LINEARLY CONSTRAINED EVOLUTIONS OF CRITICAL POINTS AND AN APPLICATION TO COHESIVE FRACTURES

M. ARTINA, F. CAGNETTI, M. FORNASIER, F. SOLOMBRINO

Abstract. We introduce a novel constructive approach to define time evolution of critical points of an energy functional. Differently from other more established approaches based on viscosity approximations in infinite dimension, our procedure is prone to efficient and consistent numerical implementations, and allows for an existence proof under very general assumptions. We consider in particular rather nonsmooth and nonconvex energy functionals, provided the domain of the energy is finite dimensional. Nevertheless, in the infinite dimensional case study of a cohesive fracture model, we prove a consistency theorem of a discrete-to-continuum limit. We show that a quasistatic evolution can be indeed recovered as limit of evolutions of critical points of finite dimensional discretizations of the energy, constructed according to our scheme. To illustrate the results, we provide several numerical experiments both in one and two dimensions. These agree with the crack initiation criterion, which states that a fracture appears only when the stress overcomes a certain threshold, depending on the material.

Keywords: quasistatic evolution, cohesive fracture, numerical approximation.

2000 Mathematics Subject Classification: 49J27, 74H10, 74R99, 74S20, 58E30.

1. Introduction

In this paper we introduce a model to study the time evolution of physical systems through linearly constrained critical points of the energy functional. Our approach is constructive, and it can be numerically implemented, as we show with an application to cohesive fracture evolution. Since we are mainly interested in numerical applications, we will consider at first finite dimensional systems. We will then also give a concrete case study showing how our results can be adapted to describe infinite dimensional systems.

Below we recall some general facts about the topic. Then we present and comment our results, also in comparison with related contributions appeared in recent literature.

1.1. Critical points evolutions in the literature. When describing the behaviour of a physical system, one can try to follow the time evolution of absolute minimizers of the energy. This idea has been used, for instance, in [5, 9, 11, 13, 15, 16, 17]. However, it is not always realistic to expect the energy to be minimized at every fixed time. In fact, global minimization may lead the system to change instantaneously in a very drastic way, and this is something which is not very often observed in nature. For this reason, several authors recently introduced time evolutions of critical points (rather than absolute minimizers) of the energy functional [1, 4, 7, 12, 18, 21].

Given a time-dependent functional \( F : Y \times [0,T] \to \mathbb{R} \), where \( Y \) is a Banach space, an evolution of critical points of \( F \) is a function \( u : [0,T] \to Y \) which satisfies

\[
0 \in \partial_u F(u(t), t), \quad \text{for a.e. } t \in [0,T],
\]  

(1.1)
where $\partial_u F$ denotes the subdifferential of $F$ with respect to $u$. Typically, the existence of such an evolution is proven by a singular perturbation method. More precisely one first considers, for every $\varepsilon > 0$, the $\varepsilon$-gradient flow

$$-\varepsilon \dot{u}^\varepsilon \in \partial_u F(u^\varepsilon(t), t)$$

with an initial datum $u^\varepsilon(0) = u_0$, where $u_0$ is a critical point of $F(\cdot, 0)$. Then, passing to the limit as $\varepsilon \to 0^+$, $u^\varepsilon$ converges to a function $u$ satisfying (1.1).

We now give a detailed description of our results and we comment on them.

1.2. Setting of the problem and main result. Let $\mathcal{E}$ and $\mathcal{F}$ be two Euclidean spaces of dimension $n$ and $m$, respectively, with $n > m$. We want to study the evolution in a time interval $[0, T]$ of a physical system, whose states are described by vectors $v \in \mathcal{E}$, and whose energy is given by a function $J : \mathcal{E} \to [0, +\infty)$. We assume that a time dependent constraint $f : [0, T] \to \mathcal{F}$ is imposed. More precisely, the only admissible states at each time $t \in [0, T]$ satisfy $Au = f(t)$, where $A : \mathcal{E} \to \mathcal{F}$ is a surjective linear operator.

Before stating the main result of the paper, let us give some definitions. At any time $t \in [0, T]$, one can consider the problem

$$\min_{Av=f(t)} J(v).$$

If $u \in \mathcal{E}$ is a minimizer for (1.3), then

$$Au = f(t) \quad \text{and} \quad \partial J(u) \cap \text{ran}(A^*) \neq \emptyset,$$

where $A^* : \mathcal{F} \to \mathcal{E}$ is the adjoint of $A$, $\text{ran}(A^*)$ denotes the range of $A^*$, and $\partial J(u)$ is the Fréchet subdifferential of $J$ at $u$. A critical point of $J$ on the affine space $A(f(t))$ is any vector $v \in \mathcal{E}$ satisfying (1.4) where, for every $f \in \mathcal{F}$, we set

$$A(f) := \{ v \in \mathcal{E} : Av = f \}.$$

A discrete quasistatic evolution with time step $\delta \in (0, 1)$, initial condition $v_0 \in \mathcal{E}$, and constraint $f$, is a right-continuous function $v_\delta : [0, T] \to \mathcal{E}$ such that

- $v_\delta(0) = v_0$;
- $v_\delta$ is constant in $[0, T] \cap [i\delta, (i + 1)\delta)$ for all $i \in \mathbb{N}_0$ with $i\delta < T$;
- $v_\delta(i\delta)$ is a critical point of $J$ on the affine space $A(f(i\delta))$ for every $i \in \mathbb{N}_0$ with $i\delta \leq T$.

Finally, we say that a measurable function $v : [0, T] \to \mathcal{E}$ is an approximable quasistatic evolution with initial condition $v_0$ and constraint $f$, if for every $t \in [0, T]$ there exists a sequence $\delta_k \to 0^+$ (possibly depending on $t$) and a sequence $\{v_{\delta_k}\}_{k \in \mathbb{N}}$ of discrete quasistatic evolutions with time step $\delta_k$, initial condition $v_0$, and constraint $f$, such that

$$\lim_{k \to +\infty} |v_{\delta_k}(t) - v(t)|_{\mathcal{E}} = 0.$$

We are now ready to state the main result of the paper (see Theorem 2.12).

Let $v_0$ be a critical point of $J$ on the affine space $A(f(0))$. Under suitable assumptions on $J$, $A$, and $f$ (see (2.1) and conditions (A0)--(A3) in Section 2), there exists bounded and measurable functions $v : [0, T] \to \mathcal{E}$ and $q : [0, T] \to \mathcal{F}$ such that:

(a) $v(\cdot)$ is an approximable quasistatic evolution with initial condition $v_0$ and constraint $f$;

(b) $A^*q(t) \in \partial J(v(t))$ for every $t \in [0, T]$;
(c) the function \( s \mapsto \langle q(s), \dot{f}(s) \rangle_F \) belongs to \( L^1(0, T) \) and for every \( t \in [0, T] \) we have
\[
J(v(t)) \leq J(v_0) + \int_0^t \langle q(s), \dot{f}(s) \rangle_F \, ds,
\]
where \( \langle \cdot, \cdot \rangle_F \) denotes the standard scalar product in \( F \).

Note that in our definition the precise value of \( v \) at every point \( t \) matters. In particular, the initial condition \( v(0) = v_0 \) has a meaning, and the energy inequality (c) needs to be satisfied at every time. Thus, we are in general not identifying functions differing on null sets, as it is usual in \( L^p \) spaces.

1.3. Comments on our result. We would like to emphasize a few aspects of our results.

(i) We prove the existence of an approximable quasistatic evolution for a large class of energy functions \( J \) and linear operators \( A \) (see (2.1) and conditions (A0)–(A3) in Section 2). In particular, these include nonsmooth and nonconvex energy functionals.

(ii) We stress the fact that \( v(t) \) is supposed to visit at different times \( t \) critical points of the cost function \( J \) over the affine space \( A(f(t)) \). This condition is rather general, if compared to the usual requirement of focusing on global minimizers of \( J \) over \( A(f(t)) \). An evolution along critical points is in general more realistic and physically sound. However, the analysis of such an evolution is usually very involved, and may allow for solutions \( v \) that are just measurable in time. While we shall be content with the generality of our results, we have to live with the fact that our solutions may not be regular.

(iii) Our approach is constructive. This is an important fact since, as we shall emphasize later, the functional \( J \) may have multiple feasible critical points at the same time \( t \). Hence, in order to promote uniqueness of evolutions, or even just their measurability, we need to design a proper selection principle. Accordingly, we shall design our selection algorithm in such a way that the selected critical point is the closest - in terms of Euclidean distance - to the one chosen at the previous instant of time, unless it is energetically convenient to perform a “jump” to another significantly different phase of the system (see formula (3.3)). This corresponds to a rather common and well-established behavior of several physical (and non physical) systems (see, for instance, [4, Section 9]). Moreover, this method can be easily implemented by means of a corresponding numerical method [2], as we will show with an application to cohesive fracture evolution, see Section 6. Thus, our evolution is the result of a constructive machinery (an algorithm), which is designed to emulate physical principles, according to which a critical point is selected in terms of a balance between neighborliness (accounting the Euclidean path length between critical points) and energy convenience. In our view, this feature is of great relevance as we provide a black box, whose outputs are physically sound solutions.

(iv) As an important remark, we stress that in general all of the constants appearing in the technical assumptions in Section 2 could depend on the dimension of the considered Euclidean spaces. Thus, our results can be applied to physical systems that can assume (a discrete or a continuum of) infinitely many states, provided all the relevant estimates obtained are dimension free. For this reason, we explain very clearly which are the parameters that can affect the constants that come into play in the crucial proofs (see Remark 3.7). We give an important application to cohesive fracture evolution (see Section 4 and Section 5), showing how an infinite dimensional system can be studied with our method. This, in particular, gives an alternative proof of the existence of
evolutions of critical points for the cohesive fracture model firstly proven in [4]. In addition, we can provide numerical simulations (see Section 6).

(v) As already mentioned, the proof of the main result (Theorem 2.12) is given only in a finite dimensional setting. This is due to the fact that, in the infinite dimensional case, the subdifferential is in general not closed with respect to the weak convergence in the domain of the energy. Such a difficulty could be overcome by requiring that the energy functional has compact sublevels, an assumption which is quite common in literature. For the model case we have in mind (see Section 4), this would amount to choosing a different domain for the energy functional. By making this choice, however, we would have that some of the other assumptions needed (see conditions (A2) and (A3) in Section 2) do not hold true anymore, see Remark 4.10 for further details. This motivates our choice of first dealing with a finite dimensional setting, and then extending the results to our model case (see Theorem 5.5) with a problem-specific technique.

(vi) The numerical simulations that we provide in Section 6 for the cohesive fracture evolutions give physically sound results. In particular, they agree with the crack initiation criterion (see [4, Theorem 4.6]), which states that a crack appears only when the maximum sustainable stress of the material is reached.

To the best of our knowledge, this is the first time that an algorithm providing critical points evolution is introduced in such a generality. However, it is worth mentioning that our approach is not completely new, and that similar methods have been exploited in recent literature.

In [23], a related scheme has been for instance investigated in order to obtain a general existence result in a nonconvex but still smooth setting. The author also takes into account viscous dissipation effects, and constructively provides a suitable time rescaling where the evolutions have a continuous dependence on time. This idea, in particular, generalises previous approaches for systems driven by nonconvex energy functionals ([8, 14, 20, 26]). Moreover, the author shows an approximation result that in spirit is close to our Theorem 5.5, and to previous results in [22, 25]. However, the results in [23] are obtained under the assumption of \( C^1 \) regularity of the energy functional, and in an unconstrained setting. In particular, stability of critical points after passing to the limit is recovered through a very strong assumption ([23, (8), Theorem 2.3]), which would seem quite unnatural for a constrained nonsmooth problem.

The plan of the paper is the following. In Section 2 we state the main result of the paper, Theorem 2.12, whose proof is given in Section 3. Section 4 is devoted to the description of the cohesive fracture model introduced in [4]. In the same section we introduce a mesh and discretize the problem. In Section 5 we pass to the limit as the size of the mesh tends to 0, thus obtaining a new proof of the result in [4, Theorem 4.4]. Finally, numerical simulations are given in Section 6.

2. Setting of the problem and main result

Throughout all the paper, we use the notation \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), and we denote by \( \mathcal{L}^1 \) the standard Lebesgue 1-dimensional measure in \( \mathbb{R} \). Let \( \mathcal{E} \simeq \mathbb{R}^n \) and \( \mathcal{F} \simeq \mathbb{R}^m \) be two Euclidean spaces which are isomorphic to \( \mathbb{R}^n \) and \( \mathbb{R}^m \) respectively, with \( m < n \). We consider an energy function \( J : \mathcal{E} \to [0, +\infty) \), a linear operator \( A : \mathcal{E} \to \mathcal{F} \), and a time dependent constraint \( f : [0, T] \to \mathcal{F} \). We will assume that \( A \) is surjective. Equivalently, we will suppose that that there exists \( \gamma > 0 \) such that the adjoint operator \( A^* : \mathcal{F} \to \mathcal{E} \) satisfies

\[
|A^*q|_\mathcal{E} \geq \gamma |q|_\mathcal{F} \quad \text{for every } q \in \mathcal{F}.
\]
Given \( u \in \mathcal{E} \), we recall that the Fréchet subdifferential \( \partial J(u) \subset \mathcal{E}' \) of \( J \) at a point \( u \in \mathcal{E} \) is defined in the following way:

\[
\xi \in \partial J(u) \iff 0 \leq \liminf_{v \to u} \frac{J(v) - (J(u) + \langle \xi, v - u \rangle_{\mathcal{E}'\mathcal{E}})}{|v - u|_{\mathcal{E}}},
\]

For every \( f \in \mathcal{F} \), we set

\[
A(f) := \{ v \in \mathcal{E} : Av = f \}.
\]

We can now give a precise definition of critical point in our setting.

**Definition 2.1.** Let \( f \in \mathcal{F} \). We say that \( u \in \mathcal{E} \) is a critical point of \( J \) on the affine space \( A(f) \) if

\[
Au = f \quad \text{and} \quad \partial J(u) \cap \text{ran}(A^*) \neq \emptyset,
\]

where \( \text{ran}(A^*) \) denotes the range of \( A^* \).

**Remark 2.2.** One can check that condition (2.2) implies, in turn, that

\[
0 \leq \liminf_{\varepsilon \to 0^+} \frac{J(u + \varepsilon w) - J(u)}{\varepsilon} \quad \forall w \in \ker(A),
\]

where \( \ker(A) \) denotes the kernel of \( A \).

**Remark 2.3.** The definition above is motivated by the fact that if \( u \in \mathcal{E} \) satisfies

\[
\min_{\mathcal{H}_v = f} J(v) = J(u),
\]

then (2.2) holds true.

We can now start stating our assumptions. We suppose that:

- (A0) the functional \( v \mapsto J(v) + |Av|^2_{\mathcal{F}} \) is coercive;
- (A1) there exists \( \eta > 0 \) such that \( v \mapsto J_\eta(v) := J(v) + \eta |v|^2_{\mathcal{E}} \) is strictly convex;
- (A2) there exist \( K, L > 0 \) such that, for every \( v \in \mathcal{E}, \xi \in \partial J(v) \implies |\xi|_{\mathcal{E}'} \leq K J(v) + L \).

Before proceeding with the setting of the problem, we make some remarks on the assumptions above.

**Remark 2.4.** Condition (A1) implies that \( J \) is a smooth perturbation of a convex function and, therefore, is locally Lipschitz continuous. From this, it follows that \( J \) is almost everywhere differentiable and its Fréchet subdifferential is nonempty at every point.

**Remark 2.5.** Let \( \eta > 0 \) be such that (A1) holds true, and let \( \overline{v} \in \mathcal{E} \). Then, the functional \( J_{\eta,\overline{v}} : \mathcal{E} \to [0, \infty) \) defined as

\[
J_{\eta,\overline{v}}(v) := J(v) + \eta |v - \overline{v}|^2_{\mathcal{E}}
\]

for every \( v \in \mathcal{E} \), is also strictly convex.

**Remark 2.6.** If \( J \) is globally Lipschitz continuous with Lipschitz constant \( L \), then

\[
\xi \in \partial J(v) \implies |\xi|_{\mathcal{E}'} \leq L,
\]

for every \( v \in \mathcal{E} \). Moreover, if \( J = J_1 + J_2 \) where \( J_1 \) is lower semicontinuous and \( J_2 \) is of class \( C^1 \), then the decomposition

\[
\partial J(v) = \partial J_1(v) + DJ_2(v)
\]

holds true at every point \( v \in \mathcal{E} \) such that \( \partial J(v) \neq \emptyset \), where \( DJ_2(v) \) denotes the gradient of \( J_2 \) at \( v \).

**Remark 2.7.** As shown in [2, Remark 2.5], it suffices to check condition (A2) only at differentiability points of \( J \), to ensure that it is satisfied at every point.
Our last assumption is the following.

(A3) There exists a positive constant $C_{J, \eta} > 0$ such that
\[ A^*q_i \in \partial J_{\eta} \pi(v_i), \quad i = 1, 2 \implies (q_1 - q_2, Av_1 - Av_2)_\mathcal{F} \leq C_{J, \eta}|v_1 - v_2|_\mathcal{F}|Av_1 - Av_2|_\mathcal{F} \quad (2.7) \]
for every $\pi \in \mathcal{E}$.

**Remark 2.8.** Although condition (A3) above might seem quite technical, it is automatically satisfied when $J \in C^{1,1}$. Indeed, in this case $\partial J(v)$ is single valued at every $v \in \mathcal{E}$, and coincides with the differential $DJ(v)$. Then, denoting by $L$ the Lipschitz constant of $DJ(\cdot)$ and using (2.1), one has
\[ |q_1 - q_2|_\mathcal{F} \leq \frac{1}{\gamma}|A^*q_1 - A^*q_2|_\mathcal{F} \leq \frac{1}{\gamma}|DJ(v_1) - DJ(v_2)|_{\mathcal{F}} \leq \frac{L}{\gamma}|v_1 - v_2|_\mathcal{F}. \]

At this point, (2.7) simply follows by the Cauchy-Schwarz inequality.

**Remark 2.9.** We will show in a concrete example that condition (A3) can also be satisfied when $J \notin C^{1,1}$ (see Section 4.3).

Before stating our main result, we give the notion of discrete and approximable quasistatic evolution, respectively.

**Definition 2.10.** Let $v_0 \in \mathcal{E}$ be a critical point of $J$ on the affine space $\mathbf{A}(f(0))$, and let $\delta > 0$. A discrete quasistatic evolution with time step $\delta$, initial condition $v_0$, and constraint $f$ is a right-continuous function $v_\delta : [0, T] \to \mathcal{E}$ such that

- $v_\delta(0) = v_0$;
- $v_\delta$ is constant in $[0, T] \cap [i\delta, (i + 1)\delta)$ for all $i \in \mathbb{N}_0$ with $i\delta \leq T$;
- $v_\delta(i\delta)$ is a critical point of $J$ on the affine space $\mathbf{A}(f(i\delta))$ for every $i \in \mathbb{N}$ with $i\delta \leq T$.

**Definition 2.11.** Let $v_0 \in \mathcal{E}$ be a critical point of $J$ on the affine space $\mathbf{A}(f(0))$. A bounded measurable function $v : [0, T] \to \mathcal{E}$ is said to be an approximable quasistatic evolution with initial condition $v_0$ and constraint $f$, if for every $t \in [0, T]$ there exists a sequence $\delta_k \to 0^+$ (possibly depending on $t$) and a sequence $\{v_\delta_k\}_{k \in \mathbb{N}}$ of discrete quasistatic evolutions with time step $\delta_k$, initial condition $v_0$, and constraint $f$, such that
\[ \lim_{k \to +\infty} |v_\delta_k(t) - v(t)|_{\mathcal{F}} = 0. \quad (2.8) \]

We are now ready to state our main result, whose proof is postponed to Section 3.

**Theorem 2.12.** Let (2.1), (A0), (A1), (A2), and (A3) be satisfied. Let $f \in W^{1,2}([0, T]; \mathcal{F})$, and let $v_0$ be a critical point of $J$ in the affine space $\mathbf{A}(f(0))$. Then, there exist bounded and measurable functions $v : [0, T] \to \mathcal{E}$ and $q : [0, T] \to \mathcal{F}$ such that:

(a) $v(\cdot)$ is an approximable quasistatic evolution with initial condition $v_0$ and constraint $f$;

(b) $A^*q(t) \in \partial J(v(t))$ for every $t \in [0, T]$;

(c) The function $s \mapsto (q(s), \dot{f}(s))_\mathcal{F}$ belongs to $L^1(0, T)$ and for every $t \in [0, T]$ we have
\[ J(v(t)) \leq J(v_0) + \int_0^t (q(s), \dot{f}(s))_\mathcal{F} \, ds. \]

**Remark 2.13.** A careful inspection of the proof of Theorem 2.12 shows that
\[ |q(t)|_\mathcal{F} \leq C_1 \quad \text{and} \quad |v(t)|_\mathcal{F} \leq C_2 \quad \text{for every} \ t \in [0, T], \]
where $C_1$ and $C_2$ are given by Theorem 3.6.
3. Proof of Theorem 2.12

In this section we prove Theorem 2.12. We start by showing that, under suitable assumptions, an approximable quasistatic evolution is automatically an evolution of critical points of the energy functional $J$.

**Proposition 3.1.** Suppose that (2.1), (A1), and (A2) are satisfied, and let $f \in W^{1,2}([0,T];\mathcal{F})$. Let $v_0 \in \mathcal{E}$ be a critical point of $J$ on the affine space $A(f(0))$, and let $v : [0,T] \to \mathcal{E}$ be an approximable quasistatic evolution with initial condition $v_0$ and constraint $f$. Then, $v(t)$ is a critical point of $J$ on the affine space $A(f(t))$ for every $t \in [0,T]$.

**Proof.** Let $\{v_{\delta_k}\}_{k \in \mathbb{N}}$ be as in (2.8), and let $t \in [0,T]$ be fixed. For every $k \in \mathbb{N}$, let $i_k \in \mathbb{N}$ be such that (to ease the notation, we do not stress the dependence of $i_k$ on $t$)

$$i_k \delta_k \leq t < (i_k + 1) \delta_k.$$

From the definition of approximate quasistatic evolution we have $A v_{\delta_k}(t) = f(i_k \delta_k)$. Then, by continuity of $f$ and (2.8) we obtain $A v(t) = f(t)$.

We thus need only to show that $\partial J(v(t)) \cap \text{ran}(A^*) \neq \emptyset$. By definition of constrained critical point, there exists $q_k \in \mathcal{F}$ (note that also $q_k$ will, in general, depend on $t$) such that

$$A^* q_k \in \partial J(v_{\delta_k}(t)) \quad \text{for every } k \in \mathbb{N}. \quad (3.1)$$

From (A1) it follows that $J$ is locally bounded and therefore, by (2.8), we have

$$\sup_{k \in \mathbb{N}} J(v_{\delta_k}(t)) < +\infty.$$

Last relation, together with (2.1), (3.1), and (A2), gives

$$\sup_{k \in \mathbb{N}} |q_k|_\mathcal{F} \leq \frac{1}{\gamma} \sup_{k \in \mathbb{N}} |A^* q_k|_\mathcal{E} \leq \frac{1}{\gamma} (K \sup_{k \in \mathbb{N}} J(v_{\delta_k}(t)) + L) < +\infty.$$

Thus, there exists $q \in \mathcal{F}$ such that, up to subsequences,

$$\lim_{k \to +\infty} |q_k - q|_\mathcal{F} = 0. \quad (3.2)$$

From (2.8), (3.1), and (3.2) we get, by the closure property of the subdifferential, that

$$A^* q \in \partial J(v(t)),$$

as required. \hfill \square

In order to construct an approximate quasistatic evolution, we first introduce an auxiliary minimum problem. Let $\delta \in (0,1)$ be a fixed time step, and let $i \in \mathbb{N}$ with $i \delta \leq T$. Suppose that $v^{i-1} \in \mathcal{E}$ is a critical point of $J$ on the affine space $A(f((i-1)\delta))$. If property (A1) is satisfied, we define the sequence $\{v^j_i\}_{j \in \mathbb{N}_0}$ by setting $v^0_i := v^{i-1}$ and

$$v^j_i := \arg\min_{A v = f(i \delta)} \left\{ J(v) + \eta |v - v^j_{i-1}|^2_\mathcal{E} : v \in \mathcal{E} \right\} \quad \text{for every } j \in \mathbb{N}. \quad (3.3)$$

**Remark 3.2.** Note that (A1) guarantees that the above minimum is unique.

The following lemma gives some properties of the sequence $\{v^j_i\}_{j \in \mathbb{N}_0}$.

**Lemma 3.3.** Let (2.1), (A0), (A1), and (A2) be satisfied, and let $f \in W^{1,2}([0,T];\mathcal{F})$. Let $\delta \in (0,1)$ and let $i \in \mathbb{N}$ with $i \delta \leq T$. Suppose that $v^{i-1}$ is a critical point of $J$ on the affine space $A(f((i-1)\delta))$, and let $\{v^j_i\}_{j \in \mathbb{N}_0}$ be as in (3.3). Then:
(1) \( \{J(v_j^i)\}_{j \in \mathbb{N}} \) is a nonincreasing converging sequence;

(2) \( \{v_j^i\}_{j \in \mathbb{N}} \) is bounded and

\[
\lim_{j \to +\infty} |v_j^i - v_{j-1}^i|_\mathcal{E} = 0; \tag{3.4}
\]

(3) any limit point of \( \{v_j^i\}_{j \in \mathbb{N}} \) is a critical point of \( J \) on the affine space \( \mathbf{A}(f(i\delta)) \).

Proof. For every \( j \geq 2 \) we have \( Av_j^i = Av_{j-1}^i = f(i\delta) \), and therefore \( v_j^i \) is a competitor for the minimum problem in (3.3). Thus,

\[
J(v_j^i) \leq J(v_{j-1}^i) - \eta|v_j^i - v_{j-1}^i|_\mathcal{E}^2, \quad \text{for every } j \geq 2. \tag{3.5}
\]

In particular, the sequence \( \{J(v_j^i)\}_{j \in \mathbb{N}} \) is nonincreasing. Since \( J \geq 0 \), the limit

\[
\lim_{j \to \infty} J(v_j^i) =: C \geq 0. \tag{3.6}
\]

exists and is nonnegative, and this shows (1). Let now \( M \in \mathbb{N} \) with \( M > 2 \). Summing up relation (3.5) for \( j = 2, \ldots, M \) we obtain

\[
\sum_{j=2}^{M} |v_j^i - v_{j-1}^i|_\mathcal{E}^2 \leq \frac{1}{\eta} (J(v_1^i) - J(v_M^i)).
\]

Sending \( M \to \infty \) we then have

\[
\sum_{j=2}^{\infty} |v_j^i - v_{j-1}^i|_\mathcal{E}^2 \leq \frac{1}{\eta} (J(v_1^i) - C) < \infty.
\]

In particular, this shows that (3.4) holds true. Note now that, by (3.6), \( \{J(v_j^i)\}_{j \in \mathbb{N}} \) is bounded. Therefore, since \( |Av_j^i|_\mathcal{E}^2 = |f(i\delta)|_\mathcal{E}^2 \) for every \( j \geq 1 \), the sequence \( \{J(v_j^i) + |Av_j^i|_\mathcal{E}^2\}_{j \in \mathbb{N}} \) is bounded. By (A0), we have that \( \{v_j^i\}_{j \in \mathbb{N}} \) is also bounded, and this concludes the proof of (2).

Let \( v^i \in \mathcal{E} \) be a limit point of \( \{v_j^i\}_{j \in \mathbb{N}} \). Up to subsequences, we can assume that

\[
\lim_{j \to \infty} v_j^i = v^i, \quad \text{in } \mathcal{E}.
\]

First of all, note that \( Av^i = f(i\delta) \). By (3.3), for every \( j \in \mathbb{N} \) there exists \( q_j^i \in \mathcal{F} \) such that

\[
A^*q_j^i \in \partial J(v_j^i) + 2\eta(v_j^i - v_{j-1}^i),
\]

where we used Remark 2.6. The previous relation can also be written as

\[
A^*q_j^i = \xi_j^i + 2\eta(v_j^i - v_{j-1}^i), \tag{3.7}
\]

for some \( \xi_j^i \in \partial J(v_j^i) \). Note that, since \( \{v_j^i\}_{j \in \mathbb{N}} \) is bounded, by (A2) we also have that \( \{\xi_j^i\}_{j \in \mathbb{N}} \) is bounded. Thanks to (3.4) and (2.1), this implies that \( \{q_j^i\}_{j \in \mathbb{N}} \) is also bounded. Thus, up to subsequences, we can assume that

\[
\lim_{j \to \infty} \xi_j^i = \xi^i \quad \text{in } \mathcal{E}', \quad \text{and} \quad \lim_{j \to \infty} q_j^i = q^i \quad \text{in } \mathcal{F},
\]

for some \( \xi^i \in \mathcal{E}' \) and \( q^i \in \mathcal{F} \). Passing to the limit in (3.7), thanks to (3.4) we conclude that

\[
A^*q^i = \xi^i.
\]

By the closure property of subdifferentials we have \( \xi^i \in \partial J(v^i) \), and thus \( \partial J(v^i) \cap \text{ran}(A^*) \neq \emptyset \). \qed
Remark 3.4. Suppose that \( v^i \) and \( z^i \) are two limit points of the sequence \( \{v^j\}_{j \in \mathbb{N}} \). By property (1) of the previous lemma, even if \( v^i \neq z^i \) we have
\[
J(v^i) = J(z^i).
\]

We state now a direct consequence of the previous lemma.

Corollary 3.5. Let (2.1), (A0), (A1), and (A2) be satisfied, and let \( f \in W^{1,2}([0,T]; F) \). Let \( \delta \in (0,1) \) and let \( v_0 \) be a critical point of \( J \) on the affine space \( A(f(0)) \). Let \( v^0 := v_0 \) and, for every \( i \in \mathbb{N} \) with \( i\delta \leq T \), let \( \{v^j_i\}_{j \in \mathbb{N}_0} \) be defined by (3.3), and let \( v^i \) be a limit point of \( \{v^j_i\}_{j \in \mathbb{N}_0} \). Then, the function \( v^i : [0, T] \to E \) defined as
\[
v^i(t) := v^i \quad \text{for every } t \in [0, T] \cap [i\delta, (i+1)\delta), \quad \text{for every } i \in \mathbb{N}_0 \text{ with } i\delta \leq T, \tag{3.8}
\]
is a discrete quasistatic evolution with time step \( \delta \), initial condition \( v_0 \), and constraint \( f \).

A key property of the discrete quasistatic evolution above is that it satisfies an approximate energy inequality, up to an error which vanishes with \( \delta \).

Theorem 3.6. Let (2.1), (A0), (A1), (A2), and (A3) be satisfied, and let \( f \in W^{1,2}([0,T]; F) \). Let \( \delta \in (0,1) \) and let \( v_0 \) be a critical point of \( J \) on the affine space \( A(f(0)) \). Let \( v^0 : [0,T] \to E \) be the discrete quasistatic evolution with time step \( \delta \), initial condition \( v_0 \), and constraint \( f \) given by (3.8). Then, there exist a piecewise constant right-continuous function \( q_\delta : [0,T] \to F \) and positive constants \( C_1, C_2 \) and \( C_3 \), independent of \( \delta \), such that
\[
\text{(i) } A^* q_\delta(t) \in \partial J(v^i(t)) \text{ for every } t \in [0,T];
\]
\[
\text{(ii) } |q_\delta(t)|_F \leq C_1 \text{ and } |v^i(t)|_F \leq C_2 \text{ for every } t \in [0,T];
\]
\[
\text{(iii) for every } t \in [0,T]
\]
\[
J(v^i(t)) \leq J(v^0) + \int_0^t \langle q_\delta(s), f(s) \rangle_F \, ds + C_3 \sqrt{\delta}.
\]

Remark 3.7. More precisely, as it appears by a careful reading of the proof of Theorem 3.6, we have
\[
C_1 = C_1(\gamma, K, L, \eta, C_{J,\eta}, \|f\|_{L^2((0,T);F)}) \quad \text{and} \quad C_3 = C_3(\gamma, K, L, \eta, C_{J,\eta}, \|f\|_{L^2((0,T);F)}).
\]
Instead, concerning the constant \( C_2 \),
\[
C_2 = C_2(\gamma, K, L, \eta, C_{J,\eta}, \|f\|_{L^\infty((0,T);F)}, \|\hat{f}\|_{L^2((0,T);F)}, J).
\]

The dependence on \( J \) above has to be intended in the sense of the coercivity assumption (A0). Therefore, if assumptions (2.1), (A1), (A2), and (A3) are satisfied by a family of functions \( \{J_\alpha\}_\alpha \), (with the same \( A, f, \gamma, \eta \), and \( C_{J,\eta} \) for all \( \alpha \)) and, in addition, the functionals \( v \mapsto J_\alpha(v) + |Av|_F^2 \) are equicoercive, then the constant \( C_2 \) is the same for all the family \( \{J_\alpha\}_\alpha \).

Proof. Since \( v^i \) is a discrete quasistatic evolution, for every \( i \in \mathbb{N} \) with \( i\delta \leq T \) there exists \( q^i \in F \) such that \( A^* q^i \in \partial J(v^i(i\delta)) \). Then, if we define \( q_\delta : [0,T] \to F \) as
\[
q_\delta(t) := q^i \quad \text{for all } t \in [0,T] \cap [i\delta, (i+1)\delta), \quad \text{for all } i \in \mathbb{N}_0 \text{ with } i\delta \leq T,
\]
property (i) is satisfied. We now divide the remaining part of the proof into three steps.

Step 1: We show that there exists a constant \( M \), depending only on \( \eta \) and \( C_{J,\eta} \), such that
\[
J(v^i(i\delta)) \leq J(v^i((i-1)\delta)) + \int_{(i-1)\delta}^{i\delta} \langle q_\delta(s), f(s) \rangle_F \, ds + M\delta \int_{(i-1)\delta}^{i\delta} |\hat{f}(s)|_F^2 \, ds, \tag{3.9}
\]
for every \( i \in \mathbb{N} \) with \( i\delta \leq T \).
To this aim, let $i \in \mathbb{N}$ with $i \delta \leq T$ be fixed, and let $\{v_j^i\}_{j \in \mathbb{N}}$ be the sequence defined by (3.3). By property (A1) and Remark 2.5, the functional $J_{h,v_0^i}$ is strictly convex. Therefore, whenever $\xi \in \partial J_{h,v_0^i}(v_1^i)$, we have

$$J_{h,v_0^i}(v) \geq J_{h,v_0^i}(v_1^i) + \langle \xi, v - v_1^i \rangle_{E',E} \quad \text{for every } v \in E.$$ 

In particular, choosing $v = v_0^i$ and recalling the definition of $J_{h,v_0^i}$ we have

$$J(v_0^i) \geq J(v_1^i) + \eta |v_1^i - v_0^i|^2 + \langle \xi, v_0^i - v_1^i \rangle_{E',E} \quad \text{for every } \xi \in \partial J_{h,v_0^i}(v_1^i). \quad (3.10)$$

By (3.3), $v_1^i$ is the global minimizer of $J_{h,v_0^i}$ on $A(f(i\delta))$. Therefore, there exists $r^i \in F$ such that $A^*r^i \in \partial J_{h,v_0^i}(v_1^i)$ so that, by (3.10),

$$J(v_0^i) \geq J(v_1^i) + \eta |v_1^i - v_0^i|^2 + \langle A^*r^i, v_0^i - v_1^i \rangle_{E',E}.$$ 

Therefore, by the absolute continuity of $f$, and recalling that $q_\delta$ is constant in the interval $[(i-1)\delta, i\delta)$, we have

$$\eta |v_1^i - v_0^i|^2 + J(v_1^i) - J(v_0^i) \leq \langle A^*r^i, v_1^i - v_0^i \rangle_{E',E}$$

$$= \langle A^*r^i - A^*q_\delta((i-1)\delta), v_1^i - v_0^i \rangle_{E',E} + \langle A^*q_\delta((i-1)\delta), v_1^i - v_0^i \rangle_{E',E}$$

$$= \langle r^i - q_\delta((i-1)\delta), Av_1^i - Av_0^i \rangle_F + \langle q_\delta^{\delta}((i-1)\delta), Av_1^i - Av_0^i \rangle_F$$

$$= \langle r^i - q_\delta((i-1)\delta), Av_1^i - Av_0^i \rangle_F + \int_{(i-1)\delta}^{i\delta} \langle q_\delta^{\delta}(s), f(s) \rangle_F ds,$$

where we used the fact that $Av_1^i = f(i\delta)$ and $Av_0^i = f((i-1)\delta)$. Observe now that, by definition of $q_\delta$, we have $A^*q_\delta((i-1)\delta) \in \partial J(v_0^i((i-1)\delta))$. Thus, recalling that $v_0^i((i-1)\delta) = v_j^{i-1} = v_0^i$, we obtain

$$A^*q_\delta((i-1)\delta) \in \partial J(v_0^i) = \partial J_{h,v_0^i}(v_0^i).$$

Thus, recalling that $A^*r^i \in \partial J_{h,v_0^i}(v_1^i)$, by property (A3) we obtain

$$\langle r^i - q_\delta((i-1)\delta), Av_1^i - Av_0^i \rangle_F \leq C_J,\eta |v_1^i - v_0^i|_E |Av_1^i - Av_0^i|_F$$

which, together with (3.11), gives

$$\eta |v_1^i - v_0^i|^2 + J(v_1^i) - J(v_0^i) \leq C_J,\eta |v_1^i - v_0^i|_E |Av_1^i - Av_0^i|_F$$

$$+ \int_{(i-1)\delta}^{i\delta} \langle q_\delta^{\delta}(s), f(s) \rangle_F ds.$$ 

Using Young’s and Hölder’s inequality, we get

$$J(v_1^i) \leq J(v_0^i) + M|f(i\delta) - f((i-1)\delta)|_F^2 + \int_{(i-1)\delta}^{i\delta} \langle q_\delta^{\delta}(s), f(s) \rangle_F ds$$

$$= J(v_0^i) + M \int_{(i-1)\delta}^{i\delta} \frac{\langle q_\delta^{\delta}(s), f(s) \rangle_F}{\mathcal{F}} ds + \int_{(i-1)\delta}^{i\delta} \langle q_\delta^{\delta}(s), f(s) \rangle_F ds$$

$$\leq J(v_0^i) + M \delta \int_{(i-1)\delta}^{i\delta} \frac{\langle q_\delta^{\delta}(s), f(s) \rangle_F}{\mathcal{F}} ds + \int_{(i-1)\delta}^{i\delta} \langle q_\delta^{\delta}(s), f(s) \rangle_F ds,$$

for a suitable constant $M > 0$ (depending only on $\eta$ and $C_J,\eta$), where we also used the fact that $f \in W^{1,2}([0,T];\mathcal{F})$. Recalling that $J(v_j^i) \leq J(v_1^i)$ for all $j \geq 2$, by (3.8) and property (1) of Lemma 3.3, we have

$$J(v_\delta(i\delta)) = J(v^i) = \lim_{j \to \infty} J(v_j^i) \leq J(v_1^i).$$
Taking into account last inequality, and recalling that \( v_0 = v^{i-1} = v_\delta((i-1)\delta) \), relation (3.12) gives (3.9).

**Step 2:** We prove (ii). Let \( t \in [0, T] \) be fixed, and let \( \tau \in \mathbb{N} \) be such that
\[
\tau \delta < t < (\tau + 1) \delta.
\]
Adding up relation (3.9) for \( i = 1, \ldots, \tau \), and recalling that \( v_\delta(t) = v_\delta(\tau \delta) \), we obtain
\[
J(v_\delta(t)) \leq J(v_0) + \int_0^{\tau \delta} \langle q_\delta(s), \dot{f}(s) \rangle_{\mathcal{F}} \, ds + M \delta \int_0^{\tau \delta} |\dot{f}(s)|^2_{\mathcal{F}} \, ds.
\]  
(3.13)

From (2.1) and (A2) we have
\[
|q_\delta(s)|_{\mathcal{F}} \leq \frac{1}{\gamma} |A^* q_\delta(s)|_{\mathcal{E}} \leq \frac{K}{\gamma} J(v_\delta(s)) + \frac{L}{\gamma} \text{ for all } s \in [0, T].
\]  
(3.14)

Therefore, (3.13) gives
\[
J(v_\delta(t)) \leq J(v_0) + \frac{K}{\gamma} \int_0^t J(v_\delta(s)) |\dot{f}(s)|_{\mathcal{F}} \, ds + \frac{L}{\gamma} \int_0^t |\dot{f}(s)|_{\mathcal{F}} \, ds + M \delta \int_0^T |\dot{f}(s)|^2_{\mathcal{F}} \, ds.
\]
Thus, from Gronwall’s inequality
\[
\sup_{\delta \in (0,1)} \sup_{t \in [0,T]} J(v_\delta(t)) \leq K_1,
\]  
(3.15)

for some positive constant \( K_1 = K_1(\gamma, K, L, \eta, C_{\mathcal{L} \mathcal{E}}, \|\dot{f}\|_{L^2([0,T],\mathcal{F})}) \). Then, from (3.14) and (3.15) we have
\[
\sup_{\delta \in (0,1)} \sup_{t \in [0,T]} |q_\delta(t)|_{\mathcal{F}} \leq C_1,
\]
for some positive constant \( C_1 = C_1(\gamma, K, L, \eta, C_{\mathcal{L} \mathcal{E}}, \|\dot{f}\|_{L^2([0,T],\mathcal{F})}) \). Thus,
\[
\sup_{\delta \in (0,1)} \sup_{t \in [0,T]} \left( J(v_\delta(t)) + \|Av_\delta(t)\|_{\mathcal{E}}^2 \right) \leq \sup_{\delta \in (0,1)} \sup_{t \in [0,T]} \left( J(v_\delta(t)) + \|\dot{f}\|_{L^2([0,T],\mathcal{F})}^2 \right) < K_2,
\]
for some positive constant \( K_2 = K_2(\gamma, K, L, \eta, C_{\mathcal{L} \mathcal{E}}, \|\dot{f}\|_{L^2([0,T],\mathcal{F})}, \|\dot{f}\|_{L^\infty([0,T],\mathcal{F})}) \). Taking into account (A0), last inequality implies that
\[
|v_\delta(t)|_{\mathcal{E}} \leq C_2, \quad \text{for every } \delta \in (0,1) \text{ and } t \in [0,T],
\]
with a constant \( C_2 \) that also depends on the coercivity of the function \( v \mapsto J(v) + |Av|_{\mathcal{E}}^2 \), see Remark 3.7.

**Step 3:** We show (iii). From (3.13) and Hölder’s inequality, taking into account that \( \delta \in (0,1) \)
\[
J(v_\delta(t)) \leq J(v_0) + \int_0^{\tau \delta} \langle q_\delta(s), \dot{f}(s) \rangle_{\mathcal{F}} \, ds + M \delta \|\dot{f}\|_{L^2([0,T],\mathcal{F})}^2
\]
\[
= J(v_0) + \int_0^t \langle q_\delta(s), \dot{f}(s) \rangle_{\mathcal{F}} - \int_0^{\tau \delta} \langle q_\delta(s), \dot{f}(s) \rangle_{\mathcal{F}} \, ds + M \delta \|\dot{f}\|_{L^2([0,T],\mathcal{F})}^2
\]
\[
\leq J(v_0) + \int_0^t \langle q_\delta(s), \dot{f}(s) \rangle_{\mathcal{F}} + C_1 \sqrt{\delta} \|\dot{f}\|_{L^2([0,T],\mathcal{F})} + M \sqrt{\delta} \|\dot{f}\|_{L^2([0,T],\mathcal{F})}^2
\]
\[
= J(v_0) + \int_0^t \langle q_\delta(s), \dot{f}(s) \rangle_{\mathcal{F}} + \sqrt{\delta} \left( C_1 \|\dot{f}\|_{L^2([0,T],\mathcal{F})} + M \|\dot{f}\|_{L^2([0,T],\mathcal{F})}^2 \right),
\]
which gives (iii). \hfill \square

Before giving the proof of Theorem 2.12, we need the following result (see [10, Lemma 3.6]).

**Lemma 3.8.** Let $X$ be a compact metric space. Let $p : [0, T] \to \mathbb{R}$, $p_k : [0, T] \to \mathbb{R}$ and $f_k : [0, T] \to X$ be measurable functions, for every $k \in \mathbb{N}$. For every $t \in [0, T]$ let us set

$$I(t) := \{ x \in X : \exists k_j \to +\infty \text{ such that } x = \lim_{j \to +\infty} f_{k_j}(t) \text{ and } p(t) = \lim_{j \to +\infty} p_{k_j}(t) \}.$$

Then

- $I(t)$ is closed for all $t \in [0, T]$;
- for every open set $U \subseteq X$ the set $\{ t \in [0, T] : I(t) \cap U \neq \emptyset \}$ is measurable.

We conclude this section with the proof of Theorem 2.12.

**Proof of Theorem 2.12.** We divide the proof into several steps.

**Step 1: Proof of (a) and (b).**

For every $\delta \in (0, 1)$, let $q_\delta : [0, T] \to \mathcal{F}$ and $v_\delta : [0, T] \to \mathcal{E}$ be given by Theorem 3.6. Let $\Lambda \subset [0, T]$ be such that $L^1(\Lambda) = 0$ and $f(t)$ is well defined for every $t \in [0, T] \setminus \Lambda$. We fix a sequence $\{\delta_k\}_{k \in \mathbb{N}}$ such that $\delta_k \to 0^+$ and define

$$\theta_k(t) := \begin{cases} (q_{\delta_k}(t), f(t))_\mathcal{F} & \text{for every } t \in [0, T] \setminus \Lambda, \\ 0 & \text{for every } t \in \Lambda, \end{cases}$$

and

$$\theta(t) := \limsup_{k \to \infty} \theta_k(t) \quad \text{for every } t \in [0, T].$$

By definition of $\theta$, for every $t \in [0, T]$ we can extract a subsequence $\{\delta_{k_j}\}_{j \in \mathbb{N}}$ (possibly depending on $t$) such that

$$\theta(t) = \lim_{j \to \infty} \theta_{k_j}(t) \quad \text{for every } t \in [0, T].$$

By (ii) of Theorem 3.6, we have

$$|q_\delta(t)|_\mathcal{F} \leq C_1, \quad |v_\delta(t)|_\mathcal{E} \leq C_2, \quad \text{for every } \delta \in (0, 1) \text{ and } t \in [0, T].$$

Thus, for every $t \in [0, T]$ we can extract a further subsequence (not relabelled) such that

$$\lim_{j \to \infty} v_{\delta_{k_j}}(t) = v(t) \quad \text{in } \mathcal{E} \quad \text{and} \quad \lim_{j \to \infty} q_{\delta_{k_j}}(t) = q(t) \quad \text{in } \mathcal{F},$$

for some $v(t) \in \mathcal{E}$ and $q(t) \in \mathcal{F}$ with $|q(t)|_\mathcal{F} \leq C_1$ and $|v(t)|_\mathcal{E} \leq C_2$. Let us now show that, for every $t \in [0, T]$, we can choose the subsequence $\{k_j\}_{j \in \mathbb{N}}$ in such a way that the maps $q : [0, T] \to \mathcal{F}$ and $v : [0, T] \to \mathcal{E}$ are measurable.

Let us denote by $B_{C_1}^\mathcal{F}$ ($B_{C_2}^\mathcal{E}$) the closed ball of $\mathcal{F}$ ($\mathcal{E}$) with center at the origin and radius $C_1$ ($C_2$). Applying Lemma 3.8 with $X = B_{C_1}^\mathcal{F} \times B_{C_2}^\mathcal{E}$, $f_k = (q_\delta, v_\delta)$ and $p_k = \theta_k$, we have that

- $I(t)$ is closed for all $t \in [0, T]$,
- for every open set $U \subseteq X$ the set $\{ t \in [0, T] : I(t) \cap U \neq \emptyset \}$ is measurable, where the set $I(t)$ is given by

$$I(t) := \{(q(t), v(t)) \in B_{C_1}^\mathcal{F} \times B_{C_2}^\mathcal{E} : \exists k_j \to +\infty \text{ such that } (q(t), v(t)) = \lim_{j \to +\infty} (q_{\delta_{k_j}}(t), v_{\delta_{k_j}}(t)) \text{ and } \theta(t) = \lim_{j \to \infty} \theta_{k_j}(t) \}. $$
Thanks to [6, Theorem III.6], for every \( t \in [0, T] \) we can select \((q(t), v(t)) \in B_{C_1}^{\mathcal{E}} \times B_{C_2}^{\mathcal{K}}\) such that \( t \to (q(t), v(t)) \) is measurable. Thus, (a) is proven. Finally, by repeating the arguments used in the proof of Proposition 3.1 we obtain (b).

**Step 2: Proof of (c).** Observe that, for every \( t \in [0, T] \setminus \Lambda \),

\[
\theta(t) = \limsup_{k \to \infty} \theta_k(t) = \lim_{j \to \infty} \theta_{k_j}(t) = \lim_{j \to \infty} \langle q_{\delta_{k_j}}(t), \hat{f}(t) \rangle_{\mathcal{F}} = \langle q(t), \hat{f}(t) \rangle_{\mathcal{F}}.
\]

Let us now show that \( \theta \in L^1(0, T) \). Since \( \theta \) is the lim sup of measurable functions, we deduce that it is measurable. Moreover, we have

\[
\int_0^T |\theta(t)| \, dt = \int_0^T |\langle q(t), \hat{f}(t) \rangle_{\mathcal{F}}| \, dt \leq C_1\int_0^T |\hat{f}(t)|_{\mathcal{F}} \, dt \leq C_1 \sqrt{T} \|\hat{f}\|_{L^2((0,T);\mathcal{F})}.
\]

In order to get the energy inequality, recall that by (iii) of Theorem 3.6 we have, for every \( j \in \mathbb{N} \),

\[
J(v_{\delta_{k_j}}(t)) \leq J(v_0) + \int_0^t \langle q_{\delta_{k_j}}(s), \hat{f}(s) \rangle_{\mathcal{F}} \, ds + C_3 \delta_{k_j}^{1/2}.
\]

Taking the limsup in \( j \) of the previous expression, using Fatou’s Lemma

\[
J(v(t)) = \lim_{j \to \infty} J(v_{\delta_{k_j}}(t)) \leq J(v_0) + \limsup_{j \to \infty} \int_0^t \langle q_{\delta_{k_j}}(s), \hat{f}(s) \rangle_{\mathcal{F}} \, ds
\]

\[
\quad \leq J(v_0) + \limsup_{j \to \infty} \int_0^t \langle q_{\delta_k}(s), \hat{f}(s) \rangle_{\mathcal{F}} \, ds \leq J(v_0) + \limsup_{k \to \infty} \int_0^t \langle q_{\delta_k}(s), \hat{f}(s) \rangle_{\mathcal{F}} \, ds
\]

\[
= J(v_0) + \int_0^t \langle q(s), \hat{f}(s) \rangle_{\mathcal{F}} \, ds,
\]

so that (c) follows. \( \square \)

4. A discrete version of fracture evolution for cohesive zone models

In the remaining part of the paper, we show how to apply Theorem 2.12 to cohesive fracture evolution. We start this section by recalling the model introduced in [4], where a critical points evolution is obtained by following the scheme described in the first part of the Introduction. We conclude the section performing a finite dimensional discretization. In the next section we will then show how it is possible to pass to the limit, thus obtaining a different proof of the existence result in [4].

4.1. A previous model for the time evolution of cohesive fractures. Let \( \Omega \subset \mathbb{R}^d \) be a bounded open set with Lipschitz boundary, with \( d \in \{1, 2\} \). We assume that the reference configuration is the infinite cylinder \( \Omega \times \mathbb{R} \subset \mathbb{R}^{d+1} \), and that the displacement \( U : \Omega \times \mathbb{R} \to \mathbb{R}^{d+1} \) has the special form

\[
U(x_1, \ldots, x_d, x_{d+1}) = (0, \ldots, 0, u(x_1, \ldots, x_d)),
\]

where \( u : \Omega \to \mathbb{R} \). This situation is referred to in the literature as generalised antiplanar shear.

We assume that the crack path in the reference configuration is contained in \( \Gamma \cap \overline{\Omega} \), where \( \Gamma \subset \mathbb{R}^d \) is a Lipschitz closed set such that \( 0 < \mathcal{H}^{d-1}(\Gamma \cap \overline{\Omega}) < \infty \) and \( \Omega \setminus \Gamma = \Omega^+ \cup \Omega^- \), where \( \Omega^+ \) and \( \Omega^- \) are disjoint open connected sets with Lipschitz boundary.

We will study the energy of a finite portion of the cylinder, obtained by intersection with two horizontal hyperplanes separated by a unit distance. Given a time interval \([0, T]\), with \( T > 0 \), we assume that the evolution is driven by a time-dependent displacement \( \omega \in H^1([0, T]; H^1(\Omega)) \),
imposed on a fixed portion \( \partial_D \Omega \) of \( \partial \Omega \). We make the assumption that \( \partial_D \Omega \) is well separated from \( \Gamma \), and that \( H^{d-1}(\partial_D \Omega \cap \partial \Omega^\pm) > 0 \).

In the framework of linearized elasticity, the stored elastic energy associated to a displacement \( u \in H^1(\Omega \setminus \Gamma) \) is given by

\[
W(u) := \frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla u|^2 \, dx. \tag{4.1}
\]

The crack in the reference configuration can be identified with the set

\[
J_u := \{ x \in \Gamma : u^+(x) \neq u^-(x) \},
\]

where \( u^\pm \) denotes the trace on \( \Gamma \) of the restriction of \( u \) to \( \Omega^\pm \). In order to take into account the cohesive forces acting between the lips of the crack, according to Barenblatt’s model [3] we consider a fracture energy of the following form:

\[
\kappa \int_{\Gamma} g(||u||) d\mathcal{H}^{d-1},
\]

where \( [u] := u^+ - u^- \) is the jump of \( u \) across \( \Gamma \), and the energy density \( g : [0, \infty) \to [0, \infty) \) is a \( C^1 \), nondecreasing, bounded, concave function with \( g(0) = 0 \) and \( \sigma := g'(0^+) \in (0, +\infty) \). We will consider here the special case (see also [4, Section 9])

\[
g(s) = \begin{cases} 
-2 \frac{s^2}{R^2} + s & \text{if } 0 \leq s < R, \\
R & \text{if } s \geq R,
\end{cases}
\tag{4.2}
\]

where the parameter \( R > 0 \) represents the range of the cohesive interactions between the lips of the crack. The parameter \( \kappa \) is the stiffness material constant and, for sake of simplicity, it will be taken equal to 1 through all the theoretical analysis. Summarizing, the energy functional \( E : H^1(\Omega \setminus \Gamma) \to [0, \infty) \) is given by

\[
E(u) = \frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla u|^2 \, dx + \int_{\Gamma} g(||u||) d\mathcal{H}^{d-1}. \tag{4.3}
\]

Let \( t \in [0, T] \), and let \( u(t) \) be a minimizer for the problem

\[
\min \{ E(v) : v \in H^1(\Omega \setminus \Gamma), v = \omega(t) \text{ on } \partial_D \Omega \}.
\]

Then [4, Proposition 3.1],

\[
\int_{\Omega \setminus \Gamma} \nabla u(t) \cdot \nabla \psi \, dx + \int_{\Gamma} \left( \frac{\psi}{|\psi|} g'(||u(t)||) \right) \text{sign}(||u(t)||)1_{J_u(t)} + ||\psi||1_{J_u(t)} d\mathcal{H}^{d-1} \geq 0, \tag{4.4}
\]

for all \( \psi \in H^1(\Omega \setminus \Gamma) \) with \( \psi = 0 \) on \( \partial_D \Omega \). One can see [4, Proposition 3.2] that \( u(t) \) satisfies (4.4) if and only if it is a weak solution of

\[
\begin{cases}
\Delta u(t) = 0 & \text{in } \Omega \setminus \Gamma, \\
u(t) = \omega(t) & \text{on } \partial_D \Omega, \\
\partial_{\nu} u(t) = 0 & \text{on } \partial \Omega \setminus \partial_D \Omega, \\
|\partial_{\nu} u(t)| \leq 1 & \text{on } \Gamma \setminus J_u(t), \\
|\partial_{\nu} u(t) = g'(||u(t)||) \text{sign}(||u(t)||) & \text{on } J_u(t), \\
\partial_{\nu} u(t) = \omega(t) & \text{on } \partial_D \Omega,
\end{cases} \tag{4.5}
\]

where \( \text{sign}(\cdot) \) denotes the sign function. Any function \( u(t) \) satisfying (4.4) or (4.5) will be referred to as a critical point of \( E \) at time \( t \).
In [4], a critical points evolution is obtained by following the general ideas given in the Introduction, by setting \( Y = L^2(\Omega) \) and
\[
F(u, t) = \begin{cases} E(u) & \text{for } u \in H^1(\Omega \setminus \Gamma), u = w(t) \text{ on } \partial_D \Omega, \\ +\infty & \text{otherwise in } L^2(\Omega). \end{cases}
\]
Given a critical point \( u_0 \) of \( E \) at time 0 and \( \varepsilon \in (0, 1) \), the author shows [4, Theorem 4.1] the existence of a function \( u^\varepsilon : [0, T] \to H^1(\Omega \setminus \Gamma) \) satisfying \( u^\varepsilon(0) = u_0 \) and (1.2), which in this case reads as
\[
\begin{align*}
\Delta u^\varepsilon(t) &= \varepsilon \dot{u}^\varepsilon(t) & \text{in } \Omega \setminus \Gamma, \\
u^\varepsilon(t) &= \omega(t) & \text{on } \partial_D \Omega, \\
\partial_v u^\varepsilon(t) &= 0 & \text{on } \partial \Omega \setminus \partial_D \Omega, \\
\partial_v u^\varepsilon(t)_{\Omega^-} &= \partial_v u^\varepsilon_{\Omega^+}(t) & \text{on } \Gamma, \\
|\partial_v u^\varepsilon(t)| &\leq 1 & \text{on } \Gamma \setminus \partial v^\varepsilon(t), \\
|\partial_v u^\varepsilon(t)| &= g(|u^\varepsilon(t)|) \text{sign}(|u^\varepsilon(t)|) & \text{on } S_{\varepsilon}(t).
\end{align*}
\]
Under the additional assumption \( g \in C^{1,1} \) (which is fulfilled by the function in (4.2)), uniqueness for the above problem also holds true [4, Theorem 4.2]. Finally [4, Theorem 4.4], there exists a bounded measurable function \( u : [0, T] \to H^1(\Omega \setminus \Gamma) \) with \( u(0) = u_0 \) such that the following properties are satisfied:

- **approximability**: for every \( t \in [0, T] \) there exists a sequence \( \varepsilon_n \to 0^+ \) such that \( u^{\varepsilon_n}(t) \to u(t) \) weakly in \( H^1(\Omega \setminus \Gamma) \);
- **stationarity**: for a.e. \( t \in [0, T] \) the function \( u(t) \) is a critical point of \( E \) at time \( t \);
- **energy inequality**: for every \( t \in [0, T] \)
\[
E(u(t)) \leq E(u(0)) + \int_0^t \int_{\partial \Omega} \nabla u(s) \cdot \nabla \omega(s) \, dx ds.
\]

4.2. **Discrete Setting.** In view of the applications of the results of this paper to the model just introduced, we need a finite dimensional version of the energy functional \( E \) in (4.3). In order to focus on the main ideas of our approach, we keep the formulation as clear as possible, considering a very simple geometry.

Let \( d = 2 \) and \( \ell > 0 \) be fixed, and define
\[
\begin{align*}
\Omega := (0, 2\ell)^2, & \quad \Omega^- := (0, \ell) \times (0, 2\ell), & \quad \Omega^+ := (\ell, 2\ell) \times (0, 2\ell), & \quad \Gamma := \{\ell\} \times [0, 2\ell].
\end{align*}
\]
We will study a fracture evolution where the deformation is imposed on the set
\[*\]
\[
\partial_D \Omega := \{\{0\} \times [0, 2\ell]\} \cup \{\{2\ell\} \times [0, 2\ell]\},
\]
see Figure 1.

For every \( h > 0 \), we assume we are given a triangulation \( T_h \) of the set \( \Omega \setminus \Gamma \), and we define \( \mathcal{E}_h \) as the finite dimensional space of continuous functions that are affine on each triangle belonging to \( T_h \). More precisely, we set
\[
\mathcal{E}_h := \{u \in C(\Omega \setminus \Gamma) \cap H^1(\Omega \setminus \Gamma) : \nabla u = \text{const. a.e. on } T, \text{ for every } T \in T_h\},
\]
and we define \( \mathcal{E}_h^{\text{reg}} \) as the set of functions of \( \mathcal{E}_h \) that do not jump across \( \Gamma \):
\[
\mathcal{E}_h^{\text{reg}} := \mathcal{E}_h \cap H^1(\Omega).
\]
We endow $E_h$ with the induced norm of $H^1(\Omega \setminus \Gamma)$
\[ |u|^2_{E_h} := \int_{\Omega} u^2 \, dx + \int_{\Omega \setminus \Gamma} |\nabla u|^2 \, dx \quad u \in E_h, \]
and we denote with $\langle u_1, u_2 \rangle_{E_h}$ the scalar product. Whenever we do not identify the dual space $E_h'$ with $E_h$, we will use the notation $\langle \cdot, \cdot \rangle_{E_h', E_h}$ for the duality pairing. Throughout this section, we convene that the equality
\[ \xi = v \]
where $\xi \in E_h'$ and $v \in E_h$, is meant in sense of the Riesz isometry. We will denote the restriction of the energy functionals $E$ and $W$ to the space $E_h$ by
\[ E_h := E |_{E_h}, \quad W_h := W |_{E_h}. \]
We denote by $A_h$ the operator which associates to every function of $E_h$ its trace on $\partial D \Omega$, and we set $\mathcal{F}_h := A_h(E_h)$. Note that $\mathcal{F}_h$ is closed, since $E_h$ is finite dimensional. Therefore, $\mathcal{F}_h$ endowed with the induced scalar product $\langle \cdot, \cdot \rangle_{\mathcal{F}_h}$ is a Hilbert subspace of $H^{1/2}(\partial D \Omega)$. We also notice that, since the operators $A_h$ are the restrictions to $\mathcal{F}_h$ of the surjective trace operator $A : H^1(\Omega \setminus \Gamma) \to H^{1/2}(\partial D \Omega)$, by the open mapping theorem we can assume that (2.1) is satisfied for all $h$ with a constant $\gamma$ independent of $h$. In the sequel this will be used tacitly.

We finally observe that, applying [27, Lemma 4.1.3] to our setting, we obtain the following version of Poincaré inequality:
\[ \|u - u_D\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega \setminus \Gamma)} \quad \text{for every } u \in E_h, \]
where the constant $C$ depends on $\Omega$ and $\partial D \Omega$, and
\[ u_D := \left( \int_{\partial D \Omega} u^2 \, d\mathcal{H}^1 \right)^{1/2}. \]
The previous inequality in particular implies that
\[ \|u\|_{L^2(\Omega)} \leq u_D + C \|\nabla u\|_{L^2(\Omega \setminus \Gamma)} \leq C \left( |A_h u|_{\mathcal{F}_h} + \|\nabla u\|_{L^2(\Omega \setminus \Gamma)} \right) \quad \text{for every } u \in E_h, \quad (4.8) \]
where with $C$ we denote different constants, all depending on $\Omega$ and $\partial D \Omega$. We conclude this subsection with an important remark that will be used later.

**Remark 4.1.** Let $v \in E_h$, $w \in E_h^{reg}$, and let $\xi \in \partial E_h(v)$. Then, from the definition of subdifferential and direct computation, one can check that the action of $\xi$ on $w$ coincides with the action of the Fréchet differential $\partial W_h(v)$ on $w$. In formulas:

$$\langle \xi, w \rangle_{E_h', E_h} = \langle \partial W_h(v), w \rangle_{E_h', E_h}.$$

We show now that the functional $E_h$ satisfies the assumptions of Theorem 2.12.

### 4.3. Assumptions (A0)–(A3) are satisfied by $E_h$ and $A_h$.

First of all, we start by observing that condition (A0) is satisfied, by using standard arguments of calculus of variations.

**Proposition 4.2.** The functional $E_h \ni v \mapsto -E_h(v) + |A_hv|_{E_h}^2$ is coercive.

**Proof.** Let $C > 0$ be fixed, and let $\{v_k\}_{k \in \mathbb{N}} \subset E_h$ be a sequence such that

$$E_h(v_k) + |A_hv_k|_{E_h}^2 \leq C.$$

Then, recalling the expression of $E_h$ and thanks to Poincaré inequality (4.8), we have

$$|v_k|_{E_h}^2 \leq C,$$

for some new constant, still denoted by $C$, depending on $\Omega, \partial D \Omega$. Then, there exists a subsequence $\{v_{k_j}\}_{j \in \mathbb{N}}$ and a function $v \in E_h$ such that

$$v_{k_j} \rightharpoonup v \quad \text{weakly in } E_h.$$

Since $E_h$ is finite dimensional, this implies that

$$v_{k_j} \to v \quad \text{in } E_h,$$

and this concludes the proof. \qed

We now show that condition (A1) is satisfied.

**Proposition 4.3.** There exists $\eta > 0$ such that the function $u \mapsto E_h(u) + \eta |u|_{E_h}^2$ is strictly convex.

**Proof.** We divide the proof into several steps.

**Step 1.** We show that there exists $\mu > 0$ such that the function $p_\mu : \mathbb{R} \to \mathbb{R}$ given by

$$p_\mu(s) := g(|s|) + \mu s^2, \quad s \in \mathbb{R}, \quad (4.9)$$

is strictly convex. To this aim, we need to find $\mu$ such that the second distributional derivative $p''_\mu$ of $p_\mu$ is a positive Radon measure. Recalling the definition of $g$, we have

$$p_\mu(s) = \begin{cases} 
|s| + \left(\mu - \frac{1}{2R}\right)s^2 & \text{if } 0 \leq |s| < R, \\
R^2 + \mu s^2 & \text{if } |s| \geq R,
\end{cases}$$

where with $C$ we denote different constants, all depending on $\Omega$ and $\partial D \Omega$. We conclude this subsection with an important remark that will be used later.
The distributional derivative \( p'_\mu \) of \( p_\mu \) is given by

\[
p'_\mu(s) = \begin{cases} 
-1 + \left(2\mu - \frac{1}{R}\right)s & \text{if } -R < s < 0, \\
1 + \left(2\mu - \frac{1}{R}\right)s & \text{if } 0 < s < R, \\
2\mu s & \text{if } |s| \geq R.
\end{cases}
\]

Note that \( p'_\mu \in L^1_{\text{loc}}(\mathbb{R}) \). We can then calculate the second distributional derivative \( p''_\mu \) of \( p_\mu \), which is the Radon measure in \( \mathbb{R} \) given by

\[
p''_\mu = \left(2\mu - \frac{1}{R}\right)L^1((-R,R)) + 2\mu L^1((-\infty, -R) \cup (R, \infty)) + 2\delta_0,
\]

where \( \delta_0 \) represents the Dirac measure concentrated at the origin. Note that \( p''_\mu(B) \geq \left(2\mu - \frac{1}{R}\right)L^1(B) \), for every Borel set \( B \subset \mathbb{R} \).

Thus, if we choose \( \mu \) such that

\[
\mu > \frac{1}{2R},
\]

\( p''_\mu \) is a positive Radon measure on \( \mathbb{R} \), and \( p_\mu \) is convex.

**Step 2.** We show that the functional \( E_h : \mathcal{E}_h \to [0, \infty) \) given by

\[
E_h(u) := E_h(u) + \mu \int_{\Gamma} ||u||^2 \, d\mathcal{H}^1,
\]

is convex. By the previous step, the function \( r_\mu : \mathcal{G}_h \to [0, \infty) \) defined as

\[
r_\mu([u]) := \int_{\Gamma} g([u]) \, d\mathcal{H}^1 + \mu \int_{\Gamma} ||u||^2 \, d\mathcal{H}^1
\]

is convex, where \( \mathcal{G}_h \) is the subset of \( L^2(\Gamma) \) given by

\[
\mathcal{G}_h := \{[u] : u \in \mathcal{E}_h\}.
\]

Note now that

\[
\overline{E}_h(u) = W_h(u) + r_\mu([u]),
\]

where \( W \) was defined in (4.1). From the fact that \( W_h : \mathcal{E}_h \to [0, \infty) \) is convex, we then obtain that also \( \overline{E}_h : \mathcal{E}_h \to [0, \infty) \) is convex.

**Step 3: conclusion.** By [4, Lemma 5.3], there exists a constant \( \overline{C} > 0 \) such that

\[
\int_{\Gamma} ||u||^2 \, d\mathcal{H}^1 \leq \overline{C} ||u||^2_{\mathcal{E}_h},
\]

(4.11)

Taking \( \eta > \mu \overline{C} \) we have

\[
E_h(u) + \eta ||u||^2_{\mathcal{E}_h} = E_h(u) + \mu \int_{\Gamma} ||u||^2 \, dx + \mu \left( \overline{C} ||u||^2_{\mathcal{E}_h} - \int_{\Gamma} ||u||^2 \, dx \right) + (\eta - \mu \overline{C}) ||u||^2_{\mathcal{E}_h}.
\]

We have already proven that \( \overline{E}_h \) is convex. On the other hand, \( \overline{E}_h \) is a quadratic form which is positive semidefinite by (4.11), and thus is convex. Since the remaining term \( (\eta - \mu \overline{C}) ||u||^2_{\mathcal{E}_h} \) is strictly convex, this concludes the proof of (A1). \( \square \)
Before passing to the proof of (A2) we need some preliminary results. First, we make a few remarks on the regularity of the elastic part and on the crack part of the energy functional.

**Remark 4.4.** Note that $W_h \in C^1(\mathcal{E}_h)$. In particular, $\partial W_h(\cdot) : \mathcal{E}_h \to \mathcal{E}_h'$ is a single-valued Lipschitz function with Lipschitz constant 1. Indeed, we have

$$\langle \partial W_h(w), v \rangle_{\mathcal{E}_h, \mathcal{E}_h} = \int_{\Omega \setminus \Gamma} \nabla w \cdot \nabla v \, dx \quad \text{for every } w, v \in \mathcal{E}_h. \quad (4.12)$$

Then, for every $w_1, w_2, v \in \mathcal{E}_h$

$$|\partial W_h(w_1) - \partial W_h(w_2)|_{E_h'} = \sup \left\{ \int_{\Omega \setminus \Gamma} (\nabla w_1 - \nabla w_2) \cdot \nabla v \, dx, \quad v \in \mathcal{E}_h \text{ with } |v|_{E_h} = 1 \right\} \leq \sup \{ \|\nabla w_1 - \nabla w_2\|_{L^2(\Omega \setminus \Gamma)} \|\nabla v\|_{L^2(\Omega \setminus \Gamma)}, \quad v \in \mathcal{E}_h \text{ with } |v|_{E_h} = 1 \} \leq |w_1 - w_2|_{E_h}. \quad (4.13)$$

Since $\partial W_h(0) = 0$, this implies

$$|\partial W(w)|_{E_h'} \leq |w|_{E_h}, \quad \text{for every } w \in \mathcal{E}_h. \quad (4.14)$$

**Remark 4.5.** From the previous remark, it also follows that

$$|\partial W_h(w_1) - \partial W_h(w_2)|_{E_h'} \leq \|\nabla w_1 - \nabla w_2\|_{L^2(\Omega \setminus \Gamma)} \quad \text{for every } w_1, w_2 \in \mathcal{E}_h.$$

**Remark 4.6.** The functional $G_h : \mathcal{E}_h \to [0, \infty)$ defined as

$$G_h(v) := \int_{\Gamma} g(||v||) \, d\mathcal{H}^1.$$

is globally Lipschitz continuous. Indeed, for every $v_1, v_2 \in \mathcal{E}_h$ we have

$$|G_h(v_1) - G_h(v_2)| \leq \int_{\Gamma} |g(||v_1||) - g(||v_2||)| \, d\mathcal{H}^1 \leq \int_{\Gamma} ||v_1|| - ||v_2|| \, d\mathcal{H}^1 \leq \int_{\Gamma} |v_1 - v_2| \, d\mathcal{H}^1 \leq (H^1(\Gamma))^{1/2} ||v_1 - v_2||_{L^2(\Gamma)} \leq C (H^1(\Gamma))^{1/2} |v_1 - v_2|_{E_h},$$

where $C$ is given by (4.11), and we used the fact that $\|g'\|_{L^\infty([0,\infty))} = 1$.

Next proposition shows condition (A2).

**Proposition 4.7.** $E_h$ satisfies condition (A2).

**Proof.** Note that

$$E_h(u) = W_h(u) + G_h(u),$$

where $G_h : \mathcal{E}_h \to [0, \infty)$ is defined in Remark 4.6. Let now $v \in \mathcal{E}$. By (2.6), every $\xi \in \partial J(v)$ can be written as

$$\xi = \xi_1 + \xi_2,$$

where $\xi_1 \in \partial W_h(v)$, and $\xi_2 \in \partial G_h(v)$. By Remark 4.5 we have

$$|\xi_1|_{E_h'} \leq \|\nabla v\|_{L^2(\Omega \setminus \Gamma)} \leq 1 + \|\nabla v\|_{L^2(\Omega \setminus \Gamma)}^2 \leq 2(1 + E_h(v)).$$
On the other hand, thanks to Remark 4.6 $G_h$ is globally Lipschitz continuous with Lipschitz constant $C (H^1(\Gamma))^{1/2}$. Therefore, by (2.5)

$$|\xi_2|_{E_h^\ast} \leq C (H^1(\Gamma))^{1/2}. $$

Thus,

$$|\xi|_{E_h^\ast} \leq |\xi_1|_{E_h^\ast} + |\xi_2|_{E_h^\ast} \leq 2E_h(v) + 2 + C (H^1(\Gamma))^{1/2}. $$

\[ \square \]

Next lemma will be used to prove (A3), and gives a bound on the norm of a regular critical point, in terms of its trace on $\partial D\Omega$.

**Lemma 4.8.** Let $w \in E_h^{reg}$, let $f \in F_h$ be such that $A_h w = f$, and suppose $\partial W_h(w) \in \text{ran}(A^\ast)$. Then, there exists a positive constant $C = C(\Omega, \gamma)$ such that

$$|w|_{E_h} \leq C |f|_{\mathcal{F}_h}. $$

**Proof.** By (4.8) and (4.12)

$$|w|_{E_h}^2 = \|w\|_{L^2(\Omega)}^2 + \|\nabla w\|_{L^2(\Omega\setminus\Gamma)}^2 \leq C \left( |A_h w|_{\mathcal{F}_h}^2 + \|\nabla w\|_{L^2(\Omega\setminus\Gamma)}^2 \right) $$

$$= C \left[ |f|_{\mathcal{F}_h}^2 + \langle \partial W_h(w), w \rangle_{E_h, E_h} \right], $$

where $C$ denotes different constants, depending only on $\Omega$ and $\partial D\Omega$. Let now $q \in F_h$ be such that $A_h^\ast q = \partial W_h(w)$. Then,

$$|\partial W_h(w)|_{E_h} = |A_h^\ast q|_{E_h} \geq \gamma |q|_{\mathcal{F}_h}. $$

Thus, taking into account (4.13) we have

$$\langle \partial W_h(w), w \rangle_{E_h, E_h} = \langle A_h^\ast q, w \rangle_{E_h} = \langle q, A_h w \rangle_{\mathcal{F}_h} = \langle q, f \rangle_{\mathcal{F}_h} \leq |q|_{\mathcal{F}_h} |f|_{\mathcal{F}_h} \leq \frac{1}{\gamma} |\partial W_h(w)|_{E_h} |f|_{\mathcal{F}_h} \leq \frac{1}{\gamma} |w|_{E_h} |f|_{\mathcal{F}_h}. $$

Using (4.15), we obtain

$$|w|_{E_h}^2 \leq C \left[ |f|_{\mathcal{F}_h}^2 + \frac{1}{\gamma} |w|_{E_h} |f|_{\mathcal{F}_h} \right]. $$

From the previous relation, the thesis follows using Young inequality. \[ \square \]

We can finally prove (A3).

**Proposition 4.9.** Let $v_1, v_2, \pi \in E_h$ and $q_1, q_2, f_1, f_2 \in F_h$ be such that

$$A_h v_i = f_i, \quad A_h^\ast q_i = \partial (E_h)_{\eta \pi}(v_i), \quad i = 1, 2,$n

where $\eta$ is given by Proposition 4.3. Then, there exists $C > 0$, depending only on $\Omega$ and $\partial D\Omega$ such that

$$\langle q_1 - q_2, A_h v_1 - A_h v_2 \rangle_{\mathcal{F}_h} \leq C |v_1 - v_2|_{E_h} |A_h v_1 - A_h v_2|_{\mathcal{F}_h}. $$

**Proof.** Let $w$ be the unique solution of the following minimization problem:

$$w = \text{argmin}_{v \in E_h^{reg}} \{ W_h(v) : A_h(v) = f_1 - f_2 \}. $$

(4.17)

By Remark 4.1 we have

$$\langle \xi_i, w \rangle_{E_h, E_h} = \langle \partial W_h(v_i), w \rangle_{E_h, E_h} \quad i = 1, 2. $$
Now, by definition of \((E_h)_{t,T}\) and Remark 2.6, there exist \(\xi_i \in \partial E_h(v_i),\) with \(i = 1, 2\) such that
\[
A^*_h q_i - 2\eta(v_i - \nu) = \xi_i, \quad i = 1, 2.
\]
Subtracting term by term we obtain
\[
A^*_h (q_1 - q_2) - 2\eta(v_1 - v_2) = \xi_1 - \xi_2.
\]
Thus, thanks to Remark 4.4
\[
\langle q_1 - q_2, A_h v_1 - A_h v_2 \rangle_{\mathcal{F}_h} = \langle q_1 - q_2, f_1 - f_2 \rangle_{\mathcal{F}_h} = \langle q_1 - q_2, A_h(w) \rangle_{\mathcal{F}_h}
\]
\[
= \langle A^*_h (q_1 - q_2), w \rangle_{\mathcal{E}_h} = \langle \xi_1 - \xi_2, w \rangle_{\mathcal{E}_h} + 2\eta(v_1 - v_2, w)_{\mathcal{E}_h}
\]
\[
= \langle \partial W_h(v_1) - \partial W_h(v_2), w \rangle_{\mathcal{E}_h} + 2\eta(v_1 - v_2, w)_{\mathcal{E}_h}
\]
\[
\leq (1 + 2\eta) |w|_{\mathcal{E}_h} |v_1 - v_2|_{\mathcal{E}_h} \leq C(1 + 2\eta) |f_1 - f_2|_{\mathcal{F}_h} |v_1 - v_2|_{\mathcal{E}_h},
\]
where we also used the fact that \(w\) satisfies the assumptions of Lemma 4.8 with \(f = f_1 - f_2.\)

We conclude this section with a remark, which clarifies why we didn’t directly prove the main theorem of the paper in the infinite dimensional setting.

**Remark 4.10.** As already explained in the Introduction, the proof of the main result of the paper (Theorem 2.12) is given only in a finite dimensional setting. This is due to the fact that, in the infinite dimensional case, the subdifferential is in general not closed with respect to the weak convergence in the domain of the energy. Such a difficulty could be overcome by requiring that the energy functional has compact sublevels, an assumption which is quite common in literature. In the cohesive fracture model case, this would amount to choosing \(L^2(\Omega)\) as domain of the energy, and considering the Dirichlet linear constraint as being encoded by an *unbounded* densely defined surjective linear operator \(A : L^2(\Omega) \to H^2(\partial D \Omega).\) This does not affect neither (2.1) nor hypothesis \((A0),\) which are still satisfied with minor modifications. Even condition \((A1)\) can be proved with a little bit more of effort. On the other hand, the key conditions \((A2)\) and \((A3),\) which we need in order to cope with the cohesive fracture energy in our model, would no longer hold true. This motivates our choice of first dealing with a finite dimensional setting, and then extend the results with a problem-specific technique.

5. Recovering an approximable quasistatic evolution

We show now that the existence of a quasistatic evolution for the functional \(E,\) in the sense of [4] and Section 4.1, can be recovered from a discrete quasistatic evolution for \(E_h,\) when the parameter \(h\) controlling the mesh size tends to 0. Before stating the main theorem of the section we need some notation. We set
\[
D := \left\{ h > 0 : h = \frac{\ell}{N} \text{ for some } N \in \mathbb{N} \right\}.
\]
Let \(\omega \in W^{1,2}([0, T], H^1(\Omega)),\) and let \(u_0\) be a constrained critical point of \(E\) at time 0, under the constraint \(u_0 = \omega(0)\) on \(\partial_D \Omega.\) By [24], there exists a sequence \(\{\omega_h\}_{h \in D} \subset W^{1,2}([0, T], H^1(\Omega))\) such that \(\omega_h \in W^{1,2}([0, T], \mathcal{E}_h)\) for every \(h \in D\) and
\[
\omega_h \xrightarrow{h \to 0^+} \omega \quad \text{in } W^{1,2}([0, T], H^1(\Omega)). \tag{5.1}
\]
For every \(t \in [0, T],\) we define \(f_h(t) := A_h \omega_h(t)\), we will denote by \(f(t)\) the trace of \(\omega(t)\) on \(\partial_D \Omega.\) Again by [24], there exists a sequence \(\{u_{0,h}\}_{h \in D} \subset H^1(\Omega \setminus \Gamma)\) such that
\[
u_{0,h} \in \mathcal{E}_h \quad \text{with} \quad A_h u_{0,h} = f_h(0) \quad \text{for every } h \in D,
\]
and

\[ u_{0,h} \xrightarrow{h \to 0^+} u_0 \quad \text{in} \quad H^1(\Omega \setminus \Gamma). \quad (5.2) \]

**Remark 5.1.** Let \( h \in D \) be fixed. Since in general \( u_{0,h} \) is not a critical point of \( E_h \) at time \( 0 \), it is not possible to consider an approximable quasistatic evolution with initial condition \( u_{0,h} \) and constraint \( f_h \). We can, however, modify Definition 2.11 in such a way that no critical point condition is required at the initial time \( 0 \), as clarified below. For our purposes, this is still sufficient. Indeed, as stated in Theorem 5.5 below, the critical point condition at \( t = 0 \) is recovered when passing to the limit \( h \to 0 \).

**Definition 5.2.** Let \( h \in D, \delta \in (0,1), \) and let \( u_{0,h} \) and \( f_h \) be given above. A **discrete quasistatic evolution** with time step \( \delta \), initial condition \( u_{0,h} \), and constraint \( f_h \) is a right-continuous function \( u_{\delta,h} : [0,T] \to \mathcal{E}_h \) such that

- \( u_{\delta,h}(0) = u_{0,h} \);
- \( u_\delta \) is constant in \([0,T] \cap [i\delta, (i+1)\delta)\) for all \( i \in \mathbb{N}_0 \) with \( i\delta \leq T \);
- \( u_\delta(i\delta) \) is a critical point of \( E_h \) on the affine space \( A(f_h(i\delta)) \) for every \( i \in \mathbb{N} \) with \( i\delta \leq T \).

**Remark 5.3.** Note that, in the definitions above, we do not require \( u_{\delta,h}(0) \) to be a critical point of \( E_h \) at \( t = 0 \).

**Definition 5.4.** Let \( h \in D, \) and let \( u_{0,h} \) and \( f_h \) be given above. We say that a measurable function \( u_h : [0,T] \to \mathcal{E}_h \) is an **approximable quasistatic evolution** with initial condition \( u_{0,h} \) and constraint \( f_h \), if for every \( t \in [0,T] \) there exists a sequence \( \delta_k \to 0^+ \) (possibly depending on \( t \)) and a sequence \( \{u_{\delta_k,h}\}_{k \in \mathbb{N}} \) of discrete quasistatic evolutions with time step \( \delta_k \), initial condition \( u_{0,h} \), and constraint \( f_h \), such that

\[
\lim_{k \to +\infty} |u_{\delta_k,h}(t) - u_h(t)|_{\mathcal{E}_h} = 0.
\]

With this choice, an approximable quasistatic evolution still satisfies \( u_h(0) = u_{0,h} \). On the other hand (see Theorem 5.6 below) in this case we will simply require that the stationarity condition for \( u_h \) holds for every \( t \in (0,T] \), while stationarity at \( 0 \) will be recovered in the limit passage \( h \to 0 \).

Our goal is now proving the following version of [4, Theorem 4.4], which is the main result of this section.

**Theorem 5.5.** Let \( \omega \in W^{1,2}([0,T],H^1(\Omega)) \), and let \( u_0 \) be a critical point of \( E \) at time \( 0 \) with \( u_0 = \omega(0) \) on \( \partial_D \Omega \). For every \( h \in D \), let \( \omega_h \) and \( u_{0,h} \) be defined as above. Then, there exists a bounded measurable function \( u : [0,T] \to H^1(\Omega \setminus \Gamma) \) with \( u(0) = u_0 \) such that the following properties are satisfied:

- **(a) approximability:** for every \( t \in [0,T] \) there exists a sequence \( h_j \to 0^+ \) such that
  \[ u_{h_j}(t) \rightharpoonup u(t) \quad \text{weakly in} \quad H^1(\Omega \setminus \Gamma) \]
  where, for every \( j \in \mathbb{N} \), \( u_{h_j} \) is an approximable quasistatic evolution of \( E_{h_j} \) with initial condition \( u_{0,h_j} \) and constraint \( f_{h_j} \), see Definition 5.2;

- **(b) stationarity:** for every \( t \in [0,T] \) the function \( u(t) \) is a critical point of \( E \) at time \( t \) under the constraint \( u(t) = \omega(t) \) on \( \partial_D \Omega \);

- **(c) energy inequality:** for every \( t \in [0,T] \)
  \[
  E(u(t)) \leq E(u(0)) + \int_0^t \int_{\Omega \setminus \Gamma} \nabla u(s) \cdot \nabla \omega(s) \, dxds.
  \]
Before proving Theorem 5.5, we need the following result, which is obtained by applying Theorem 2.12 to $E_h$.

**Theorem 5.6.** Let $h \in D$ be fixed, and let $u_{0,h}$ and $f_h$ given above. Then, there exists a measurable bounded mapping $u_h: [0,T] \to E_h$ such that

(a') $u_h(\cdot)$ is an approximable quasistatic evolution with initial condition $u_{0,h}$ and constraint $f_h$;

(b') stationarity: for every $t \in (0,T]$ we have

$$
\int_{\Omega \Gamma} \nabla u_h(t) \cdot \nabla \psi \, dx + \int_{\Gamma} \left( [\psi] g'(||u_h(t)||) \text{sign}(||u_h(t)||) 1_{J_{u_h(t)}} + [\psi] 1_{J_{u_h(t)}}' \right) dH^{d-1} \geq 0, 
$$

for all $\psi \in E_h$ with $\psi = 0$ on $\partial_D \Omega$.

(c') energy inequality: The function $s \mapsto \int_{\Omega \Gamma} \nabla u_h(s) \cdot \nabla \hat{\omega}_h(s) \, dx$ belongs to $L^1(0,T)$ and

$$
E_h(u_h(t)) \leq E(u_{0,h}) + \int_0^t \int_{\Omega \Gamma} \nabla u_h(s) \cdot \nabla \hat{\omega}_h(s) \, dx ds 
$$

for every $t \in [0,T]$.

(d') Uniform bound: There exists a constant $\overline{C}_2$, independent of $h$, such that

$$
\|u_h(t)\|_{E_h} \leq \overline{C}_2 
$$

for every $t \in [0,T]$.

**Remark 5.7.** As already observed, property (a') has to be intended in the sense of Definition 5.4.

**Proof.** As proven in the previous subsection, (2.1) and assumptions (A0)–(A3) are satisfied. We now need to check that the proof of Theorem 2.12 can be adapted to the present situation, where Definition 5.2 and Definition 5.4 substitute Definition 2.10 and Definition 2.11, respectively.

**Step 1:** We construct a discrete quasi static evolution. Let $\delta \in (0,1)$ be a fixed time step, and let $i \in \mathbb{N}$ with $i\delta \leq T$. We set $u^0_h := (u_{0,h})$ while, for $i \geq 2$, we suppose that $u^{i-1}_h \in E_h$ is a critical point of $E_h$ on the affine space $A(f_h((i-1)\delta))$. Then, analogously to (3.3), we define the sequence $\{u_{h,j}\}_{j \in \mathbb{N}}$ by setting $u_{h,0} := u^{i-1}_h$ and

$$
u_{h,j} := \arg \min_{A_h v = f_h(i\delta)} \{ E_h(v) + \eta \|v - u^{i-1}_h\|_{E_h}^2 : v \in E_h \} 
$$

for every $j \in \mathbb{N}$.

Let now $i \in \mathbb{N}$ with $i\delta \leq T$. One can check that Lemma 3.3 and Corollary 3.5 still hold true. In particular, for every $i \in \mathbb{N}$ with $i\delta \leq T$ we can find a function $u^i_h \in E_h$ with $A_h u^i_h = f_h(i\delta)$ such that, up to subsequences,

$$
\lim_{j \to \infty} u_{h,j} = u^i_h 
$$

in $E_h$.

Moreover, the function $u_{h,\delta}: [0,T] \to E_h$ defined as

$$
u_{h,\delta}(t) := u^*_h \quad \text{for every } t \in [0,T] \cap [i\delta, (i+1)\delta), 
$$

for every $i \in \mathbb{N}$ with $i\delta \leq T$, is a discrete quasistatic evolution with time step $\delta$, initial condition $(u_{0,h})$, and constraint $f_h$, in the sense of Definition 5.2.

At this point, we need to check that Theorem 3.6 can still be proven. In particular, we want to define a function $g_{h,\delta}: [0,T] \to F_h$ such that an approximate energy inequality (as (iii) of Theorem 3.6) holds true. The main problem consists in defining $g_{h,\delta}$ in the interval $[0,\delta]$. Indeed, since $u_{h,\delta}(0)$ is not a critical point, we do not have a natural choice available. We then modify the proof of Theorem 3.6 in the following way.

Let $i = 1$. Since $(E_h)_{\eta,u_0} h$ is strictly convex, for every $\xi \in \partial(E_h)_{\eta,u_0} h(u^1_{h,1})$, we have

$$(E_h)_{\eta,u_0} h(v) \geq (E_h)_{\eta,u_0} h(u^1_{h,1}) + \langle \xi, v - u^1_{h,1} \rangle_{E_h^*,E_h} \quad \text{for every } v \in E_h.$$
In particular, choosing \( v = (u_0)_h \) and recalling the definition of \((E_h)_{\eta,(u_0)_h}\), we have
\[
E_h((u_0)_h) \geq E_h(u_{h,1}^1) + \eta u_{h,1}^1 - (u_0)_h \delta^2 \| h \|_{E_h}^2 + \langle \xi, (u_0)_h - u_{h,1}^1 \rangle \varepsilon_h \forall \xi \in \partial (E_h)_{\eta,(u_0)_h}(u_{h,1}^1). \tag{5.5}
\]
Recall now that \( u_{h,1} \) is the global minimizer of \((E_h)_{\eta,(u_0)_h}\) on \( A(f_h(\delta)) \). Therefore, there exists \( r^1 \in \mathcal{F}_h \) such that \( A_h^1 r^1 \in \partial (E_h)_{\eta,(u_0)_h}(u_{h,1}^1) \). Therefore, by (5.5),
\[
E_h((u_0)_h) \geq E_h(u_{h,1}^1) + \eta u_{h,1}^1 - (u_0)_h \delta^2 \| h \|_{E_h}^2 + \langle A_h^1 r^1, (u_0)_h - u_{h,1}^1 \rangle \varepsilon_h.
\]
At this point we can finally define the function \( q_{h,\delta} : [0,T] \to \mathcal{F}_h \). Since \( u_{h,\delta} \) is a discrete quasi static evolution, for every \( i \in \mathbb{N} \) with \( i\delta \leq T \) there exists \( q^i \in \mathcal{F}_h \) such that \( A^* q^i \in \partial (E_h)(u_{h,\delta}(i\delta)) \). Then, we define \( q_{h,\delta} : [0,T] \to \mathcal{F}_h \) as
\[
q_{h,\delta}(t) := \begin{cases} \begin{align*}
r^1 & \quad t \in [0,\delta), \\
q^i & \quad t \in [0,T] \cap [i\delta, (i+1)\delta), \quad \text{for } i \in \mathbb{N} \text{ with } i\delta \leq T.
\end{align*} \end{cases}
\]
Thus, repeating the proof of Theorem 3.6 for \( i = 1 \) we have
\[
\eta u_{h,1}^1 - (u_0)_h \delta^2 \| h \|_{E_h}^2 + E_h((u_0)_h) \leq (r^1 - q_{h,\delta}(0), A_h u_{h,1}^1 - A_h (u_0)_h)_{\mathcal{F}_h} + \int_0^\delta \langle q^\delta(s), \dot{f}_h(s) \rangle_{\mathcal{F}_h} ds
\]
\[
= \int_0^\delta \langle q^\delta(s), \dot{f}_h(s) \rangle_{\mathcal{F}_h} ds.
\]
Recalling that Lemma 3.3 holds true also in this case, we have
\[
E_h(u_{h,\delta}(\delta)) \leq E_h(u_{h,1}^1) \leq E_h((u_0)_h) + \int_0^\delta \langle q^\delta(s), \dot{f}_h(s) \rangle_{\mathcal{F}_h} ds.
\]
Since for \( i \geq 2 \) the proof of (3.9) can be repeated with no modifications, this shows Step 1 of the proof of Theorem 3.6.

In order to prove condition (ii) of Theorem 3.6, we proceed in the following way. By the minimality property of \( u_{h,1} \), we have
\[
E_h(u_{h,1}^1) \leq E_h(u_{h,1}^1) + \eta u_{h,1}^1 - (u_0)_h \delta^2 \| h \|_{E_h}^2 \leq E_h(\omega_h(\delta)) + \eta |w_h(\delta) - (u_0)_h| \| h \|_{E_h}^2 \leq C,
\]
where, by (5.1) and (5.2), \( C \) is a constant depending only on \( \eta, \omega \) and \( u_0 \) (and not on \( h \) and \( \delta \)). Then, thanks to (5.1)
\[
E_h(u_{h,1}^1) + |A_h u_{h,1}^1|_{\mathcal{F}_h}^2 = E_h(u_{h,1}^1) + |f_h(\delta)|_{\mathcal{F}_h}^2 \leq C,
\]
where \( C \) is again a (possibly different) constant, depending only on \( \eta, \omega \) and \( u_0 \) (and not on \( h \) and \( \delta \)). Thus, by condition (A0) and by equicoercivity of the family of functionals \( \{ E_h \}_{h \in D} \) we have that
\[
|u_{h,1}^1| \| h \|_{E_h}^2 \leq C
\]
for some constant independent of \( h \). Therefore, thanks to property (A2) and recalling that \( A_h^1 r^1 \in \partial (E_h)_{\eta,(u_0)_h}(u_{h,1}^1) \)
\[
|r^1|_{\mathcal{F}_h} \leq \frac{1}{\gamma} |A_h^1 r^1| \| h \|_{E_h} \leq \frac{K}{\gamma} E_h(u_{h,1}^1) + \frac{L}{\gamma} + \frac{2\eta}{\gamma} |u_{h,1}^1 - (u_0)_h| \| h \|_{E_h} \leq C,
\]
for some constant \( C \) independent of \( h \). From this, in particular, we obtain that
\[
|q_{h,\delta}(0)|_{\mathcal{F}_h} = |r^1|_{\mathcal{F}_h} \leq C \leq \frac{K}{\gamma} E_h(u_{h,\delta}(0)) + C,
\]
which gives relation (3.14) in the interval \([0, \delta]\). At this point, the proof of Theorem 3.6 can be repeated without any modifications.

**Step 2:** We apply Theorem 2.12.

By Step 1, properties (a)–(c) of Theorem 2.12 hold true. In particular, (a) implies (a’). By (b) of Theorem 2.12, there exists a bounded measurable function \(q_h : (0, T] \to F_h\) such that

\[
A^* q_h(t) \in \partial E_h(u_h(t)) \quad \text{for every } t \in (0, T].
\]

(5.6)

Let now \(t \in (0, T]\) be fixed. Thanks to Remark 2.2, we have

\[
0 \leq \liminf_{\varepsilon \to 0^+} \frac{E_h(u_h(t) + \varepsilon w) - E_h(u_h(t))}{\varepsilon} \quad \text{for every } w \in \ker(A_h^*).
\]

A careful inspection of the proof of [4, Proposition 3.1] shows that last inequality implies (b’).

Let us now show (c’). From (c) of Theorem 2.12, the function \(s \mapsto \langle q_h(s), f_h(s) \rangle_{F_h}\) belongs to \(L^1(0, T)\), and for every \(t \in [0, T]\) we have

\[
E_h(u_h(t)) \leq E((u_0)_h) + \int_0^t \langle q_h(s), f_h(s) \rangle_{F_h} ds.
\]

Recalling that \(f_h(s) = A_h \omega_h(s)\) and that the linear operator \(A_h\) is independent of time, we have

\[
E_h(u_h(t)) \leq E((u_0)_h) + \int_0^t \langle q_h(s), f_h(s) \rangle_{F_h} ds
= E((u_0)_h) + \int_0^t \langle q_h(s), A_h \omega_h(s) \rangle_{F_h} ds
= E((u_0)_h) + \int_0^t \langle A_h^* q_h(s), \omega_h(s) \rangle_{E_h} ds.
\]

By (5.6), for every \(s \in (0, T)\) we have \(A^* q_h(s) \in \partial E_h(u_h(s))\). Since \(\omega_h(s) \in \mathcal{E}^\text{reg}_h\) for every \(s \in (0, T)\), by Remark 4.1 we have

\[
\langle A_h^* q_h(s), \omega_h(s) \rangle_{E_h} = \int_{\Omega \setminus \Gamma} \nabla u_h(s) \cdot \nabla \omega_h(s) dx \quad \text{for every } s \in (0, T).
\]

Therefore,

\[
E_h(u_h(t)) \leq E((u_0)_h) + \int_0^t \int_{\Omega \setminus \Gamma} \nabla u_h(s) \cdot \nabla \omega_h(s) dx ds,
\]

which gives (c). Finally, property (d’) directly follows from Remark 2.13 and Remark 3.7. \(\square\)

We can now pass to the limit as \(h \to 0^+\).

**Proof of Theorem 5.5.** Let \(t \in [0, T]\). We will use argument similar to those used in the proof of Theorem 2.12.

**Step 1:** Proof of (a) and (c).

First of all, we fix the subsequence \(\{h_k\}_{k \in \mathbb{N}}\) given by

\[
h_k := \frac{\ell}{2^k}, \quad k \in \mathbb{N},
\]

so that

\[
\mathcal{E}_{h_l} \subset \mathcal{E}_{h_m} \quad \text{for every } m > l.
\]

(5.7)
By (5.1), we have
\[ \omega_{h_k} \to \omega \quad \text{strongly in } L^2([0, T]; H^1(\Omega)) \quad \text{as } k \to \infty. \]
Thus, there exists a set \( \Lambda_2 \subset [0, T] \) with \( \mathcal{L}^1(\Lambda_2) = 0 \) such that \( \omega(t) \) is well defined for every \( t \in [0, T] \setminus \Lambda \) and
\[ \omega_{h_k}(t) \to \omega(t) \quad \text{strongly in } H^1(\Omega) \quad \text{for every } t \in [0, T] \setminus \Lambda \quad \text{as } k \to \infty. \] (5.8)
For every \( h \in D \), let \( u_h : [0, T] \to \mathcal{E}_h \) be given by Theorem 5.6. We define
\[ \theta_k(t) := \begin{cases} \int_{\Omega \setminus \Gamma} \nabla u_{h_k}(t) \cdot \nabla \omega_{h_k}(t) \, dx & \text{for every } t \in [0, T] \setminus \Lambda, \\ 0 & \text{for every } t \in \Lambda, \end{cases} \]
and
\[ \theta(t) := \lim_{k \to \infty} \sup \theta_k(t) \quad \text{for every } t \in [0, T]. \]
By definition of \( \theta \), for every \( t \in [0, T] \) we can extract a subsequence \( \{h_{k_j}\}_{j \in \mathbb{N}} \) (possibly depending on \( t \)) such that
\[ \theta(t) = \lim_{j \to \infty} \theta_{k_j}(t) \quad \text{for every } t \in [0, T]. \] (5.4)
for every \( t \in [0, T] \) we can extract a further subsequence (not relabelled) such that
\[ u_{h_{k_j}}(t) \to u(t) \quad \text{weakly in } H^1(\Omega \setminus \Gamma) \quad \text{as } j \to \infty. \] (5.9)
for some \( u(t) \in H^1(\Omega \setminus \Gamma) \) with \( \|u(t)\|_{H^1(\Omega \setminus \Gamma)} \leq C_2 \). By repeating what was done in the proof of Theorem 2.12, we can show that the subsequence \( \{k_j\}_{j \in \mathbb{N}} \) can be chosen in such a way that the map \( u : [0, T] \to H^1(\Omega \setminus \Gamma) \) is measurable, and this shows (a).

Let us now show the energy inequality. By (5.8) and (5.9) we have that, for every \( t \in [0, T] \setminus \Lambda \),
\[ \theta(t) = \lim_{k \to \infty} \sup \theta_k(t) = \lim_{j \to \infty} \theta_{k_j}(t) = \lim_{j \to \infty} \int_{\Omega \setminus \Gamma} \nabla u_{h_{k_j}}(t) \cdot \nabla \omega_{h_{k_j}}(t) \, dx = \int_{\Omega \setminus \Gamma} \nabla u(t) \cdot \nabla \omega(t) \, dx. \]
In order to prove that \( \theta \in L^1(0, T) \) we first observe that \( \theta \) is measurable, since it is the lim sup of a sequence of measurable functions. Moreover, we have
\[
\int_0^T \theta(t) \, dt = \int_0^T \int_{\Omega \setminus \Gamma} \nabla u(t) \cdot \nabla \omega(t) \, dx \, dt \leq \int_0^T \|\nabla u(t)\|_{L^2(\Omega \setminus \Gamma)} \|\nabla \omega(t)\|_{L^2(\Omega \setminus \Gamma)} \, dt
\leq C_2 \int_0^T \|\nabla \omega(t)\|_{L^2(\Omega \setminus \Gamma)} \, dt \leq C_2 \sqrt{T} \|\omega\|_{L^2((0, T); H^1(\Omega))}. \]
By (c)’ of Theorem 5.6 we have, for every \( j \in \mathbb{N} \) and for every \( t \in [0, T] \)
\[ E(u_{h_{k_j}}(t)) = E_{h_{k_j}}(u_{h_{k_j}}(t)) \leq E((u_0)_{h_{k_j}}) + \int_0^t \int_{\Omega \setminus \Gamma} \nabla u_{h_{k_j}}(s) \cdot \nabla \omega_{h_{k_j}}(s) \, dx \, ds \]
\[ = E((u_0)_{h_{k_j}}) + \int_0^t \int_{\Omega \setminus \Gamma} \nabla u_{h_{k_j}}(s) \cdot \nabla \omega_{h_{k_j}}(s) \, dx \, ds. \] (5.10)
Note now that the energy \( E(\cdot) \) is lower semicontinuous w.r.t. weak convergence in \( H^1(\Omega \setminus \Gamma) \). Moreover, since \( (u_0)_{h_{k_j}} \to u_0 \) strongly in \( H^1(\Omega \setminus \Gamma) \), we have
\[ \lim_{j \to \infty} E((u_0)_{h_{k_j}}) = E(u_0). \]
Therefore, taking the limsup in $j$ of the (5.10), and using Fatou’s Lemma

\[
E(u(t)) \leq \liminf_{j \to \infty} E(u_{h_{k_j}}(t)) \leq E(u_0) + \limsup_{j \to \infty} \int_0^t \int_{\Omega \setminus \Gamma} \nabla u_{h_{k_j}}(s) \cdot \nabla \omega_{h_{k_j}}(s) \, dxds
\]

\[
\leq E(u_0) + \limsup_{k \to \infty} \int_0^t \int_{\Omega \setminus \Gamma} \nabla u_{k_j}(s) \cdot \nabla \omega_{k_j}(s) \, dxds
\]

\[
\leq E(u_0) + \int_0^t \limsup_{k \to \infty} \int_{\Omega \setminus \Gamma} \nabla u_{k_j}(s) \cdot \nabla \omega_{k_j}(s) \, dxds
\]

\[
= E(u_0) + \int_0^t \int_{\Omega \setminus \Gamma} \nabla u(s) \cdot \nabla \omega(s) \, dxds,
\]

so that (c) follows.

We finally prove the stationarity. Since $u_h(0) = u_{0,h} \to u_0$ as $h \to 0$, we only have to prove the condition at a point $t \in (0, T]$. Let $\psi \in H^1(\Omega \setminus \Gamma)$ with $\psi = 0$ on $\partial D \Omega$. Then [24], we can find a sequence \( \{\psi_{h_{k_j}}\}_{j \in \mathbb{N}} \) such that

\[
\psi_{h_{k_j}} \to \psi \quad \text{strongly in } H^1(\Omega \setminus \Gamma) \quad \text{as } j \to \infty
\]

and $\psi_{h_{k_j}} \in \mathcal{E}_{h_{k_j}}$ with $\psi_{h_{k_j}} = 0$ on $\partial D \Omega$, for every $j \in \mathbb{N}$. Note that, by (5.7), we have

\[
\psi_{h_{k_j}} \in \mathcal{E}_{h_{k_j}} \quad \text{with } \psi_{h_{k_j}} = 0 \quad \text{on } \partial D \Omega \quad \text{for every } j > l.
\]

Therefore, by (5.3)

\[
\int_{\Omega \setminus \Gamma} \nabla u_{h_{k_j}}(t) \cdot \nabla \psi_{h_{k_j}} \, dx
\]

\[
\geq \int_{\Gamma} \left( -[\psi_{h_{k_j}}]g'([u_{h_{k_j}}(t)]) \text{sign}([u_{h_{k_j}}(t)]) \mathbb{1}_{J_{u_{h_{k_j}}}(t)} - ||\psi_{h_{k_j}}||1_{J_{u_{h_{k_j}}}(t)} \right) d\mathcal{H}^{d-1},
\]

for every $j > l$. By (5.9) we have

\[
\lim_{j \to \infty} \int_{\Omega \setminus \Gamma} \nabla u_{h_{k_j}}(t) \cdot \nabla \psi_{h_{k_j}} \, dx = \int_{\Omega \setminus \Gamma} \nabla u(t) \cdot \nabla \psi_{h_{k_j}} \, dx.
\]

Define now, for every $t \in [0, T]$ and for every $j > l$, the function $f_j(t) : \Gamma \to \mathbb{R}$ as

\[
f_j(t) := -[\psi_{h_{k_j}}]g'([u_{h_{k_j}}(t)]) \text{sign}([u_{h_{k_j}}(t)]) \mathbb{1}_{J_{u_{h_{k_j}}}(t)} - ||\psi_{h_{k_j}}||1_{J_{u_{h_{k_j}}}(t)}.
\]

We want to prove that for every $t \in [0, T]$

\[
\liminf_{j \to \infty} f_j(t) \geq -[\psi_{h_{k_j}}]g'([u(t)]) \text{sign}([u(t)]) \mathbb{1}_{J_{u(t)}} - ||\psi_{h_{k_j}}||1_{J_{u(t)}} \quad \mathcal{H}^1\text{-a.e. in } \Gamma.
\]

Up to extracting a further subsequence, we can assume that

\[
\liminf_{j \to \infty} f_j(t) = \lim_{j \to \infty} f_j(t) \quad \mathcal{H}^1\text{-a.e. in } \Gamma,
\]

and

\[
\lim_{j \to \infty} [u_{h_{k_j}}(t)] = [u(t)] \quad \mathcal{H}^1\text{-a.e. in } \Gamma.
\]

Now, let us fix $x \in J_{u(t)}$ such that (5.14) and (5.15) hold true. Then, for $j \in \mathbb{N}$ large enough we have

\[
x \in J_{u_{h_{k_j}}(t)} \quad \text{and} \quad \text{sign}([u_{h_{k_j}}(t)](x)) = \text{sign}([u(t)](x)).
Therefore,

\[
\liminf_{j \to \infty} f_j(t)(x) = \lim_{j \to \infty} f_j(t)(x)
\]

\[
= \lim_{j \to \infty} -[\psi_{h_{kJ}}](x)g'([u_{h_{kJ}}(t)](x))\text{sign}([u_{h_{kJ}}(t)](x))1_{J_{u_{h_{kJ}}}(t)}(x) - ||\psi_{h_{kJ}}||1_{J_{u_{h_{kJ}}}(t)}(x)
\]

\[
= -[\psi_{h_{kJ}}](x)g'([u(t)](x))\text{sign}([u(t)](x))1_{J_{u(t)}}(x) - ||\psi_{h_{kJ}}||1_{J_{u(t)}}(x)
\]

(5.16)

for \(\mathcal{H}^1\text{-a.e. } x \in \Gamma \cap J_{u(t)}\). If, instead, \(x \in J_{u(t)}^c\), then recalling that \(0 \leq g' \leq 1\) we have

\[
\liminf_{j \to \infty} f_j(t)(x) = \lim_{j \to \infty} f_j(t)(x)
\]

\[
= \lim_{j \to \infty} -[\psi_{h_{kJ}}](x)g'([u_{h_{kJ}}(t)](x))\text{sign}([u_{h_{kJ}}(t)](x))1_{J_{u_{h_{kJ}}}(t)}(x) - ||\psi_{h_{kJ}}||1_{J_{u_{h_{kJ}}}(t)}(x)
\]

\[
\geq -||\psi_{h_{kJ}}||1_{J_{u(t)}}(x).
\]

(5.17)

Combining (5.16) and (5.17) we obtain (5.13). Thanks to (5.12) and (5.13) we can pass to the limit in (5.11), obtaining

\[
\int_{\Omega \setminus \Gamma} \nabla u(t) \cdot \nabla \psi_{h_{kJ}} \, dx = \lim_{j \to \infty} \int_{\Omega \setminus \Gamma} \nabla u_{h_{kJ}}(t) \cdot \nabla \psi_{h_{kJ}} \, dx
\]

\[
\geq \liminf_{j \to \infty} \int_{\Gamma} \left(-[\psi_{h_{kJ}}]g'([u_{h_{kJ}}(t)])\text{sign}([u_{h_{kJ}}(t)](x))1_{J_{u_{h_{kJ}}}(t)} - ||\psi_{h_{kJ}}||1_{J_{u_{h_{kJ}}}(t)}\right) \, d\mathcal{H}^{d-1}
\]

\[
\geq \int_{\Gamma} \liminf_{j \to \infty} \left(-[\psi_{h_{kJ}}]g'([u_{h_{kJ}}(t)])\text{sign}([u_{h_{kJ}}(t)](x))1_{J_{u_{h_{kJ}}}(t)} - ||\psi_{h_{kJ}}||1_{J_{u_{h_{kJ}}}(t)}\right) \, d\mathcal{H}^{d-1}
\]

\[
\geq \int_{\Gamma} \left(-[\psi_{h_{kJ}}]g'([u(t)])\text{sign}([u(t)](x))1_{J_{u(t)}} - ||\psi||1_{J_{u(t)}}\right) \, d\mathcal{H}^{d-1}
\]

Finally, passing to the limit as \(l \to \infty\) we have

\[
\int_{\Omega \setminus \Gamma} \nabla u(t) \cdot \nabla \psi \, dx \geq \int_{\Gamma} \left(-[\psi]g'([u(t)])\text{sign}([u(t)](x))1_{J_{u(t)}} - ||\psi||1_{J_{u(t)}}\right) \, d\mathcal{H}^{d-1},
\]

and we conclude.

\[\Box\]

6. Numerical Experiments

The scope of this section is to practically show that the procedure illustrated in the previous sections can be effectively implemented and produces the desired quasi-static evolution, according to the one described in [4]. We refer the reader to Section 4 for the notations used here.

6.1. Numerical simulations in 1 dimension. We first analyze the results obtained for a one dimensional problem, when \(\Omega \subset \mathbb{R}\). Despite its simplicity, the one dimensional setting allows to give a detailed comparison between numerical results and analytic predictions, since in this case the explicit solutions of (4.5) are known. We consider the following geometry:

\[
\Omega = [0, 2\ell], \quad \ell = 0.5, \quad \Gamma = \{\ell\}, \quad \partial D \Omega = \{0, 2\ell\}.
\]

We follow the evolution in the time interval \([0, T] = [0, 1]\), and the external load applied to the endpoints \(\partial D \Omega = \{0, 2\ell\}\) is given by

\[
f(t)(x) = 2(x - \ell)t, \quad \text{for every } x \in [0, 1] \text{ and } t \in [0, 1].
\]
We uniformly discretize the domain into $2N = 80$ intervals, so that the spatial discretization step is given by $h = \ell/N$. Finally, we choose a time step $\delta = 0.02$, so that the total evolution is concluded after 50 time steps. In our specific case, by Proposition 4.3 and a direct calculation we have that the parameter $\eta$ in condition (A1) can be taken as 

$$\eta = \frac{1}{2R} + \max\{4, 4\sqrt{\ell}\},$$

where the constant $R$ is the one appearing in the definition of the function $g$, see (4.2).

From a practical viewpoint, the computational time needed to solve the minimization problem (3.3) could grow without any control. Hence, for any $i \in \{0, \ldots, 50\}$ and $j \in \mathbb{N}$ fixed, we stop the minimization loop as soon as $\|Av - f(i\delta)\| < 10^{-6}$. That is, $v_j^i$ is chosen in such a way that $\|Av_j^i - f(i\delta)\| < 10^{-6}$. Instead, we stop the external loop (that is, the limit of $v_j^i$ as $j \to \infty$), as soon as $\|v_j^i - v_j^{i-1}\| < 10^{-13}$. The main reason for these choices is that the quasi-static evolution generated by the algorithm is extremely sensitive to any perturbation. Thus, a larger time step $\delta$, or a too badly approximated critical point at each time step, could lead to nonphysical results.

We observe that the analytic evolutions discussed in [4, Section 9] depend on the size of the parameter $R$. Therefore, in order to compare our results with those in [4], we distinguish two cases.

**Case $R \geq 2\ell$.** When $R$ is chosen large with respect to the size $2\ell$ of the elastic body, the evolution found by numerical simulations evolves along *global* minimizers of the energy, and we can observe the three phases of the cohesive fracture formation: non-fractured, pre-fractured (that is when the opening of the crack is smaller than $R$ and cohesive forces appear), and completely fractured (when the opening of the crack is larger than $R$ and the cohesive forces disappear), see Figure 2.

![Figure 2](image_url)

**Figure 2.** The evolution of the quasi-static cohesive fracture for $R \geq 2\ell$ at time instances $t = 0, 0.5, 0.52, 1$.

Note that in the time interval $[0, 0.5]$ the evolution follows the elastic deformation. After $t = 0.5$ a fracture appears, since the elastic deformation is not any more a critical point of the energy functional (see [4, Section 9]). Then, the pre-fracture phase starts, showing a bridging force acting on the two lips of the crack. At time $t = 1$ the cohesive energy reaches its maximum, and the body is completely fractured. It is worth observing that this evolution coincides with the one analytically calculated in [4, Section 9].

We can also investigate what happens from the energy point of view, see Figure 3. We have a smooth transition between the different phases, and the total energy has a nondecreasing profile. The beginning of the pre-fractured phase can be observed at the $25^{th}$ time step (i.e. at time
Case $R < 2\ell$. The evolution of the system changes radically when $R < 2\ell$. In this case (see Figure 4) the failure happens instantaneously, without a bridging phase, and thus the body exhibits what in literature is known as brittle behavior. More precisely, in the time interval $[0, 0.5]$ the evolution follows again the elastic deformation, and a crack appears at $t = 0.5$. However, immediately after $t = 0.5$ the body is completely fractured, and no cohesive forces appear. It is important to observe that in this case we actually observe an evolution along critical points that are not global minimizers. Indeed, the evolution is elastic until $t = 0.5$, although it would be energetically convenient to completely break the body at some earlier time $\bar{t} < 0.5$ (see [4, Section 9] for a detailed description of all critical points). Thus, we see that
the algorithm chooses the critical point which is the closest to the initial configuration, even if other options are available, which are more convenient from an energetic point of view. This evolution is particularly supported by the idea that in nature a body does not completely change its configuration crossing high energetic barriers if a stable configuration can be found with less energetic effort.

![Figure 5](image)

**Figure 5.** The total, fracture, and elastic energy evolution of the quasi-static cohesive fracture for $R < 2\ell$.

Also in this case, we can observe the evolution from the energetic viewpoint, see Figure 5. At time $t = 0.5$, when the elastic deformation ceases to be a critical point, the domain breaks and the total energy decreases up to the value of $R/2$, so that no bridging force is keeping the two lips together. As we already observed, the evolution along global minimizers would instead lead to a fracture way before the critical load is reached.

Again, the evolution found with our numerical simulation coincides with that one given in [4, Section 9]. In particular, our simulations agree with the *crack initiation criterion* (see [4, Theorem 4.6]), which states that a crack appears only when the maximum sustainable stress along $\Gamma$ is reached. In this case, this happens at $t = 0.5$, when the slope of the elastic evolution reaches the value $g'(0) = 1$.

6.2. **Numerical simulations in 2 dimensions.** Having a first analytical validation of the numerical minimization procedure, we can now challenge the algorithm in the simulation of two dimensional evolutions. We now consider the domain introduced in Section 4.2 setting $\ell = 0.5$, $2N = 8$, and $\kappa = 1/2$. Within this choice, the crack initiation time is reduced exactly of a factor $1/2$, allowing us to speed up the failure process. Since all the computations are performed on a MacBook Pro equipped with a 2.6GHz Intel Core i7 processor, 8GB of RAM, 1600MHz DDR3, the two dimensional simulations are performed only for a qualitative purpose. Indeed, we are mainly interested in showing that our algorithm produces physically sound evolutions also in dimension 2, and when the external displacement $f$ is non-trivial. The very sparse discretization of the domain $\Omega$ is due to the fact that the minimization in (3.3) requires a huge computational effort, both in terms of time and memory. Indeed, in order to implement more realistic experiments, with a finer discretization, we would need to modify the architecture of the minimization algorithm, in such a way that it may run on parallel cores.
We perform two different series of experiments, one with boundary datum
\[
f_1(t)(\mathbf{x}) = 2(x_1 - \ell)t, \quad \text{for every } t \in [0, 1] \text{ and } \mathbf{x} \in \Omega,
\]
see Figure 6, and the other one with boundary datum
\[
f_2(t)(\mathbf{x}) = 2t \cos \left(2\frac{x_2 - \ell}{\ell}(x_1 - \ell)\right), \quad \text{for every } t \in [0, 1] \text{ and } \mathbf{x} \in \Omega,
\]
see Figure 7. Here, we denoted by \( \mathbf{x} = (x_1, x_2) \) the generic point of \( \Omega = (0, 1) \times (0, 1) \). We now need to reduce the tolerance of the termination condition of the outer loop of the Algorithm, setting it to \( 5 \cdot 10^{-14} \). Indeed, we experimented that for bigger values of this tolerance some instabilities in the solution were introduced, leading to an asymmetric evolution, also in the case of \( f_1 \) as external displacement, where we expect an invariant behavior with respect to the space variable \( x_2 \).

**Case \( R \geq 2\ell \).** In Figure 6 and 7 we report 4 different instances of the evolution for the two different boundary data, when \( R \geq 2\ell \). When the external displacement is \( f_1 \), which is constant with respect to the second coordinate \( x_2 \), we observe that the evolution is also constant with respect to \( x_2 \). For both boundary data, the failure of the body undergoes the three phases of deformation, as it happened in the one dimensional case.

**Case \( R < 2\ell \).** When the boundary datum is \( f_1 \), see Figure 8, the specimen breaks in a brittle fashion, without showing any cohesive intermediate phase. This simulation is actually an evidence that the algorithm still characterizes the correct critical points, following the principle that the domain should not fracture as long as a non-fractured configuration is still a critical point. We conclude commenting the simulation where the boundary datum is \( f_2 \) with \( R < 2\ell \), see Figure 9. By setting a displacement highly varying with respect to the \( x_2 \) coordinate, we observe that the different phases of the fracture formation can coexist. At time \( t = 0.24 \) the domain still presents no fracture, as expected by the previous numerical experiments. Then, at
$t = 0.34$, a pre-fracture appears, but only at those points where the external load is bigger, i.e. around $x_2 = \ell$. In fact, even at the final time $t = 1$, the domain is not completely fractured.

Note that, when the boundary datum is $f_1$, the evolution coincides with the one obtained analytically [4, Section 9]. In particular, the fracture appears at $t = 0.25$, when the slope of the elastic evolution reaches the value $\kappa g'(0) = 1/2$ and thus the crack initiation criterion is satisfied.

![Figure 8](image_url1)

**Figure 8.** The evolution of the quasi-static cohesive fracture for $R < 2\ell$ at time instances $t = 0.04, 0.24, 0.26, 0.5$ with external displacement $f_1$.

![Figure 9](image_url2)

**Figure 9.** The evolution of the quasi-static cohesive fracture for $R < 2\ell$ at time instances $t = 0.02, 0.22, 0.24, 0.5$ with external displacement $f_2$.

**Acknowledgments**

Marco Artina and Massimo Fornasier acknowledge the financial support of the International Research Training Group IGDK 1754 “Optimization and Numerical Analysis for Partial Differential Equation with Nonsmooth Structures” of the German Science Foundation.

Francesco Solombrino acknowledges the financial support of the ERC under Grant No. 290888 “Quasistatic and Dynamic Evolution Problems in Plasticity and Fracture” (P.I. Prof. G. Dal Maso).

**References**


