Rightarrow \|f - f_k\|_{L_p(\Omega)} \lesssim 2^{-ksd}.$$

Linear and Adaptive approximation by wavelets (ii)

Adaptive approximation f_k of f (order k : $\sim 2^{kd}$ terms):

$$f_k^D = P_0(g) + \sum_{(I,\eta) \in D} \langle g, \eta_{I,p'} \rangle \eta_{I,p} \text{ with } |D| = 2^{kd}$$

best m -term approximation: choose D to minimize

$$\|f - f_k^D\|_{L_p(\Omega)}: \text{ take } 2^{kd} \text{ largest wavelet coefficients!}$$

Let $\frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p}$, in particular $\tau < 1$ possible. It holds

$$f \in B_{\tau,\tau}^\sigma(\Omega) \Rightarrow \|f - f_k\|_{L_p(\Omega)} \sim 2^{-k\sigma d}$$

Besov regularity is the **maximal possible convergence rate of an adaptive algorithm** \Rightarrow how much higher than Sobolev regularity?

Sobolev regularity for $p = 2$, the linear case

Theorem (Jerison, Kenig 1995)

Positive: Lipschitz domain $\Omega \in \mathbb{R}^d$, $f \in L_2(\Omega)$. Then the solution u of

$$\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

belongs to $W_2^{3/2}(\Omega)$.

Negative: For any $s > 3/2$ there exists a Lipschitz domain Ω and smooth f s.t. u with

$$\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

does not belong to $W_2^s(\Omega)$.

D. Jerison, C.E. Kenig. *The Inhomogeneous Dirichlet Problem in Lipschitz Domains*. *J. Funct. Anal.* 130, 161–219, 1995.

Besov regularity for $p = 2$

Theorem (Dahlke, DeVore '97; Jerison, Kenig '95; Hansen 2013)
Lipschitz domain $\Omega \in \mathbb{R}^d$, $f \in W_2^\gamma(\Omega)$ for $\gamma \geq \max\left(\frac{4-d}{2d-2}, 0\right)$. Then the solution u of

$$\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

belongs to $B_{\tau,\tau}^\sigma(\Omega)$, $\frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p}$, for any $\sigma < \frac{3}{2} \cdot \frac{d}{d-1}$.

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- Besov reg. always better than $3/2$, the maximal Sobolev regularity
- proof by a general embedding:
small global Sobolev regularity + better local (weighted) Sobolev regularity (Babuska-Kondratiev) result in better Besov regularity!

S. Dahlke, R.A. DeVore. *Besov regularity for elliptic boundary value problems*. *Comm. Partial Differential Equations*, 22(1-2), 116, 1997.

M. Hansen, *n-term approximation rates and Besov regularity for elliptic PDEs on polyhedral domains*, to appear in *J. Found. Comp. Math.*

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Sobolev regularity of the p -Laplace

Theorem (Ebmeyer 2001,2002)

$\Omega \subset \mathbb{R}^d$ bounded polyhedral domain, $d \geq 2$, $1 < p < \infty$, $f \in L_{p'}(\Omega)$.
If $\Delta_p u = f$ and $u = 0$ on $\partial\Omega$, then

$$V := |\nabla u|^{\frac{p-2}{2}} \nabla u \in W_2^{1/2-\varepsilon}(\Omega) \text{ for all } \varepsilon > 0.$$

Furthermore

$$|\nabla u| \in L_q(\Omega) \text{ for } q < \frac{pd}{d-1}$$

and

$$u \in \begin{cases} W_{\tilde{p}}^{3/2-\varepsilon}(\Omega), & \text{if } 1 < p \leq 2, \\ W_p^{1+1/p-\varepsilon}(\Omega), & \text{if } p \geq 2, \end{cases} \quad \tilde{p} = \frac{p}{1 - \frac{2-p}{2d}} > p.$$

C. Ebmeyer. *Nonlinear elliptic problems with p -structure under mixed boundary value conditions in polyhedral domains*. *Adv. Diff. Equ.*, 6:873–895, 2001.

Local Hölder regularity of the homogen. p -Laplace

Replacement for the local (weighted) Sobolev regularity ($p = 2$)

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Replacement for the local (weighted) Sobolev regularity ($p = 2$)

Theorem (Lewis 1983; Ural'ceva; Evans; DiBenedetto; ...)

$\Omega \subset \mathbb{R}^d$ bounded open set, $d \geq 2$, $1 < p < \infty$. There exists $\alpha \in (0, 1]$ s.t. u with $\Delta_p u = 0$ fulfils: $\forall x_0 \in \Omega, r > 0$ s.t. $B(x_0, 64r) \subset \Omega$

$$\max_{x \in B(x_0, r)} |\nabla u(x)| \leq C \left(\int_{B(x_0, 32r)} |\nabla u|^p dx \right)^{1/p} \leq C \cdot r^{-d/p},$$

$$\max_{x, y \in B(x_0, r)} |\nabla u(x) - \nabla u(y)| \leq C \cdot r^{-\alpha} \left(\int_{B(x_0, 32r)} |\nabla u|^p dx \right)^{1/p} |x - y|^\alpha.$$

\Rightarrow local (weighted) Hölder regularity for homogeneous p -Laplace

J. Lewis. Regularity of the derivatives of solutions to certain degenerate elliptic equations. Indiana Univ. Math. J., 32(6):849–858, 1983.

Local Hölder regularity of the inhomog. p -Laplace

We can transfer the local Hölder regularity from the homogeneous case to the inhomogeneous p -Laplace equation:

Theorem (Kuusi, Mingione 2013; Diening, Kaplicky, Schwarzacher)

Ω, d, p as before. Let

$$\alpha^* = \sup\{\alpha : \text{Theorem of Lewis holds including the estimates}\}.$$

Then for u with $\Delta_p u = f \in C^{1,\beta(\alpha)}$:

u is locally α -Hölder continuous for $\alpha < \min(\alpha^*, 1/(p-1))$.

Analog estimates hold for local Hölder-seminorm of u .

Local Hölder regularity of the **inhomog.** p -Laplace

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Then for u with $\Delta_p u = f \in C^{1, \beta(\alpha)}$:

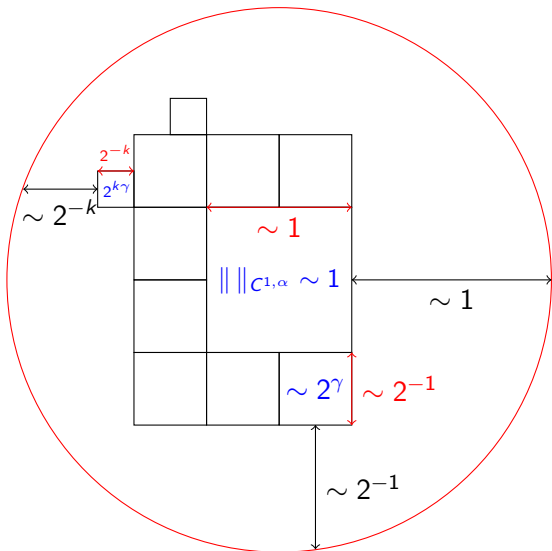
u is locally α -Hölder continuous for $\alpha < \min(\alpha^*, 1/(p-1))$.

Analog estimates hold for local Hölder-seminorm of u .

Problem: $\alpha^* \in (0, 1]$ is unknown for $d \geq 3$. (later: case $d = 2$)

T. Kuusi and G. Mingione. *Guide to Nonlinear Potential Estimates*. *Bull. Math. Sci*, 4(1):1–82, 2014.

Locally weighted Hölder spaces $C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)$



$C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)$. . . Hölder space,
locally weighted, with

ℓ . . . number of derivatives

α . . . Hölder exponent of
derivatives of order ℓ

γ . . . growth of Hölder exp.
with distance to $\partial\Omega$

Local Hölder regularity of the p -Laplace

Although the optimal local Hölder regularity of the solution of the p -Poisson is unknown ($d \geq 3$), we can estimate γ by Lewis' Theorem

$$\begin{aligned} \max_{x,y \in B(x_0,r)} |\nabla u(x) - \nabla u(y)| &\leq C \cdot r^{-\alpha} \left(\int_{B(x_0,32r)} |\nabla u|^p dx \right)^{1/p} |x - y|^\alpha \\ &\leq C \cdot r^{-\alpha-d/q} \cdot \|\nabla u\|_{L_q(\Omega)} \cdot |x - y|^\alpha, \quad p \leq q. \end{aligned}$$

Hence, using the result of Ebmeyer

$$|\nabla u| \in L_q(\Omega) \text{ for } q < \frac{pd}{d-1},$$

we are allowed to choose

$$\gamma = \alpha + (d-1)/p + \varepsilon \text{ for all } \varepsilon > 0.$$

The case $d = 2$: Hölder regularity of the p -Poisson

Theorem (Lindgren, Lindqvist 2013; (DDHSW 2014))

$\Omega \subset \mathbb{R}^2$ bounded polygonal domain, $1 < p < \infty$, $f \in L_\infty(\Omega)$. If $\Delta_p u = f$, $u = 0$ on $\partial\Omega$, then u is locally α -Hölder continuous for all

$$\alpha < \begin{cases} 1, & \text{if } 1 < p \leq 2, \\ \frac{1}{p-1}, & \text{if } 2 < p < \infty. \end{cases}$$

Furthermore, for the same α 's, it holds

$$u \in C_{\gamma, \text{loc}}^{1, \alpha}(\Omega) \text{ for } \gamma = \alpha + 1/p + \varepsilon.$$

- The regularity $\frac{1}{p-1}$ is a natural bound, take $v(x) = |x|^{p/(p-1)}$.
- homogen. case: Iwaniec, Manfredi (1989) proved $u \in C_{\text{loc}}^{\ell, \alpha}(\Omega)$ with

$$\ell + \alpha = 1 + \frac{1}{6} \left(1 + \frac{1}{p-1} + \sqrt{1 + \frac{14}{p-1} + \frac{1}{(p-1)^2}} \right) > \max \left(2, \frac{p}{p-1} \right)$$

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From $B_{p,p}^s(\Omega)$ and $C_{\gamma,\text{loc}}^{\ell,\alpha}(\Omega)$ to $B_{\tau,\tau}^\sigma(\Omega)$

Theorem (Dahlke, Diening, Hartmann, S., Weimar(DDHSW) '14)

$\Omega \subset \mathbb{R}^d$ bound. Lipschitz dom., $d \geq 2$, $s > 0$, $1 < p < \infty$, $\alpha \in (0, 1]$,

$$\sigma^* = \begin{cases} \ell + \alpha, & \text{if } 0 < \gamma < \frac{\ell + \alpha}{d} + \frac{1}{p}, \\ \frac{d}{d-1} \left(\ell + \alpha + \frac{1}{p} - \gamma \right), & \text{if } \frac{\ell + \alpha}{d} + \frac{1}{p} \leq \gamma < \ell + \alpha + \frac{1}{p}, \end{cases}$$

then for all

$$0 < \sigma < \min \left\{ \sigma^*, \frac{d}{d-1} s \right\} \quad \text{and} \quad \frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p}$$

we have the continuous embedding

$$B_{p,p}^s(\Omega) \cap C_{\gamma,\text{loc}}^{\ell,\alpha}(\Omega) \hookrightarrow B_{\tau,\tau}^\sigma(\Omega).$$

If γ not too bad and local Hölder regularity $\ell + \alpha$ is higher than Sobolev regularity s , Besov regularity σ is higher than Sobolev reg. !

The case $d = 2$: Besov regularity of the p -Poisson

1. By Ebmeyer's result

$$u \in \begin{cases} B_{p,p}^{3/2-\varepsilon}(\Omega), & \text{if } 1 < p \leq 2, \\ B_{p,p}^{1+1/p-\varepsilon}(\Omega), & \text{if } p \geq 2, \end{cases}$$

2. Lindgren, Lindqvist:

$$u \in C_{\gamma, \text{loc}}^{1, \alpha}(\Omega), \quad \gamma = \alpha + 1/p + \varepsilon, \quad \alpha < \begin{cases} 1, & \text{if } 1 < p \leq 2, \\ \frac{1}{p-1}, & \text{if } 2 < p < \infty. \end{cases}$$

3. γ not too bad? $\alpha + \frac{1}{p} + \varepsilon = \gamma \stackrel{?}{<} \frac{\ell + \alpha}{d} + \frac{1}{p} = \frac{1 + \alpha}{2} + \frac{1}{p}$? Yes, $\alpha < 1$

4. general embedding theorem, $\frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p}$

$$u \in B_{\tau, \tau}^{\sigma}(\Omega) \text{ for all } \sigma < \begin{cases} 2, & \text{if } 1 < p \leq 2, \\ 1 + \frac{1}{p-1}, & \text{if } 2 < p < \infty. \end{cases}$$

Summary: Besov regularity of the p -Poisson

- For $d = 2$ results on Besov regularity beat Sobolev regularity:

$$\text{it holds } \begin{cases} 2 > 3/2, & \text{if } 1 < p \leq 2, \\ 1 + \frac{1}{p-1} > 1 + \frac{1}{p} & \text{if } 2 < p < \infty. \end{cases}$$

- For $d \geq 3$ the optimal α is unknown, $\alpha \geq 1/3$ is an open conj.
- For $d \geq 3$ to beat Sobolev regularity we need

$$\alpha > \begin{cases} \frac{1}{2}, & \text{if } 1 < p < 2, \\ \frac{1}{p}, & \text{if } p > 2, \end{cases}$$

and γ not too large depending on d . This implies

$$p \in (p_d, \infty) \text{ with } p_d \rightarrow \infty \text{ for } d \rightarrow \infty.$$

E. Lindgren and P. Lindqvist. Regularity of the p -poisson equation in the plane. arXiv:1311.6795v2, 2013.

T. Iwaniec and J. Manfredi. Regularity of p -harmonic functions on the plane. Rev. Mat. Iberoamericana, 5(1-2):119, 1989.

Open problems

- $d = 2$, can one do better, in dependency of the angles of the boundary?
- Is the chosen γ optimal?
- non-polyhedral domains
- measure Besov regularity in L_q for the p -Laplace ($q \neq p$)

work in progress...

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work in progress...

Thank you for your attention

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