Wavelets for reinforced function spaces on cubes

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September 19, 2012

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The spaces $W_p^k(\mathbb{R}^n)$ and $W_p^k(\Omega)$

Definition

Let $1 < p < \infty$ and $k \in \mathbb{N}_0$. Then

$$W_p^k(\mathbb{R}^n) := \{f \in D'(\mathbb{R}^n) : D^\alpha f \in L_p(\mathbb{R}^n) \text{ for } |\alpha| \leq k\}$$

with

$$\|f|W_p^k(\mathbb{R}^n)\| = \sum_{|\alpha|\leq k} \|D^\alpha f|L_p(\mathbb{R}^n)\|.$$

Let Q be the unit cube in $\mathbb{R}^2$ (more general $\mathbb{R}^n$). Then

$$W_p^k(Q) := \{f \in D'(Q) : f = g|Q \text{ for some } g \in W_p^k(Q)\},$$

$$\|f|W_p^k(Q)\| = \inf \|g|W_p^k(Q)\|$$

where the infimum is taken over all $g \in W_p^k(\mathbb{R}^n)$ with $g|Q = f$.

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where the infimum is taken over all $g \in W_p^k(\mathbb{R}^n)$ with $g|Q = f$.

An equivalent characterization is given by

$$f \in W_p^k(Q) \Leftrightarrow \sum_{|\alpha| \leq k} \|D^\alpha f|L_p(Q)\| < \infty \text{ (equivalent norms)}$$

Wavelets for cubes (i)

Theorem (Triebel 2008 - Theorem 6.30 for spaces $F^s_{p,q}(Q)$)

Let

$$1 < p < \infty, u, k \in \mathbb{N}_0, u > k \text{ and } k - \frac{m}{p} \notin \mathbb{N}_0$$

for $m = 1, \dots, n$.

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$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r(f) 2^{-\frac{jn}{2}} \Phi^j_r \tag{1}$$

with λ *in the sequence space* $w^k_p(Q)$ *and the convergence is unconditional in* $W^k_p(Q)$*. The representation (1) is unique and*

$$\text{essentially } \lambda^j_r(f) \sim 2^{jn/2}(f, \Phi^j_r).$$

Wavelets for function spaces on domains Ω (i)

Let $u \in \mathbb{N}_0$.
Then

$$\Phi = \left\{ \Phi_\ell^j : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\} \subset C^u(\Omega)$$

is called a u-wavelet system in $\bar{\Omega}$ (adapted to $\mathbb{Z}^\Omega$) if it fulfils

- support conditions: For some $c_3 > 0$ it holds

$$supp\, \Phi_\ell^j \subset B(x_\ell^j, c_3 2^{-j}) \cap \bar{\Omega},\ j \in \mathbb{N}_0,\ \ell = 1, \ldots, N_j,$$

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- derivative conditions: For some $c_4 > 0$ and all $\alpha \in \mathbb{N}_0^n$ with $0 \leq |\alpha| \leq u$ we have

$$\left| D^\alpha \Phi_\ell^j(x) \right| \leq c_4 2^{j\frac{n}{2} + j|\alpha|}, x \in \Omega,\, j \in \mathbb{N}_0,\, \ell = 1, \dots, N_j.$$

Wavelets for function spaces on domains Ω (ii)

Additionally, the u-wavelet system is called oscillating if it fulfils

- (substitute) moment conditions: Let c_5 and $c_6 < c_7$ be constants such that

$$\mathrm{dist}(B(x^0_\ell, c_3), \partial\Omega) \geq c_6, \text{ for } \ell = 1, \ldots, N_0 \text{ and}$$

$$\left| \int_\Omega \psi(x) \Phi^j_\ell(x) \, dx \right| \leq c_5 2^{-j\frac{n}{2} - ju} \, \|\psi | C^u(\Omega)\| \text{ for all } \psi \in C^u(\Omega)$$

  for all j and ℓ with $dist(B(x^j_\ell, c_3), \partial\Omega) \notin \left(c_6 2^{-j}, c_7 2^{-j}\right)$.

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for all j and ℓ with $dist(B(x_\ell^j, c_3), \partial\Omega) \notin \left(c_6 2^{-j}, c_7 2^{-j}\right)$.

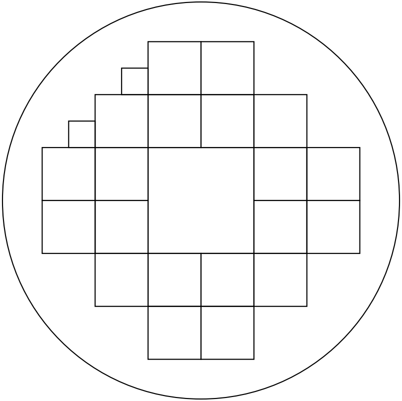
An oscillating u-wavelet system is called interior if it fulfils

- (further) interior support conditions, namely

$$\text{dist}(B(x_\ell^j, c_3 2^{-j}), \partial\Omega) \geq c_6 2^{-j}, \, j \in \mathbb{N}_0, \, \ell = 1, \ldots, N_j.$$

Interior wavelets on Ω

The support of the first interior wavelets (here Ω is a ball):



Interior wavelet bases in $L_2(\Omega)$ - the starting point

Theorem (Triebel 2008)

Let Ω be an arbitrary domain in $\mathbb{R}^n$. For any $u \in \mathbb{N}_0$ there is a

$$\Phi = \left\{ \Phi_\ell^j : j \in \mathbb{N}_0, \ell = 1, \dots, N_j \right\} \subset C^u(\Omega)$$

which is

1. *an orthonormal basis in $L_2(\Omega)$,*
2. *an interior u-wavelet system*

simultaneously.

For $u = 0$ one can take the Haar Wavelet suitably restricted to Ω.

Wavelets for cubes (ii)

Let Q be the open unit cube in $\mathbb{R}^n$.

Theorem (Triebel 2008 - Theorem 6.30 for spaces $F^s_{p,q}(Q)$)

Let

$$1 < p < \infty, u, k \in \mathbb{N}_0, u > k \text{ and } k - \frac{m}{p} \notin \mathbb{N}_0$$

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Then there is an oscillating u-wavelet system Φ *which is a Riesz basis in the Sobolev space* $W^k_p(Q)$*, i.e.: An element* $f \in D'(Q)$ *belongs to* $W^k_p(Q)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r(f) 2^{-\frac{jn}{2}} \Phi^j_r \qquad (2)$$

with λ *in the sequence space* $w^k_p(Q)$ *and the convergence is unconditional in* $W^k_p(Q)$*. The representation (2) is unique with*

$$\lambda^j_r(f) = 2^{jn/2}(f, \Phi^j_r).$$

Traces on the boundary of cubes (i)

Let $Q = \{x \in \mathbb{R}^n : x = (x_1, \ldots, x_n), 0 < x_m < 1, m = 1, \ldots, n)$. The boundary $\Gamma = \partial Q$ of Q can be represented as

$$\Gamma = \bigcup_{\ell=0}^{n-1} \Gamma_\ell \text{ with } \Gamma_\ell \cap \Gamma_{\ell'} = \emptyset \text{ for } \ell \neq \ell',$$

where $\Gamma_\ell = \bigcup_{j=0}^{n_\ell} \Gamma_{\ell,j}$ consists of all ℓ-dimensional faces $\Gamma_{\ell,j}$ of Q, which are disjoint cubes of dimension ℓ.

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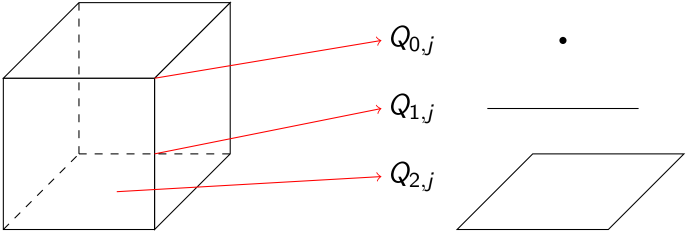

Traces on the boundary of cubes (ii)

Let $\mathrm{tr}_{\ell,j}$ be the restriction of $f \in W_p^k(\mathbb{R}^n)$ to $\Gamma_{\ell,j}$ and

$$\mathrm{tr}_\ell^r : f \mapsto TR_\ell^r(f) := \prod \left\{ \mathrm{tr}_{\ell,j}\ D_\gamma^\alpha f : |\alpha| \leq r, j = 0, \ldots, n_\ell \right\},$$

where only derivatives perpendicular to $Q_{\ell,j}$ are admitted.

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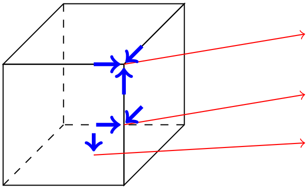
where only derivatives perpendicular to $Q_{\ell,j}$ are admitted. Then we consider the composite mapping for $\bar{r} = \left(r^0, \ldots r^{n-1}\right)$ with $r^\ell = \lfloor k - \frac{n-\ell}{p} \rfloor$ (as many traces as existing!)

$$\mathrm{tr}_\Gamma^{\bar{r}} : f \mapsto \prod_{\ell=\ell_0}^{n-1} TR_\ell^{\bar{r}}(f).$$



$Q_{0,j}$: 3 directions, $\lceil k - \frac{3}{p} \rceil$ derivatives

$Q_{1,j}$: 2 directions, $\lceil k - \frac{2}{p} \rceil$ derivatives

$Q_{2,j}$: 1 direction, $\lceil k - \frac{1}{p} \rceil$ derivatives

Traces on the boundary of cubes (iii)

A u-wavelet Riesz basis cannot be interior for $k \geq 1$: Then $W_p^k(Q)$ has boundary values on the faces of Q (traces), interior wavelets do not.

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Proposition

Let

$$1 < p < \infty, k \neq 0 \text{ and } k - \frac{m}{p} \notin \mathbb{N}_0 \text{ for } m = 1, \ldots, n.$$

Then it holds

$$\tilde{W}_p^k(Q) = \left\{ f \in W_p^k(Q) : \mathrm{tr}_\Gamma^{\bar{r}} = 0 \right\},$$

(all existing traces have to vanish!)

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Then it holds

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(all existing traces have to vanish!)

Here

$$\tilde{W}_p^k(Q) := \{ f \in W_p^k(\mathbb{R}^n) : \mathit{supp}\, f \subset \overline{Q} \}.$$

The spaces on the left have interior u-wavelet Riesz bases.

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The situation for cubes Q in the exceptional cases

Roughly speaking:

- a C^∞-domain Ω has only boundaries of dimension $n-1 \rightarrow$ exceptional values for $k - \frac{1}{p} \in \mathbb{N}_0$
- the cube Q has boundaries of dimension 0 to $n-1 \rightarrow$ exceptional values for $k - \frac{m}{p} \in \mathbb{N}_0$ for $m = 1, \ldots, n$

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- the cube Q has boundaries of dimension 0 to $n-1 \rightarrow$ exceptional values for $k - \frac{m}{p} \in \mathbb{N}_0$ for $m = 1, \ldots, n$

Example (Grisvard '85, '92)

The space $W_2^1(Q) = F_{2,2}^1(Q)$ is exceptional: $k - \frac{2}{p} = 2 - 1$. Let $\Gamma = \partial\Omega = I_1 \cup I_2 \cup I_3 \cup I_4$. Then the trace space $\mathrm{tr}_\Gamma\, W_2^1(Q)$ is the collection of all tuples $g = (g_1, g_2, g_3, g_4)$ with

$$g_\ell \in H^{\frac{1}{2}}(I_\ell), \quad \ell = 1, 2, 3, 4$$

and

$$\int_0^{1/2} \frac{|g_1(t) - g_2(t)|^2}{t}\, dt < \infty, \text{ etc.}$$

Reinforced function spaces for cubes Q

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Let

$$d(x) = \text{dist}(x, \partial\Omega) \text{ and } \Omega_\varepsilon := \{x \in \Omega : d(x) < \varepsilon\}.$$

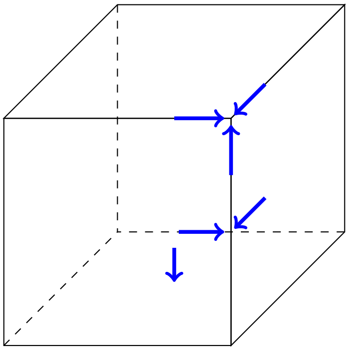
Let $\Gamma_{\ell,j}$ be the ℓ-dimensional faces of the cube Q. With $\mathbb{N}^n_{\ell,j}$ we denote the multi-indices with directions perpendicular to $\Gamma_{\ell,j}$.

Definition

We say that $f \in W^k_p(\mathbb{R}^n)$ has the reinforced property $R^{r,p}_\ell$ if, and only if,

$$d^{-\frac{n-\ell}{p}} \cdot D^\alpha f \in L_p((\mathbb{R}^n \setminus \Gamma_{\ell,j})_\varepsilon) \text{ for all } \alpha \in \mathbb{N}^n_{\ell,j}, |\alpha| = r \text{ and } j = 1, \ldots, n_\ell.$$

Reinforced function spaces for cubes Q (ii)



Reinforced spaces for cubes Q (iii)

Definition

Let $1 < p < \infty$, $k \neq 0$. Let

$$W_p^{k,\mathrm{rinf}}(Q) := W_p^{k,\mathrm{rinf}}(\mathbb{R}^n \setminus \Gamma)_{|Q}$$

with inf-norm and

$$W_p^{k,\mathrm{rinf}}(\mathbb{R}^n \setminus \Gamma) :=$$

$$\{f \in W_p^k(\mathbb{R}^n) : \forall\, 0 \leq \ell \leq n-1 : f \text{ fulfils } R_\ell^{r,p} \text{ if } r = k - \frac{n-\ell}{p} \in \mathbb{N}_0\}$$

Check for every dimension ℓ if the values are exceptional and if so, add reinforce property!

Wavelet bases for reinforced function spaces on cubes Q

Theorem (S. (2012) - for $F^s_{p,q}(Q)$-spaces)

Let

$$1 < p < \infty,\ k \in \mathbb{N}_0,\ u > k.$$

Then there is an oscillating u-wavelet system which is a Riesz basis in $W_p^{k,\mathrm{rinf}}(Q)$ - i. e.

$$f \in W_p^k(Q) \quad \Leftrightarrow \quad f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda_r^j(f) 2^{-\frac{jn}{2}} \Phi_r^j$$

with $\lambda \in w_p^k(Q)$ (linear functionals).

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with $\lambda \in w_p^k(Q)$ (linear functionals).

- In the special case $k = 0$ ($L_p(\Omega)$) we can choose an interior u-wavelet system, for instance the Haar wavelet system - otherwise not.
- This theorem is a generalization of the wavelet theorem for $k - \frac{m}{p} \notin \mathbb{N}_0$ (Triebel 2008) since then there are no extra conditions.

An example - $W_2^{1,\mathrm{rinf}}(Q) = F_{2,2}^{1,\mathrm{rinf}}(Q)$ for $n = 2$ (i)

We have $k - \frac{2}{p} = r = 0$ - faces of dimension 0 are problematic. Hence

$$W_2^{1,\mathrm{rinf}}(Q) = F_{2,2}^{1,\mathrm{rinf}}(Q) = \left\{ f \in W_2^1(Q) : \int_Q |f(x)|^2 \frac{dx}{d(x)^2} < \infty \right\},$$

where d is the distance from the corner points (Γ_0) of Q.

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where d is the distance from the corner points (Γ_0) of Q.

Then by the theorem we find a (non-interior) oscillating u-wavelet basis which is a Riesz basis. This is not possible (in this way) for $W_2^1(Q)$ - wavelet bases (at least using the definition here) for $W_2^1(Q)$ were not found yet.

The end

Thank you for your attention

Questions?