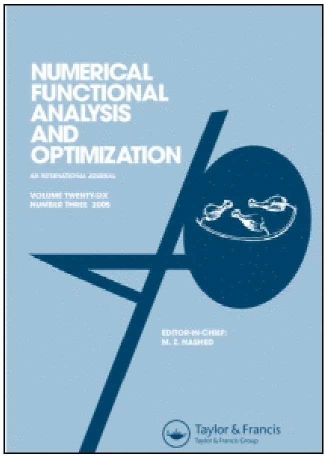

Numerical Functional Analysis and Optimization

Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597287

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Online Publication Date: 06 January 2003
To cite this Article: Fornasier, Massimo (2003) 'Function Spaces Inclusions and Rate of Convergence of Riemann-Type Sums in Numerical Integration', Numerical Functional Analysis and Optimization, 24:1, 45 - 57
To link to this article: DOI: 10.1081/NFA-120020243
URL: http://dx.doi.org/10.1081/NFA-120020243

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NUMERICAL FUNCTIONAL ANALYSIS AND OPTIMIZATION
Vol. 24, Nos. 1 & 2, pp. 45–57, 2003

Function Spaces Inclusions and Rate of Convergence of Riemann-Type Sums in Numerical Integration

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ABSTRACT

In signal processing, discrete convolutions are usually involved in fast calculating coefficients of time-frequency decompositions like wavelet and Gabor frames. Depending on the regularity of the mother analyzing functions, one wants to detect the right resolution in order to achieve good approximations of coefficients. Local–global conditions on functions in order to get the convergence rate of Riemann-type sums to their scalar products in L^2 are presented. Wiener amalgam spaces, in particular $W(C^0, l^2)$ for the space-time domain and $W(L^2, l^1)$ for the frequency domain, give natural norms in order to estimate errors. In particular, relations between the rate of convergence of these series to integrals by increasing resolution and the (minimal) required Besov regularity are presented by means of functional and harmonic analysis techniques.

Key Words: Decomposition spaces; Discrete convolutions; Function spaces inclusions; Sampling theory; Wavelets.

Mathematics Subject Classifications 2000: 46S30; 94A20; 65T50; 65T60.

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DOI: 10.1081/NFA-120020243 | 0163-0563 (Print); 1532-2467 (Online)
 www.dekker.com

1. INTRODUCTION

The increasing investigation in Gabor analysis (Feichtinger and Strohmer, 1998, 2003; Gröchenig, 2001) and wavelets (Christensen, 2001; Cohen et al., 1999; Daubechies, 1990, 1992; Dahlke, 1997) decompositions came to define these expansions in more general function spaces, such as Modulation spaces (Feichtinger, 1989; Gröchenig, 2001) and Besov spaces (Dahlke 1997; Triebel, 1983) respectively. In many applications, such as time–frequency and signal analysis (Feichtinger and Strohmer, 1998, 2003) or numerical solving partial differential equations (Cohen et al., 1999; Dahlke, 1997), the mother generating function might be non-polynomial or non-refinable because of requirement of special properties. In this case, the L^2-coefficients integration problem arises. Let us mention that L^2-coefficients are useful in order to characterize also Banach functions spaces (Gröchenig, 2002). Many integration methods based on polynomial approximation or refinable equations (Dahmen and Micchelli, 1993), also for multivariate functions, can be applied, but the most *practicable* tool in *general cases* seems the use of the FFT algorithm: specially in digital signal analysis, functions are usually sampled on regular square grids and people are willing to approximate coefficients by Riemann-type sums which can be efficiently calculated by the FFT algorithm by discrete convolutions. It is also intuitive that the smoothness of the involved functions is related to the rate of convergence of these Riemann type sums approximation. In the following we investigate the relations between smoothness and rate of convergence by increasing resolution and, in particular, which is the minimal (fractional) regularity, in the sense of the Besov space $B^s_{2,\infty}$, required by a given rate of convergence. An outline of the article is as follows: Sec. 2 just recalls some of the tools we need; Sec. 3 presents the error estimates results, based on sampling theory, of the approximation of L^2-scalar products by means of Riemann sums. In particular, Theorems 3.10 and 3.14 describe respectively sufficient and necessary conditions in order to achieve a given rate of convergence.

2. SOME USEFUL TOOLS

As we write $\tau > 0$ we mean $\tau = (\tau_1, \ldots, \tau_d) \in \mathbb{R}^d_+$; let we set also $w = (w_1, \ldots, w_n)$ where $w_i = C\tau_i^{-1}$ and $C > 0$ (C depends on the definition of the Fourier transform: usually $C = 1$ or $C = 2\pi$), $\det(\tau) = \tau_1, \ldots, \tau_n$ and $Q_w = \prod_{i=1}^n w_i[-(1/2), 1/2)$. We write $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ for the Schwartz space. If $f \in \mathcal{S}'$ (the temperate distributions space), $\hat{f}$ or $\mathcal{F}(f)$ stands for the Fourier transform of f. We consider the functions or distributions $\hat{\eta_f} = \hat{f}$ on Q_w and $\eta_f = 0$ identically on $Q_w^c = \mathbb{R}^d \backslash Q_w$, $\hat{\varepsilon_f} = \hat{f} - \hat{\eta}_f$ and η_f, ε_f their respective inverse Fourier transforms. As we have fixed τ, we will write l^2 instead of $l^2(\tau\mathbb{Z}^d)$ and we assume it endowed with the scalar product $\langle f, g\rangle_{l^2} := \det(\tau)\sum_{k\in\mathbb{Z}^d} f(\tau k)\overline{g(\tau k)}$. For a continuous function f sometimes we write $\|f\|_{l^2}$ instead of $\|f_{|\tau\mathbb{Z}^d}\|_{l^2}$. A function f is said *band-limited* when $spec(f) := \mathrm{supp}(\hat{f}) \subset \Omega$, for some Ω relatively compact subset of $\mathbb{R}^d$. We will write $f(x) \lesssim g(x)$ (for $x \to x_0$) whenever $f(x) \leq O(g(x))$ for $x \to x_0$, where x_0 can be ∞ or 0. When $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$ then we will write $f(x) \asymp g(x)$ (for $x \to x_0$).

As an introduction on general *decomposition spaces* refer to Feichtinger (1981, 1987, 1989, 1991), Feichtinger and Gröbner (1985). We recall anyway some of the properties of the *decomposition function spaces* that we will use in the following without introducing them in full generality.

We use decomposition spaces because of their important capability of highlighting the relations between continuous and discrete behaviors of functions. As we will see in the following, in particular some Wiener amalgam norms are able to estimate l^p-norms of L^p sampled continuous functions (Fournier and Stewart, 1985), independently on the resolution, giving a powerful tool for estimating discretizations. Moreover, they are related, by inclusions or by application of Fourier transforms, with the most common function spaces, like L^p, Bessel potential, Modulation and Besov-Triebel spaces.

A useful tool for an approach to discrete domains are BAPU's:

Definition 2.1. Given any Banach algebra $(A, \|\cdot\|)$ of bounded regular functions on $\mathbb{R}^d$ and a family $\Omega = (\Omega_i)_{i\in I}$ of compact subsets of $\mathbb{R}^d$ such that

(i) $\bigcup_{i\in I} \Omega_i = \mathbb{R}^d$;
(ii) there exists $N \in \mathbb{N}$ such that $\#\{ j \in I : \Omega_i \bigcap \Omega_j \neq \emptyset\} \leq N$, $\forall i \in I$, a family $\Psi = (\psi_i)_{i\in I}$ in A is called a Ω-BAPU (Bounded Admissible Partition of Unity) of size Ω if the following properties hold:
(iii) $\sup_{i\in I} \|\psi_i\|_A = C_0 < \infty$;
(iv) $\text{supp}(\psi_i) \subset \Omega_i$, $\forall i \in I$;
(v) $\sum_{i\in I} \psi_i(x) = 1$.

Definition 2.2. Assume that $(A, \|\cdot\|)$ is a Banach algebra of regular bounded functions on $\mathbb{R}^d$. Set A_0 the space of compactly supported functions of A endowed with the inductive limit topology. A Banach space $(B, \|\cdot\|_B)$ is in standard situation if it is continuously embedded in the dual A_0' of A_0 and it is a module over A with respect to pointwise multiplication, i.e., we suppose

$$\|hf\|_B \leq \|h\|_A \|f\|_B, \ \forall h \in A, \ f \in B$$

Definition 2.3. Assume that $(A, \|\cdot\|)$ is a Banach algebra of regular bounded functions on $\mathbb{R}^d$. Given a Banach space $(B, \|\cdot\|_B)$ in standard situation with respect to A, $\Psi = (\psi_i)_{i\in I} \subset A$ an Ω-BAPU and w a discrete weight function, we define the relative decomposition space as:

$$D(\Omega, B, l^q_w)(\mathbb{R}^d) = \{f \in A_0' : f\psi_i \in B \quad \forall i \in I, (\|f\psi_i\|_B)_{i\in I} \in l^q_w(I)\}.$$

Moreover, one can define for $1 \leq q < \infty$ the natural norm:

$$\|f\,|D\| := \left(\sum_{i\in I} \|f\psi_i\|_B^q w(i)^q\right)^{1/q},$$

with the usual modification for $q = \infty$.

With this norm the space $(D(\Omega, B, l^q_w), \|\cdot|D\|)$ is a Banach space. In the case the Ω-BAPU is a BUPU (Bounded Uniform Partition of Unity), i.e., there exists $(x_i)_{i\in I} \subset \mathbb{R}^d$ such that $\Omega_i = x_i + Q$, with $0 \in Q \subset \mathbb{R}^d$ compact, we call $D(\Omega, B, l^q_w)$

a Wiener amalgam space and we will write $W(B, l^q_w) := D(\Omega, B, l^q_w)$ as the definition does not depend on the particular BUPU Ψ taken when the weight function w is assumed moderate, that is $w(x+y) \leq w(x)w(y)$ for all $x, y \in \mathbb{R}^d$. In particular, different choices of BUPUs arise equivalent norms. The properties of Wiener amalgams with respect to inclusions, Fourier transform, convolutions, were described and studied by Feichtinger (1981, 1989, 1991). An important and useful example of Wiener amalgam space is $S_0(\mathbb{R}^d) = W(\mathcal{F}L^1, l^1)(\mathbb{R}^d)$, called the Feichtinger algebra (Feichtinger, 1981).

As a notation, we will write w_s the weight function $w_s(x) = (1+|w|^2)^{s/2}$ and $L^p_s := L^p_{w_s}$. In particular, the weight w_s is moderate.

Theorem 2.4. (*Useful properties*)

(i) *If* $B^1_{\text{loc}} \subset B^2_{\text{loc}}$ *and/or* $l^{q_1}_{w_1} \subseteq l^{q_2}_{w_2}$ *then* $D(\Omega, B^1, l^{q_1}_{w_1}) \subset D(\Omega, B^2, l^{q_2}_{w_2})$ *continuously*;

(ii) $\mathcal{F}(W(L^p, l^q)) \subset W(L^{q'}, l^{p'})$ *continuously, for* $1 \leq p, q \leq 2$, $1/p + 1/p' = 1$ *and* $1/q + 1/q' = 1$.

3. APPROXIMATION OF L^2 SCALAR PRODUCTS

The *aliasing* error measures the norm of the *tails* $\hat{\varepsilon}_f$ of the Fourier transform of a function f and it stands for the uncertainty one has in identifying the function defined on a continuum from its samples on a regular or irregular set of nodes. Anyway it is not absolutely a sharp way to estimate this uncertainty: in fact, for example compactly supported B-spline functions can be completely reconstructed by polynomial interpolation theory from a set of samples, but they can also have important aliasing errors. Hence one can consider aliasing errors as a large estimate of the ambiguity with which one manages functions or samples.

Theorem 3.1. (*Whittacker-Shannon*) *For any band-limited* $f \in L^2$ *there exists* $\tau_0 > 0$ *such that for all* $0 < \tau \leq \tau_0$

$$f(x) = \sum_{k \in \mathbb{Z}^d} f(\tau k) \prod_{i=1}^{d} \operatorname{sinc}(\tau_i^{-1} x_i - k_i). \tag{1}$$

Corollary 3.2. (*Aliasing error*) *From the Whittacker-Shannon theorem is immediate to show that for all* $f \in L^2 \cap C^0$ *and any* $\tau > 0$ *such that* $f_{|\tau\mathbb{Z}^d} \in l^2$:

$$f(x) = \sum_{k \in \mathbb{Z}^d} (f(\tau k) - \varepsilon_f(\tau k)) \prod_{i=1}^{d} \operatorname{sinc}(\tau_i^{-1} x_i - k_i) + \varepsilon_f(x). \tag{2}$$

The quantities ε_f *are the uncertainty on the reconstruction of* f *directly from its samples.*

In particular in $L^2(\mathbb{R}^d)$ the reconstruction ambiguity on functions propagates on their scalar products as we will show in the following results.

Lemma 3.3. (*Useful inclusions*)

(i) *The identity is continuous from* $W(C^0, l^2)$ *to* $L^2 \cap C^0$, *i.e.*, $\forall f \in W(C^0, l^2)$ *there exists* $C > 0$ *such that* $\|f\|_2 \leq C \cdot \|f\,|W(C^0, l^2)\|$.
(ii) *The sampling operator is continuous from* $W(C^0, l^2)$ to l^2, *i.e., for each* $\tau > 0$ *there exists* $C_\tau > 0$ *such that*, $\forall f \in W(C^0, l^2)$, $\|f_{|\tau Z^d}\|_{l^2} \leq C_\tau \cdot \|f\,|W(C^0, l^2)\|$.

Proof. Observe that $L^2 = W(L^2, l^2)$ and $C^0_{\text{loc}} \subset L^2_{\text{loc}}$. Hence by Theorem 2.4 (i), one has immediately (i). The proof of (ii) is analogous to that of the following Proposition 3.6.

Theorem 3.4. (*Propagation of the aliasing errors to* L^2*-scalar products*). *Assume* $\hat{f}, \hat{g} \in W(L^2, l^1)$ *and* $\tau > 0$ *fixed. Then* $f, g \in W(C^0, l^2)$, *and we can approximate the* L^2 *scalar product of* f *and* g *as follows*:

$$|\langle f, g\rangle_{L^2} - \langle f_{|\tau Z^d}, g_{|\tau Z^d}\rangle_{l^2}| \leq C_{\tau,1}\|f\|\cdot\|\epsilon_g\| + C_{\tau,2}\|g\|\cdot\|\epsilon_f\| + C_{\tau,3}\|\epsilon_f\|\cdot\|\epsilon_g\| \tag{3}$$

where the norms on the right-hand side are taken in $W(C^0, l^2)$.

Proof. Observe that $W(L^2, l^1) \subset L^1$. Hence, by Theorem 2.4 (ii), $f, g \in W(L^\infty, l^2) \cap C^0 = W(C^0, l^2) \subset L^2 \cap C^0$. Moreover, by Lemma 3.3 (ii), f and g are such that $f_{|\tau\mathbb{Z}^d} \in l^2(\tau\mathbb{Z}^d)$ and $g_{|\tau\mathbb{Z}^d} \in l^2(\tau\mathbb{Z}^d)$ for all $\tau > 0$. By Corollary 3.2

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{R}^d} f(x)\overline{g(x)}\,dx = \int_{\mathbb{R}^d}\left(\sum_{k\in\mathbb{Z}^d}(f(\tau k) - \varepsilon_f(\tau k))\prod_{i=1}^d \operatorname{sinc}(\tau_i^{-1}x_i - k_i) + \varepsilon_f(x)\right)$$
$$\times\left(\overline{\sum_{h\in\mathbb{Z}^d}(g(\tau h) - \varepsilon_g(\tau h))\prod_{i=1}^d \operatorname{sinc}(\tau_i^{-1}x_i - h_i) + \varepsilon_g(x)}\right)dx. \tag{4}$$

Since the functions of the type η_* belong to an orthogonal subspace of L^2 with respect to that of functions of the type ε_*, one has:

$$\langle f, g\rangle_{L^2} = \sum_{h,k\in\mathbb{Z}^d}(f(\tau k) - \varepsilon_f(\tau k))\overline{(g(\tau h) - \varepsilon_g(\tau h))}$$
$$\times\left(\int_{\mathbb{R}^d}\prod_{i=1}^d \operatorname{sinc}(\tau_i^{-1}x_i - k_i)\operatorname{sinc}(\tau_i^{-1}x_i - h_i)\,dx\right) + \langle\varepsilon_f, \varepsilon_g\rangle_{L^2}$$

by orthogonality of sinc's

$$= \det(\tau)\cdot\sum_{k\in\mathbb{Z}^d}(f(\tau k) - \varepsilon_f(\tau k))\overline{(g(\tau k) - \varepsilon_g(\tau k))} + \langle\varepsilon_f, \varepsilon_g\rangle_{L^2}$$
$$= \langle f_{|\tau Z^d}, g_{|\tau Z^d}\rangle_{l^2} + \langle f_{|\tau Z^d}, (\varepsilon_g)_{|\tau Z^d}\rangle_{l^2} + \langle(\varepsilon_f)_{|\tau Z^d}, g_{|\tau Z^d}\rangle_{l^2}$$
$$+ \langle(\varepsilon_f)_{|\tau Z^d}, (\varepsilon_g)_{|\tau Z^d}\rangle_{l^2} + \langle\varepsilon_f, \varepsilon_g\rangle_{L^2}$$

Hence, by using the Cauchy-Schwarz inequality, one has the following inequality:

$$|\langle f,g\rangle_{L^2} - \langle f_{|\tau Z^d}, g_{|\tau Z^d}\rangle_{l^2}| \\ \leq \|\varepsilon_f\|_{l^2}\|g\|_{l^2} + \|\varepsilon_g\|_{l^2}\|f\|_{l^2} + \|\varepsilon_f\|_{l^2}\|\varepsilon_g\|_{l^2} + \|\varepsilon_f\|_{L^2}\|\varepsilon_f\|_{L^2}. \tag{5}$$

Now, observe that also $\hat{\varepsilon}_f$ and $\hat{\varepsilon}_g$ are in $W(L^2, l^1)$ and hence ε_f and ε_g are in $W(C^0, l^2)$ and apply the inequalities of Lemma 3.3. □

Since the definition of $W(B, l^p_w)$ does not depend on the particular lattice taken, i.e., on the particular BUPU, one can show that the constants $C_{\tau,\cdot}$ do not depend on the resolution $\tau = 2^{-k} = (2^{-k_1}, \ldots, 2^{-k_d})$, $k \in \mathbb{Z}^d_+$. It will be enough to find a *good* BUPU. Consider, for example, the box function

$$\psi(x) = \begin{cases} 1, & 0 \leq x = (x_1, \ldots, x_d) \leq 1 \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

and $\psi_{h,k}(x) = \psi(2^k x - h)$, where $2^k x := (2^{-k_1}x_1, \ldots, 2^{-k_d}x_d)$ and $h \in \mathbb{Z}^d, k \in \mathbb{Z}^d_+$. For all $k > 0$, $\{\psi_{h,k}\}_{h\in\mathbb{Z}^d}$ is a BUPU.

Lemma 3.5. *The following sentences hold*:

(i) $\text{supp}(\psi_{h,k}) \subset \text{supp}(\psi_{m,0})$ *if and only if* $0 \leq h - 2^k m \leq 2^k - 1$.
(ii) *If* $f \in C^0$ *and* $k \in \mathbb{N}$ *then* $\|f\psi_{h,k}\|_\infty \leq \|f\psi_{m,0}\|_\infty$ *for all* $h \in \mathbb{Z}$ *such that* $0 \leq h - 2^k m \leq 2^k - 1$.

Proof. One has $\text{supp}(\psi_{h,k}) = \{x \in \mathbb{R}^d : 2^{-k}h \leq x \leq 2^{-k}(h+1)\}$. Hence, $\text{supp}(\psi_{h,k}) \subset \text{supp}(\psi_{m,0})$ if and only if $m \leq 2^{-k}h \leq 2^{-k}(h+1) \leq m+1$ if and only if $0 \leq h - 2^k m \leq 2^k - 1$.

Proposition 3.6. *Assume* $k > 0$ *and* $\tau = 2^{-k}$. *Then for any* $f \in W(C^0, l^2)$:

$$\|f_{|\tau\mathbb{Z}}\|_{l^2} \leq C\|f\,|W(C^0, l^2)\|,$$

where the constant C *does not depend on the particular resolution* τ.

Proof. One has:

$$\|f_{|\tau\mathbb{Z}^d}\|^2_{l^2} = 2^{-\left(\sum_{i=1}^d k_i\right)} \sum_{h\in\mathbb{Z}^d} |f(2^{-k}h)|^2 \cdot 1 \leq 2^{\left(-\sum_{i=1}^d k_i\right)} \sum_{h\in\mathbb{Z}^d} |f(2^{-k}h)\psi_{h,k}(2^{-k}h)|^2$$

$$\leq 2^{\left(-\sum_{i=1}^d k_i\right)} \sum_{h\in\mathbb{Z}^d} \|f\psi_{h,k}\|^2_\infty \leq 2^{\left(-\sum_{i=1}^d k_i\right)} \sum_{m\in\mathbb{Z}^d} \sum_{h=2^k m}^{2^k m + 2^k - 1} \|f\psi_{h,k}\|^2_\infty$$

(Lemma 3.5)

$$\leq \left(\frac{\prod_{i=1}^d (2^{k_i} - 1)}{2^{\left(\sum_{i=1}^d k_i\right)}}\right) \sum_{m\in\mathbb{Z}} \|f\psi_{m,0}\|^2_\infty \leq C_0^2\|f\,|W(L^\infty, l^2)\|^2 \leq C^2\|f\,|W(C^0, l^2)\|^2.$$

□

Remark 3.7. Under the assumptions of Theorem 3.4 and by using formula (5) one also has

$$|\|f\|_{L^2}^2 - \|f\|_{l^2}^2| \leq 2\|f\|_{l^2} \cdot \|\varepsilon_f\|_{l^2} + \|\varepsilon_f\|_{l^2}^2 + \|\varepsilon_f\|_{L^2}^2. \tag{7}$$

Solving the inequality in $\|f\|_{l^2}$ one obtains

$$\sqrt{\|f\|_{L^2}^2 - \|\varepsilon_f\|_{L^2}^2} - \|\varepsilon_f\|_{l^2} \leq \|f\|_{l^2} \leq \sqrt{\|f\|_{L^2}^2 + \|\varepsilon_f\|_{L^2}^2 + 2\|\varepsilon_f\|_{l^2}^2} + \|\varepsilon_f\|_{l^2}^2. \tag{8}$$

Applying this inequality and formula (5) one also has

$$\begin{aligned} 1|\langle f, g\rangle_{L^2} - \langle f, g\rangle_{l^2}| &\leq \left(\sqrt{\|f\|_{L^2}^2 + \|\varepsilon_f\|_{L^2}^2 + 2\|\varepsilon_f\|_{l^2}^2} + \|\varepsilon_f\|_{l^2}^2\right)\|\varepsilon_g\|_{l^2} \\ &\quad + \left(\sqrt{\|g\|_{L^2}^2 + \|\varepsilon_g\|_{L^2}^2 + 2\|\varepsilon_g\|_{l^2}^2} + \|\varepsilon_g\|_{l^2}^2\right)\|\varepsilon_f\|_{l^2} \\ &\quad + \|\varepsilon_f\|_{l^2}\|\varepsilon_g\|_{l^2} + \|\varepsilon_f\|_{L^2}\|\varepsilon_g\|_{L^2}. \end{aligned} \tag{9}$$

If $0 \approx \|\varepsilon_\star|W(C^0, l^2)\|$, $\star = f, g$, then, with approximation up to the first order, one has:

$$|\langle f, g\rangle_{L^2} - \langle f, g\rangle_{l^2}| \lesssim \|f\|_{L^2}\|\varepsilon_g\|_{l^2} + \|g\|_{L^2}\|\varepsilon\|_{l^2}$$

Keeping this approximation one can express a upper bound for the error $\delta\theta = \theta_{L^2} - \theta_{l^2}$, where $\langle f, g\rangle_{L^2} = \rho_{L^2}e^{i\theta_{L^2}}$, $\langle f, g\rangle_{l^2} = \rho_{l^2}e^{i\theta_{l^2}}$ by

$$\delta\theta \leq \arcsin\left(\frac{\|f\|_2\|\varepsilon_g\|_{l^2} + \|g\|_{L^2}\|\varepsilon_f\|_{l^2}}{|\langle f, g\rangle_{L^2}|}\right). \tag{10}$$

Hence, if $|\langle f, g\rangle_{L^2}|$ is large enough then the error on the angle can be controlled and be quite small. On the other hand, no bound is guaranteed when $|\langle f, g\rangle_{L^2}| \approx 0$. This observation turns to be crucial for angle detection in 2D pattern recognition applications (Fornaisier and Toniolo, 2002).

The estimate in Eq. (3) can be used for calculating the rate of convergence of the Riemann-type series for interesting classes of functions.

Lemma 3.8. $f \in H^\alpha(\mathbb{R}^d)$, the (*fractional*) *Sobolev space of the temperate distributions such that* $\hat{f}(\xi)w_\alpha(\xi) \in L^2$, *and assume* $\alpha > s + d/2$. *Then* $\hat{f} \in W(L^2, l_{w_s}^1)$.

Proof. $f \in H^\alpha(\mathbb{R}^d)$ implies $\hat{f} \in L_{w_\alpha}^2 = W(L_{w_\alpha}^2, l^2) = W(L^2, l_{w_\alpha}^2)$. If $c = (c_i)_{i\in I} \in l_{w_\alpha}^2$ then:

$$\sum_{i\in I}|c_i|w_s(i) = \sum_{i\in I}|c_i|w_\alpha(i)\frac{w_s(i)}{w_\alpha(i)} \leq C\sum_{i\in I}|c_i|w_\alpha(i)w_{s-\alpha}(i) \leq C\|c\|_{l_{w_\alpha}^2}\|w_{s-\alpha}\|_{l^2}.$$

This implies that $c \in l_{w_s}^1$. Hence, by Theorem 2.4 (i) one has $W(L^2, l_{w_\alpha}^2) \subset W(L^2, l_{w_s}^1)$ and hence $\hat{f} \in W(L^2, l_{w_s}^1)$. □

Lemma 3.9. *If* $s > 0$, $\tau > 0$ *and* $\hat{f} \in W(L^2, l^1_{w_s})$, *then* $\hat{f} \in W(L^2, l^1)$ *and*

$$\|\hat{\varepsilon}_f | W(L^2, l^1)\| \leq C_f\, |\tau|^s.$$

Proof. Given $\Psi = (\psi_k)_{k\in\mathbb{N}}$ a countable BUPU, there exists $k_0 \in \mathbb{N}$, $k_0 \asymp |\tau|^{-1}$ such that

$$\begin{aligned}\|\hat{\varepsilon}_f | W(L^2, l^1)\| &= \sum_{k\geq k_0} \|\hat{f}\psi_k\|_2 = \sum_{k\geq k_0} \|\hat{f}\psi_k\|_2 w_s(k)\frac{1}{w_s(k)} \\ &\leq \frac{1}{w_s(k_0)}\|f\,|W(L^2, l^1_{w_s})\| \leq C_f^0 k_0^{-s} \leq C_f\,|\tau|^s.\end{aligned}$$ □

Theorem 3.10. *If* $f \in H^{\alpha_1}$, $g \in H^{\alpha_2}$, $\alpha_1 > s_1 + d/2$ *and* $\alpha_2 > s_2 + d/2$, *then for* $\tau > 0$ *one has*:

$$|\langle f, g\rangle_{L^2} - \langle f_{|\tau\mathbb{Z}^d}, g_{|\tau\mathbb{Z}^d}\rangle_{l^2}| \leq C_1\|f\|\,|\tau|^{s_2} + C_2\|g\|\,|\tau|^{s_1} + C_3|\tau|^{s_1+s_2}, \tag{11}$$

where the norms are taken in $W(C^0, l^2)$.

Proof. By Theorem 3.4 and previous Lemmas. □

This last result shows the relation between the regularity of the involved functions and the rate of convergence.

Remark 3.11. By the Sobolev embedding theorem, if $f \in H^\alpha(\mathbb{R}^d)$ with $\alpha > s + d/2$ then $f \in C^{[s]}(\mathbb{R}^d)$. This is, as we have seen, a sufficient condition to have a rate of convergence of order s of the Riemann-type sum to the scalar product. In particular, we have required that $\hat{f} \in W(L^2, l^1_{w_s})$. One could ask now the following questions:

(i) Is $\hat{f} \in W(L^2, l^1_{w_s})$ a necessary condition to have a rate of convergence of order s?
(ii) Which is the minimal regularity of functions required to have a rate of convergence of order s?

The answer to the first question is no as one deduces from the following:

Proposition 3.12. *If* $c = (c_k)_{k\in\mathbb{N}}$ *is such that* $\sum_{k\geq k_0} |c_k| \asymp k_0^{-s}$ *then there exists* Ω *as in Definition 2.3 such that* $c \in D(\Omega, l^1, l^\infty_s)$. *In particular*, $l^1_{w_s} = D(\Omega, l^1, l^1_{w_s}) \subsetneq D(\Omega, l^1, l^\infty_{w_s})$.

Proof. $\sum_{k\geq k_0} |c_k| = O(k_0^{-s})$ implies $\sum_{k=2^l}^{2^{l+1}} |c_k| \leq C2^{-ls}, \forall l \in \mathbb{N}$ which holds if and only if $\sum_{k=2^l}^{2^{l+1}} |c_k| k^s$ is bounded. Chosen $\Omega = ([2^l, 2^{l+1}])_{l\in\mathbb{N}}$ and $\Phi = (\varphi_l)_{l\in\mathbb{N}}$, $\varphi_l = 1_{[2^l, 2^{l+1}]}$ which is an Ω-BAPU, then one has $\sum_{k=2^l}^{2^{l+1}} |c_k| k^s \in l^\infty(\mathbb{N})$ if and only if $c \in D(\Omega, l^1, l^\infty_{w_s})$. By (i) of Theorem 2.4 one has $l^1_{w_s} = D(\Omega, l^1, l^1_{w_s}) = D(\Omega, l^1_{w_s}, l^1) \subsetneq D(\Omega, l^1_{w_s}, l^\infty) = D(\Omega, l^1, l^\infty_{w_s})$. □

Definition 3.13. Given $\varphi \in S(\mathbb{R}^d)$ such that $\varphi(x) = 1$, $|x| < 1$, and $\varphi(x) = 0$, $|x| > 3/2$, we can define $\varphi_0 = \varphi$, $\varphi_1(x) = \varphi(x/2) - \varphi(x)$ and $\varphi_k(x) = \varphi(2^{1-k}x)$ for all $k \in \mathbb{N}\backslash\{0\}$. The sequence $\Phi = (\varphi_k)_{k\in\mathbb{N}}$ is called a dyadic BAPU.

We partially answer now the second question of the last remark.

Theorem 3.14. *Assume $s > 0$ and $\tau > 0$. If $f \in L^2$ is such that $\|\hat{\varepsilon}_f\,|W(L^2, l^1)\| \leq C|\tau|^s$, then $f \in B^s_{2,\infty}(\mathbb{R}^d)$ (Besov space (Triebel, 1983)). In particular, if $s > d/2$ then $f \in C^{[s-d/2]}(\mathbb{R}^d)$.*

Proof. Given $\Psi = (\psi_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ BUPU such that $\psi_{\mathbf{k}}$ is centered in $\mathbf{k}$, as in the previous proof, f satisfies equivalently the condition:

$$\sum_{2^l \leq |k| \leq 2^{l+1}} \|\hat{f}\psi_k\|_2 |k|^s \in l^\infty(\mathbb{N}), \tag{12}$$

which implies also the following condition. Given $\Phi = (\varphi_l)_{l\in\mathbb{N}}$ a dyadic BAPU:

$$\sup_{l\in\mathbb{N}} \|\hat{f}\varphi_l\|_{W(L^2,l^1)} 2^{ls} < \infty. \tag{13}$$

In fact, for $l \in \mathbb{N}$ fixed, one has:

$$\begin{aligned}\|\hat{f}\varphi_l\|_{W(L^2,l^1)} &= \sum_{k\in\mathbb{Z}^d} \|\hat{f}\varphi_l\psi_k\|_2 \\ &\leq \sum_{2^l \leq |k| \leq 2^{l+2}} \|\hat{f}\psi_k\|_2 \|\varphi_l\|_\infty \leq \sum_{2^l \leq |k| \leq 2^{l+2}} \|\hat{f}\psi_k\|_2.\end{aligned}$$

This means, in particular, that $\hat{f} \in D(\Omega, W(L^2, l^1), l^\infty_{2^{s\cdot}})$. Moreover Eq. (13) implies, by means of (ii) of Theorem 2.4, that

$$\sup_{l\in\mathbb{N}} \|F^{-1}(\hat{f}\varphi_l)\|_{W(C^0,l^2)} 2^{ls} < \infty \tag{14}$$

Since $W(C^0, l^2) \subset W(L^2, l^2) = L^2$ and by definition of $B^s_{2,\infty}$ (Triebel, 1983) finally one has $f \in B^s_{2,\infty}$. The last statement follows by inclusions of Besov spaces in C^m spaces.

Remark 3.15. Even if the result of Theorem 3.14 tells us that the required regularity in dimension $d > 1$ is less than that assumed in Theorem 3.10, in dimension $d = 1$, as a necessary condition in order to have a rate of convergence of $s \in \mathbb{N}$, one needs at least for $f \in C^{s-1}$, a very regular function. Anyway the results tell us also that we are able to calculate scalar products by Riemann-type sums with the precision we want up to a sufficient regularity. It is also clear that we can extend Theorem 3.10 in a natural way: if $f, g \in H^\alpha$, with $\alpha > 0$ large enough, we can approximate also scalar products in the Sobolev space H^β, $0 < \beta < \alpha$, $\beta \in \mathbb{N}$. In fact, the scalar product of H^β is the sum of the scalar products of the derivatives, whenever $\beta \in \mathbb{N}$.

Example 3.16. We propose now an example which indicates the existence of functions in $B^s_{2,\infty}$, but not in any H^α, for $\alpha > s + d/2$, having a rate of convergence in Eq. (11) of order s.

Cardinal B-splines can be constructed by convolution from the box function ψ previously defined in Eq. (6). In particular, one has:

$$N_m(x) = \begin{cases} \psi(x+1/2), & m = 1 \\ N_{m-1} * N_1(x), & m > 1 \end{cases} \tag{15}$$

They are used in order to construct compactly supported wavelets (Daubechies, 1992) and in many other applications in approximation theory.

It is well known that $N_m \in C^{m-2}$, but it is also $N_m \in B^{m-1}_{2,\infty}$ and $N_m \notin H^\alpha$, for all $\alpha > (m-1) + 1/2$ as one can deduce by means of the following relation:

$$\hat{N}_m(\xi) = \left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^m \asymp \xi^{-m}, \tag{16}$$

$\xi \to \infty$. On the other hand, there exists $\beta > (m-2) + 1/2$ such that $N_m \in H^\beta$. By definition (15) one also has:

$$N_{2m}(y) = \int_{\mathbb{R}} N_m(x) N_m(y-x)\, dx. \tag{17}$$

By expression (17) one can calculate analytically the scalar product $\langle N_m(\cdot), N_m(y-\cdot)\rangle$. Hence, we can compare the approximation of Eq. (17) by Riemann-type sums and its real value. Consider, then, the error function $\gamma_{m,y}$ defined on $\mathbb{R}_+$ by:

$$\gamma_{m,y}(\tau) = N_{2m}(y) - \det(\tau) \sum_{k\in\mathbb{Z}} N_m(\tau k) N_m(y - \tau k). \tag{18}$$

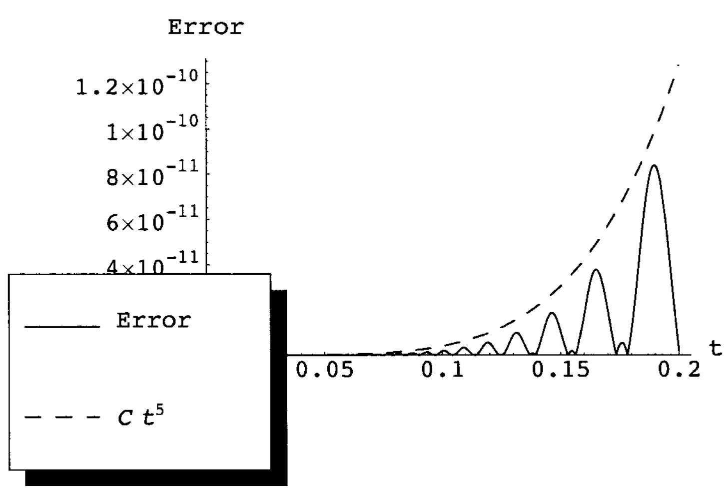

Figure 1. Rate of convergence for B-splines.

Let $s = m - 1$. From Eq. (11) and since $N_m \in H^\beta$, one could expect that $|\gamma_{m,y}|$ moves to zero at least as $O(\tau^{s-1})$. On the other hand, Theorem 3.14 gives the hope of a better rate of the type $O(\tau^s)$, as $N_m \in B^s_{2,\infty}$.

Figure 1 shows the graphic of function $|\gamma_{m,y}|$ for $m = 6$ and $y = 1.3$. In this case the rate seems to be in fact at least of order $s = 5$. Moreover, by Eq. (16), it is not hard to show, calculating explicitly the norm, that

$$\|\hat{\varepsilon}_{N_m} | W(L^2, l^1)\| \lesssim |\tau|^s. \tag{19}$$

In fact, Theorem 3.14 confirms that $N_m \in B^s_{2,\infty}$.

Usually the problem of numerical integration, the construction of quadrature formulas, is treated by polynomial interpolation on special nodes. On the other hand, the approach carried in this article shows we are able to approximate integrals in a efficient way even by Riemann-type sums up to a sufficient smoothness, well described by previous results. Moreover, as the tools we have used in this article (Fourier analysis, Wiener amalgam spaces, (ir)regular sampling theory (Feichtinger and Gröchenig, 1992; Feichtinger et al., 1995) are developed over general locally compact Abelian (LCA) groups, one could extend our approach and results in that context, building directly a correspondent numerical integration method and estimates on LCA groups. This fact is one of the advantage of this Fourier approach with respect to the classical polynomial interpolation one. The computational advantage comes from the possibility to use the FFT algorithm to calculate Riemann-type sums: in Gabor (Feichtinger and Strohmer, 1998, 2003) and wavelets (Daubechies, 1990, 1992) expansions and frames of translates (Casazza et al., 2001), for example, the coefficients are scalar products with functions translated by integer coordinates, i.e., they are sampled convolutions. It is well known that a discrete convolution is equivalent to apply the formula $\Im^{-1}(\Im(f_{|\tau\mathbb{Z}^d})\Im(g_{|\tau\mathbb{Z}^d}))$, where $\Im()$ is the discrete Fourier transform.

Other different uses of the scalar products approximation as convolutions in digital signal processing and pattern recognition can be found in (Fornaiser and Toniolo, 2002). In this case, the estimate (5) is used in order to guarantee that special eigenfunctions of two-dimensional rotation operators, the circular harmonics basis on the circle, can approximatively keep, at least weakly, their invariance up to rotations even when they were sampled at a fixed resolution. In this way, one can construct efficient (accurate and fast) methods for comparing and representing digital images up to mutual rotations and translations.

Finally one can deduce simply the rate of convergence of Riemann sums to integrals of (single) functions.

Remark 3.17. If $f \in L^2$, $\hat{f} \in W(L^2, l^1)$ and f is compactly supported one can choose $\theta \in \mathcal{S}$ such that:

$$\left| \int f - \int f\theta \right| = 0.$$

Just take, $\theta_{|\mathrm{supp}(f)} \equiv 1$. Hence, the approximation of the integral by a Riemann sum depends on the convergence rate of

$$\left|\int f - \det(\tau)\sum_{k\in Z^d} f(\tau k)\right| = \left|\int f\theta - \det(\tau)\sum_{k\in Z^n} f(\tau k)\theta(\tau k)\right| \to 0, \quad \tau \to 0.$$

One can use Theorem 3.10 to estimate the quantities, just depending on the regularity of f. In particular, if $f \in H^\alpha$, $\alpha > s + d/2$ then

$$\left|\int f - \det(\tau)\sum_{k\in Z^d} f(\tau k)\right| \lesssim |\tau|^s. \tag{20}$$

4. CONCLUSIONS

Sampling theory and function spaces (in particular Wiener amalgams) inclusions appear effective tools in order to describe discretizations and error estimates in numerical harmonic analysis. In particular, by means of these methods, one can avoid the use of polynomial approximations and complex analysis, usually considered the most powerful tools. Minimal fractional regularity in the sense of the Besov space $B^s_{2,\infty}$ is investigated in order to achieve some given rate of convergence of Riemann type sums to scalar products in Sobolev spaces H^α by increasing resolution, with possible applications in numerical treatment of digital signals and PDEs.

ACKNOWLEDGMENTS

The author wants to thank Giuseppe De Marco (Department of Pure and Applied Mathematics, University of Padua, Italy), Hans G. Feichtinger (Department of Mathematics, University of Vienna, Austria), Walter Gautschi (Department of Computer Science, Purdue University, West Lafayette, Indiana) for the fruitful discussions, suggestions, and encouragement. The author has been supported by the Doctorate program in Computational Mathematics at the University of Padna, Italy.

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