

Pointwise multipliers and diffeomorphisms in function spaces

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The problem setting

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We want to observe the behaviour of the linear mappings

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where f is an element of a function space (Besov, Triebel-Lizorkin type) and φ is a suitably smooth function.

The aim:

If φ fulfils . . . , then P_φ resp. D_φ maps the function space A into A .

The spaces C^k

Let C^k be the space of all k -times differentiable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\|f\|_{C^k} := \sum_{|\alpha| \leq k} \sup |D^\alpha f(x)| < \infty.$$

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Then

$$f, g \in C^k \Rightarrow f \cdot g \in C^k \text{ and } \|f \cdot g|C^k\| \leq c_k \|f|C^k\| \cdot \|g|C^k\|$$

and

$$(\forall f \in C^k : f \cdot g \in C^k) \Rightarrow g \in C^k \text{ and } \|P_g : C^k \rightarrow C^k\| \geq \|g|C^k\|.$$

Proof: Leibniz rule and $1 \in C^k$.

The Hölder spaces C^k

Let $0 < \sigma \leq 1$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous. We define

$$\|f|lip^\sigma\| := \sup_{x,y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\sigma},$$

Let $s > 0$ and $s = [s] + \{s\}$ with $[s] \in \mathbb{Z}$ and $\{s\} \in (0, 1]$. Then the Hölder space with index s is given by

$$C^s = \left\{ f \in C^{[s]} : \|f|C^s\| := \|f|C^{[s]-}\| + \sum_{|\alpha|=[s]} \|D^\alpha f|lip^{\{s\}}\| < \infty \right\}.$$

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It holds

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Proof: Leibniz rule for Hölder spaces and $1 \in C^s$.

The Lebesgue spaces $L_p(\mathbb{R}^n)$

Let $0 < p \leq \infty$ and $L_p(\mathbb{R}^n)$ the usual set of equivalence classes of measurable functions f with finite

$$\|f\|_{L_p(\mathbb{R}^n)} := \begin{cases} \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}} & , 0 < p < \infty \\ \text{ess sup } |f(x)| & , p = \infty \end{cases}$$

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Then

$f \in L_p(\mathbb{R}^n), g \in L_\infty(\mathbb{R}^n) \Rightarrow f \cdot g \in L_p(\mathbb{R}^n)$ and

$$\|f \cdot g\|_{L_p(\mathbb{R}^n)} \leq \|f\|_{L_p} \cdot \|g\|_{L_\infty(\mathbb{R}^n)}$$

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$$\|P_g : L_p(\mathbb{R}^n) \rightarrow L_p(\mathbb{R}^n)\| \geq \|g\|_{L_\infty(\mathbb{R}^n)}$$

The Sobolev spaces $W_p^k(\mathbb{R}^n)$ (i)

Let $1 < p < \infty$, $k \in \mathbb{N}_0$ and $W_p^k(\mathbb{R}^n)$ the set of equivalence classes of measurable functions f with finite

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Then

$$f \in W_p^k(\mathbb{R}^n), g \in C^k \Rightarrow f \cdot g \in W_p^k(\mathbb{R}^n) \text{ and } \|f \cdot g\|_{W_p^k(\mathbb{R}^n)} \leq \|f\|_{W_p^k} \cdot \|g\|_{C^k}$$

The converse is not true!

The Sobolev spaces $W_p^k(\mathbb{R}^n)$ (ii)

Theorem (Sobolev embedding)

Let $k_1 < k_2$ and $k_1 - \frac{n}{p_1} \leq k_2 - \frac{n}{p_2}$. Then

$$W_{p_2}^{k_2}(\mathbb{R}^n) \hookrightarrow W_{p_1}^{k_1}(\mathbb{R}^n).$$

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Theorem (Multiplier algebra)

If $k > \frac{n}{p}$, then

$$\|f \cdot g\|_{W_p^k(\mathbb{R}^n)} \leq \|f\|_{W_p^k(\mathbb{R}^n)} \cdot \|g\|_{W_p^k(\mathbb{R}^n)}.$$

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Proof: We start with

$$\|D^\alpha(f \cdot g)\|_{L_p(\mathbb{R}^n)} \leq c \sum \|(D^\beta f) \cdot (D^{\alpha-\beta} g)\|_{L_p(\mathbb{R}^n)}$$

The Sobolev spaces $W_p^k(\mathbb{R}^n)$ (iii)

$$\begin{aligned}
 \|D^\alpha(f \cdot g)\|_{L_p(\mathbb{R}^n)} &\leq c \sum \| (D^\beta f) \cdot (D^{\alpha-\beta} g) \|_{L_p(\mathbb{R}^n)} \\
 &\leq c \sum \| (D^\beta f) \|_{L_{p_1}} \cdot \| (D^{\alpha-\beta} g) \|_{L_{p_2}} \\
 &\leq c \sum \| f \|_{W_{p_1}^{|\beta|}(\mathbb{R}^n)} \cdot \| g \|_{W_{p_2}^{|\alpha|-|\beta|}(\mathbb{R}^n)} \\
 &\leq c' \| f \|_{W_p^k(\mathbb{R}^n)} \cdot \| g \|_{W_p^k(\mathbb{R}^n)}.
 \end{aligned}$$

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 &\leq c \sum \| f \|_{W_{p_1}^{|\beta|}(\mathbb{R}^n)} \cdot \| g \|_{W_{p_2}^{|\alpha|-|\beta|}(\mathbb{R}^n)} \\
 &\leq c' \| f \|_{W_p^k(\mathbb{R}^n)} \cdot \| g \|_{W_p^k(\mathbb{R}^n)}.
 \end{aligned}$$

Here ($|\alpha| \leq k$)

$$\begin{aligned}
 \frac{1}{p_1} + \frac{1}{p_2} &= \frac{1}{p} \\
 |\beta| - \frac{n}{p_1} &\leq k - \frac{n}{p} \\
 |\alpha| - |\beta| - \frac{n}{p_2} &\leq k - \frac{n}{p}
 \end{aligned}$$

This is possible, if $k > \frac{n}{p}$.

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Resolution of unity

Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ such that $\text{supp } \varphi_0 \subset \{|x| \leq \frac{3}{2}\}$ and $\varphi_0(x) = 1$ for $|x| \leq 1$. We define

$$\varphi(x) := \varphi_0(x) - \varphi_0(2x) \text{ and } \varphi_j(x) := \varphi(2^{-j}x) \text{ for } j \in \mathbb{N}.$$

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Then we have

$$\begin{aligned} \sum_{j=0}^{\infty} \varphi_j(x) &= 1. \\ |D^\alpha \varphi_j(x)| &\leq c_\alpha 2^{-j|\alpha|}, \\ \text{supp } \varphi_j &\subset \{2^{j-1} \leq |x| \leq 2^{j+1}\}, \end{aligned} \tag{1}$$

A sequence of functions $\{\varphi_j\}_{j=0}^{\infty}$ with (1), $\varphi_j \in \mathcal{S}(\mathbb{R}^n)$ and φ_0 as above will be called resolution of unity.

The definition of $B_{p,q}^s(\mathbb{R}^n)$

Let $\{\varphi_j\}_{j=0}^\infty$ be a resolution of unity. Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we define

$$\|f|B_{p,q}^s(\mathbb{R}^n)\|^\varphi := \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^\vee|L_p(\mathbb{R}^n)\|^q \right)^{\frac{1}{q}}$$

(modified in case $q = \infty$) and

$$B_{p,q}^{s,\varphi}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f|B_{p,q}^s(\mathbb{R}^n)\|^\varphi < \infty\}.$$

Then $(B_{p,q}^{s,\varphi}(\mathbb{R}^n), \|\cdot|B_{p,q}^s(\mathbb{R}^n)\|^\varphi)$ is a quasi-Banach space. It does not depend on the choice of the resolution of unity $\{\varphi_j\}_{j=0}^\infty$ in the sense of equivalent norms. So we denote it shortly by $B_{p,q}^s(\mathbb{R}^n)$.

The definition of $F_{p,q}^s(\mathbb{R}^n)$

Let $\{\varphi_j\}_{j=0}^\infty$ be a resolution of unity. Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we define

$$\|f\|_{F_{p,q}^s(\mathbb{R}^n)}^\varphi := \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \hat{f})^\vee|^q \right)^{\frac{1}{q}} \right\|_{L_p(\mathbb{R}^n)}$$

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Function spaces as multiplier algebras

Definition

A function space $A_{p,q}^s(\mathbb{R}^n)$ is said to be a multiplier algebra iff there is a bounded bilinear symmetric mapping

$$P : A_{p,q}^s(\mathbb{R}^n) \cdot A_{p,q}^s(\mathbb{R}^n) \rightarrow A_{p,q}^s(\mathbb{R}^n)$$

such that

$$P(f, g) = f \cdot g$$

for $f \in \mathcal{S}(\mathbb{R}^n)$, $g \in A_{p,q}^s(\mathbb{R}^n)$.

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for $f \in \mathcal{S}(\mathbb{R}^n)$, $g \in A_{p,q}^s(\mathbb{R}^n)$.

If such a mapping exists, then it is uniquely determined. This follows by completion resp. local arguments.

Known necessary results for function spaces (i)

Theorem (see Runst and Sickel 1996, 4.3.2)

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned} \|P_\varphi|A_{p,q}^s(\mathbb{R}^n) \rightarrow A_{p,q}^s(\mathbb{R}^n)\| &:= \sup_{f \in A_{p,q}^s(\mathbb{R}^n), f \neq 0} \frac{\|\varphi \cdot f|A_{p,q}^s(\mathbb{R}^n)\|}{\|f|A_{p,q}^s(\mathbb{R}^n)\|} \\ &\geq c \cdot \|\varphi|L_\infty(\mathbb{R}^n)\|, \end{aligned}$$

where $c > 0$ does not depend on f .

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where $c > 0$ does not depend on f .

Hence, if $A_{p,q}^s(\mathbb{R}^n)$ is a multiplier algebra, then

$$\|\varphi|_{L_\infty(\mathbb{R}^n)}\| \leq c' \|P_\varphi|_{A_{p,q}^s(\mathbb{R}^n)} \rightarrow A_{p,q}^s(\mathbb{R}^n)}\| \leq c'' \|\varphi|_{A_{p,q}^s(\mathbb{R}^n)}\|$$

for $\varphi \in \mathcal{S}(\mathbb{R}^n)$. So

$$A_{p,q}^s(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n).$$

Known necessary results for function spaces (ii)

More general:

Theorem

If there is a (Besov or Triebel-Lizorkin) function space A such that

$$\|\varphi \cdot f\|_{A_{p,q}^s(\mathbb{R}^n)} \leq c \|\varphi\|_A \cdot \|f\|_{A_{p,q}^s(\mathbb{R}^n)},$$

then

$$A \hookrightarrow L_\infty.$$

Known sufficient results for function spaces

Theorem (see e.g. Runst and Sickel 1996)

For $s > \sigma_p$ the spaces $A_{p,q}^s(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ are multiplier algebras, even

$$\begin{aligned} \|f \cdot g|_{A_{p,q}^s(\mathbb{R}^n)}\| \\ \leq c \left(\|f|_{A_{p,q}^s(\mathbb{R}^n)}\| \cdot \|g|_{L_\infty}\| + \|g|_{A_{p,q}^s(\mathbb{R}^n)}\| \cdot \|f|_{L_\infty}\| \right). \end{aligned}$$

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Theorem (see e.g. Triebel 2008)

If $F_{p,q}^s(\mathbb{R}^n)$ is a multiplier algebra, then φ is a pointwise multiplier for $F_{p,q}^s(\mathbb{R}^n)$ iff

$$\sup_{m \in \mathbb{Z}} \|\psi(\cdot - m) \cdot \varphi|_{F_{p,q}^s(\mathbb{R}^n)}\| < \infty,$$

where ψ is a nonnegative C_0^∞ -function with

$$\sum_m \psi(x - m) = 1 \text{ for } x \in \mathbb{R}^n.$$

Atomic characterization of $B_{p,q}^s(\mathbb{R}^n)$

Theorem

Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. Let $K, L \geq 0$, $K > s$ and $L > \sigma_p - s$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B_{p,q}^s(\mathbb{R}^n)$ if and only if it can be represented as

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \cdot a_{\nu,m} \quad \text{with convergence in } \mathcal{S}'(\mathbb{R}^n).$$

Here $a_{\nu,m}$ are $(s, p)_{K,L}$ -atoms located at $Q_{\nu,m}$ and $\|\lambda|b_{p,q}\| < \infty$. Furthermore, we have in the sense of equivalence of norms

$$\|f|B_{p,q}^s(\mathbb{R}^n)\| \sim \inf \|\lambda|b_{p,q}\|,$$

where the infimum on the right-hand side is taken over all admissible representations of f .

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Treatment of products using atomic decompositions

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$$\varphi \cdot f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \cdot \varphi \cdot a_{\nu,m}$$

Treatment of products using atomic decompositions

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$$\downarrow$$

If $\varphi \cdot a_{\nu,m}$ are atoms: $\varphi \cdot f \in A_{p,q}^s(\mathbb{R}^n)$

The definition of atoms

A function $a : \mathbb{R}^n \rightarrow \mathbb{R}$ is called classical $(s, p)_{K,L}$ -atom located at $Q_{\nu,m}$ if
 $\text{supp } a \subset d \cdot Q_{\nu,m}$

$$|D^\alpha a(x)| \leq C \cdot 2^{-\nu \left(s - \frac{n}{p} \right) + |\alpha| \nu} \text{ for all } |\alpha| < K + 1, \quad (2)$$

$$\int_{\mathbb{R}^n} x^\beta a(x) dx = 0 \text{ for all } |\beta| < L. \quad (3)$$

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A function $a : \mathbb{R}^n \rightarrow \mathbb{R}$ is called $(s, p)_{K,L}$ -atom located at $Q_{\nu,m}$ if instead of (2) and (3) it holds (for all $\psi \in \mathcal{C}^L$)

$$\|a(2^{-\nu} \cdot) | \mathcal{C}^K \| \leq C \cdot 2^{-\nu(s - \frac{n}{p})}$$

$$\left| \int_{d \cdot Q_{\nu,m}} \psi(x) a(x) dx \right| \leq C \cdot 2^{-\nu(s + L + n(1 - \frac{1}{p}))} \| \psi | \mathcal{C}^L \|$$

Atomic representations revisited

Every classical $(s, p)_{K,L}$ -atom is an $(s, p)_{K,L}$ -atom.

Theorem

The atomic representation theorem for $B_{p,q}^s(\mathbb{R}^n)$ and $F_{p,q}^s(\mathbb{R}^n)$ is valid with both forms of atoms. Hence every f which can be represented as a linear combination of classical $(s, p)_{K,L}$ -atom resp. $(s, p)_{K,L}$ -atom belongs to $B_{p,q}^s(\mathbb{R}^n)$ resp. $F_{p,q}^s(\mathbb{R}^n)$. Hereby

$$K > s \quad \text{and}$$

$$L > \sigma_p - s = \sigma_p = n \left(\frac{1}{p} - 1 \right)_+ - s \quad \text{resp.}$$

$$L > \sigma_{p,q} - s = n \left(\frac{1}{\min(p, q)} - 1 \right)_+ - s$$

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The proof for classical atoms goes back to Triebel '97. The modifications were suggested by Skrzypczak '98, Triebel/Winkelvoss '96.

The pointwise multiplier theorem (i)

Now we get

Lemma

There exists a constant c with the following property: For all $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}$, all $(s, p)_{K,L}$ -atoms $a_{\nu,m}$ with support in $d \cdot Q_{\nu,m}$ and all $\varphi \in C^\rho$ with $\rho \geq \max(K, L)$ the product

$$c \cdot \|\varphi\|_{C^\rho}^{-1} \cdot \varphi \cdot a_{\nu,m}$$

is an $(s, p)_{K,L}$ -atom with support in $d \cdot Q_{\nu,m}$.

The pointwise multiplier theorem (i)

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Lemma

There exists a constant c with the following property: For all $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}$, all $(s, p)_{K,L}$ -atoms $a_{\nu,m}$ with support in $d \cdot Q_{\nu,m}$ and all $\varphi \in \mathcal{C}^\rho$ with $\rho \geq \max(K, L)$ the product

$$c \cdot \|\varphi\|_{\mathcal{C}^\rho}^{-1} \cdot \varphi \cdot a_{\nu,m}$$

is an $(s, p)_{K,L}$ -atom with support in $d \cdot Q_{\nu,m}$.

Proof: Use that \mathcal{C}^ρ is a multiplication algebra.

This does not work for classical atoms $(s, p)_{K,L}$ -atoms with $L \geq 1$, since in general moment conditions are destroyed when multiplying by φ !

The pointwise multiplier theorem (ii)

We get as a corollary

Theorem

Let $s \in \mathbb{R}$ and $0 < q \leq \infty$.

(i) Let $0 < p \leq \infty$ and $\rho > \max(s, \sigma_p - s)$. Then there exists a positive number c such that

$$\|\varphi \cdot f\|_{B_{p,q}^s(\mathbb{R}^n)} \leq c \|\varphi\|_{\mathcal{C}^\rho} \cdot \|f\|_{B_{p,q}^s(\mathbb{R}^n)}$$

for all $\varphi \in \mathcal{C}^\rho$ and all $f \in B_{p,q}^s(\mathbb{R}^n)$.

(ii) Let $0 < p < \infty$ and $\rho > \max(s, \sigma_{p,q} - s)$. Then there exists a positive number c such that

$$\|\varphi \cdot f\|_{F_{p,q}^s(\mathbb{R}^n)} \leq c \|\varphi\|_{\mathcal{C}^\rho} \cdot \|f\|_{F_{p,q}^s(\mathbb{R}^n)}$$

for all $\varphi \in \mathcal{C}^\rho$ and all $f \in F_{p,q}^s(\mathbb{R}^n)$.

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- Non-smooth atomic representation theorems

The diffeomorphism theorem (i)

In the same way we can treat the mapping D_φ :

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \cdot a_{\nu,m} \Rightarrow f \circ \varphi = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \cdot (a_{\nu,m} \circ \varphi).$$

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Hence we have to investigate if $a_{\nu,m} \circ \varphi$ is an $(s, p)_{K,L}$ -atom when $a_{\nu,m}$ is an $(s, p)_{K,L}$ -atom.

Definition

Let $\rho \geq 1$. We say that the one-to-one mapping $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a ρ -diffeomorphism if the components of $\varphi(x) = (\varphi_1(x), \dots, \varphi_n(x))$ have classical derivatives up to order $[\rho]$ with $\frac{\partial \varphi}{\partial x_j} \in \mathcal{C}^{\rho-1}$ and if $|\det \varphi_*| \geq c > 0$ for some c and all $x \in \mathbb{R}^n$. Here φ_* stands for the Jacobian matrix.

The diffeomorphism theorem (ii)

Theorem

(i) Let $0 < p \leq \infty$, $\rho \geq 1$ and $\rho > \max(s, \sigma_p - s)$. If φ is a ρ -diffeomorphism, then there exists a constant c such that

$$\|f(\varphi(\cdot))\|_{B_{p,q}^s(\mathbb{R}^n)} \leq c \|f\|_{B_{p,q}^s(\mathbb{R}^n)}.$$

for all $f \in B_{p,q}^s(\mathbb{R}^n)$. Hence D_φ maps $B_{p,q}^s(\mathbb{R}^n)$ onto $B_{p,q}^s(\mathbb{R}^n)$.

(ii) Let $0 < p < \infty$, $\rho \geq 1$ and $\rho > \max(s, \sigma_{p,q} - s)$. If φ is a ρ -diffeomorphism, then there exists a constant c such that

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Proof: Show that $a_{\nu,m}$ is an $(s, p)_{K,L}$ -atom and control the support of the atoms.

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The characteristic function and the Haar wavelet

It is known that

$$\chi_{[0,1]^n} \in A_{p,q}^s(\mathbb{R}^n)$$

for $0 < s < \frac{1}{p}$ and $p \geq 1$. But it can't be understood as a (classical) atom for these spaces! This would be of interest in connection with Haar wavelets.

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The question: Is there a more general “non-smooth” atomic representation theorem?

The end

Thank you for your attention

Questions?