Proof Search and Co-NP Completeness for Many-Valued Logics

Mattia Bongini
Department of Mathematics, Munich University of Technology, Germany

Agata Ciabattoni
Department of Computer Languages, Vienna University of Technology, Austria

Franco Montagna
Department of Information Engineering and Mathematics, University of Siena, Italy

Abstract

We provide a methodology to introduce relational hypersequent calculi for a large class of many-valued logics, and a sufficient condition for their Co-NP completeness. Our results apply to many well known logics including Gödel, Łukasiewicz and Product Logic, as well as Hajek’s Basic Fuzzy Logic.

1 Introduction

The invertibility of rules in a proof system is an important feature for guiding proof search and turns out to be very useful to settle the computational complexity of the formalized logic. For many-valued logics, calculi with invertible rules (proof search oriented calculi) have been provided for all finite-valued logics. These calculi are defined by suitably generalizing Gentzen sequents $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$ to many placed (or labelled) sequents, each corresponding to a truth value of the logic, see e.g. the survey [9] (or [12], for the non-deterministic case). The construction of these calculi, out of the truth

Email addresses: mattia.bongini@ma.tum.de (Mattia Bongini),
agata@logic.at (Agata Ciabattoni), montagna@unisi.it (Franco Montagna).

The premises are derivable whenever their conclusions are.
tables of the connectives, is even computerized [10]. This does not apply to infinite-valued logics where, excepting Gödel logic [8,5], proof search oriented calculi – when available – are introduced on a logic by logic basis and their construction requires some ingenuity; this is for instance the case of the calculi for Łukasiewicz and Product logic, see e.g. [26]. Among them, the calculi in [28,27] are defined using hypersequents, which are finite “disjunctions” of Gentzen sequents [4,3].

An important step towards the automated construction of proof search oriented calculi for many-valued logics was done in [8], with the introduction of sequents of relations that are disjunctions of semantic predicates over formulas, and of a methodology to construct such calculi for all projective logics. Intuitively a logic is projective if for each connective □, the value of □(x₁,...,xₙ) is equal to a constant or to one of the x₁,...,xₙ. The methodology was extended in [18] to handle semi-projective logics where the value of each □(x₁,...,xₙ) can also be a term of the form p(xᵢ) with p unary function symbol. Projective logics are quite interesting, but perhaps not general enough: among many-valued logics, only the finite-valued logics and Gödel logic are projective. Semi-projective logics constitute a slightly larger class, and they capture, for instance, Nilpotent and Weak Nilpotent Minimum logic [20], the relevant logic RM [2] and, by allowing conservative extensions, n-contractive BL-logics [13] (i.e. Hajek’s Basic Fuzzy Logic BL extended with n-contraction). All semi-projective logics have a locally finite variety as their equivalent algebraic semantics while important many-valued logics such as Łukasiewicz logic, Product Logic and BL do not, despite of the fact that they have suitable calculi with invertible rules [17,29,15] and are Co-NP complete. The calculi in [17,29,15] are defined using relational hypersequents (r-hy persequents for short) that generalize hypersequents by considering finite disjunctions of two different types of sequents, where Gentzen’s sequent arrow ⇒ is replaced in one by < and in the other by ≤.

In this paper we consider r-hypersequents that generalize both hyipsequents and sequents of relations. They indeed consist of disjunctions of arbitrary semantic predicates (not only < and ≤) over multisets of formulas, rather than single formulas as in the case of sequents of relations.

We introduce a methodology to define r-hyipsequent calculi for a large class of many-valued logics (hyperprojective logics) and use the resulting calculi to prove complexity results for the formalized logics. We indeed identify sufficient conditions on r-hyipsequent calculi that guarantee Co-NP completeness of the formalized logics. Our methodology applies to projective and semi-projective logics as well as to Łukasiewicz, Product Logic and BL; it subsumes existing results on sequent of relations and on r-hyipsequent calculi (e.g. [8,6,18,17,29,15]) and puts most of the known complexity results for many-valued logics under a common umbrella. Moreover, our methodology can be
applied to new logics (or, e.g., to logics not having yet a proof search oriented calculus), provided that they are hyperprojective.

In a hyperprojective logic the value of each connective \( \Box(x_1, \ldots, x_n) \) is defined by cases this time expressed by relations on multisets of constants, of terms \( x_1, \ldots, x_n \) and of \( p(x_i) \), for \( p \) unary functions, and the main formula is reduced to multisets of such terms.

We illustrate the idea behind hyperprojective logics and the way we define \( r \)-hypersequent calculi for them with the example of Product Logic. For this logic, as in (the projective presentation of) Gödel Logic \([8,6,7]\) it is natural to consider the relations \( < \) and \( \leq \). Product Logic is neither projective nor semi-projective, because if \( x, y \notin \{0, 1\} \) then the product \( x \& y \) depends on both \( x \) and \( y \). The idea is to represent the product by a monoidal operation \( \oplus \) standing for the union of multisets, i.e. \( x \& y = x \oplus y \) (here there is no case distinction).

In general, to define (invertible) rules for a connective \( \Box(x_1, \ldots, x_n) \) of a hyperprojective logic \( L \) we will consider “reductions” (based on the relations in the semantic theory of \( L \)) that act on multisets of formulas, i.e., on \( \mu \oplus \Box(x_1, \ldots, x_n) \), where \( \mu \) stands for a multiset.

In the particular case of the connective \( x \& y \) of Product Logic we consider the following “reduction cases”: for \( \mu \oplus x \& y \triangleleft \mu' \) as \( \mu \oplus x \oplus y \triangleleft \mu' \) and for \( \mu \triangleleft \mu' \oplus x \& y \) as \( \mu \triangleleft \mu' \oplus x \oplus y \), where \( \triangleleft \) denotes either \( < \) or \( \leq \). Our calculus for Product Logic will then contain \( r \)-hypersequents consisting of disjunctions of sequents of the form \( \Gamma < \Delta \) or \( \Gamma \leq \Delta \), where \( \Gamma \) and \( \Delta \) are multisets of formulas. As in the case of hypersequents \([4,3]\) the disjunction will be denoted by “\( | \)” and the union of multisets by “\( , \)”. With this notation we have that \( \phi \& \psi, \Gamma \triangleleft \Delta \) reduces (and it is indeed equivalent to) to \( \phi, \psi, \Gamma \triangleleft \Delta \) and \( \Gamma \triangleleft \Delta, \phi \& \psi \) reduces to \( \Gamma \triangleleft \Delta, \phi, \psi \), which naturally lead to the left and right rules for the connective \& w.r.t. the relation \( \triangleleft \). Now since

\[
x \land y = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } y < x \end{cases} \quad x \lor y = \begin{cases} y & \text{if } x \leq y \\ x & \text{if } y < x \end{cases}
\]

we have:

- \( \Gamma, \phi \lor \psi \triangleleft \Delta \) reduces to \( \psi \triangleleft \phi \lor \Gamma, \psi \triangleleft \Delta \) and to \( \psi \leq \phi \lor \Gamma, \phi \triangleleft \Delta \);
- \( \Gamma \triangleleft \Delta, \phi \lor \psi \) reduces to \( \psi \triangleleft \phi \lor \Gamma \triangleleft \Delta, \psi \) and to \( \phi \leq \psi \lor \Gamma \triangleleft \Delta, \phi \);
- \( \Gamma, \phi \land \psi \triangleleft \Delta \) reduces to \( \psi \triangleleft \phi \land \Gamma, \psi \triangleleft \Delta \) and to \( \phi \leq \psi \land \Gamma, \psi \triangleleft \Delta \);
- \( \Gamma \triangleleft \Delta, \phi \land \psi \) reduces to \( \psi \triangleleft \phi \land \Gamma \triangleleft \Delta, \phi \) and to \( \phi \leq \psi \land \Gamma \triangleleft \Delta, \psi \).
Finally, recalling that in Product Logic

\[ x \rightarrow y = \begin{cases} \frac{y}{x} & \text{if } y < x \\ 1 & \text{if } x \leq y \end{cases} \]

and that “,” represents the product, we have the following reductions for \( \rightarrow \):

- \( \Gamma, \phi \rightarrow \psi < \Delta \) reduces to \( \psi < \phi | \Gamma, 1 < \Delta \) and to \( \phi \leq \psi | \Gamma, \psi < \Delta, \phi \), to be read as: either \( \phi \leq \psi \) or \( (\psi < \phi \text{ and then} \) \( \Gamma, \frac{\psi}{\phi} < \Delta \), which is equivalent to \( \Gamma, \psi < \Delta, \phi \); moreover, either \( \psi < \phi \) or \( (\phi \leq \psi \text{ and hence} \) \( \phi \rightarrow \psi = 1 \), and \( \Gamma, 1 < \Delta \).
- \( \Gamma < \Delta, \phi \rightarrow \psi \) reduces to \( \psi < \phi | \Gamma, 1 < \Delta \) and \( \phi \leq \psi | \Gamma, \phi < \Delta, \psi \), whose explanation is similar.

Note that the above reductions are nothing but (invertible) rules introducing a connective \( \star \in \{\rightarrow, \land, \lor\} \) in the left position \( (\Gamma, \phi \star \psi < \Delta) \) or in the right position \( (\Gamma < \Delta, \phi \star \psi) \).

In this paper we provide a methodology to introduce relational hypersequent calculi for all hyperprojective logics, and a sufficient condition for their Co-NP completeness. The paper is organized as follows: Section 2 introduces hyperprojective logics. Similarly to the case of projective and semi-projective logics the definition is based on the shape of their underlying first-order semantic theory. Section 3 connects hyperprojective logics to r-hypersequent calculi. Section 3.1 shows how to transform (the semantic theory behind) a hyperprojective logic into invertible r-hypersequent rules. Soundness and completeness of the resulting calculi is contained in Section 3.2. Proving that a logic is hyperprojective is however a non trivial task. Section 3.3 shows how to automate this process for a large class of many-valued logics (algebraizable and whose equivalent semantics is generated by a single algebra whose first-order theory satisfies suitable properties), which includes Łukasiewicz, Product and Basic Logic. Section 4 proves the co-NP completeness of the validity problem of hyperprojective logics whose r-hypersequent calculi satisfy suitable and easily checkable properties. As case studies we apply our methodology to Gödel logic, RM, Product logic , Łukasiewicz logic and Hajek’s BL by presenting for them r-hypersequent calculi and alternative proofs (w.r.t., e.g., [11,23,1]) of Co-NP completeness.

2 Hyperprojective logics

We define the class of many-valued logics we consider in this paper and begin this section recalling some basic properties of multisets. For all concepts of
universal algebra we refer to [16] while for many-valued logics to [22].

A finite multiset on a set $S$ is a map $\mu$ from $S$ into $\mathbb{N}$ such that the set $S(\mu) = \{ s \in S \mid \mu(s) > 0 \}$, called the support of $\mu$, is finite. Moreover for every $s \in S(\mu)$, $\mu(s)$ is called the multiplicity of $s$. Notice that a multiset $\mu$ is completely determined by its support $S(\mu)$ and the multiplicity of each $s \in S(\mu)$.

The union $\mu_1 \oplus \mu_2$ of two finite multisets $\mu_1$ and $\mu_2$ of $S$ is defined as $(\mu_1 \oplus \mu_2)(s) = \mu_1(s) + \mu_2(s)$. Note that $\oplus$ is commutative and associative and its neutral element is the zero function on $S$, indicated by $\varepsilon$. In addition, $S(\mu_1 \oplus \mu_2) = S(\mu_1) \cup S(\mu_2)$. Given two sequences of multisets $\overline{\mu} = \mu_1, \ldots, \mu_n$ and $\overline{\nu} = \nu_1, \ldots, \nu_n$, both with length $n$, then $\overline{\mu} \oplus \overline{\nu}$ denotes the sequence of multisets $\mu_1 \oplus \nu_1, \ldots, \mu_n \oplus \nu_n$. Finally, we define $\Theta^n_i(\nu)$ as the sequence $\varepsilon, \ldots, \nu, \ldots, \varepsilon$ of multisets, such that each multiset is the empty one except at position $i$, where it is $\nu$. Thus, given a sequence of multisets $\overline{\mu} = \mu_1, \ldots, \mu_n$, we have $\overline{\mu} \oplus \Theta^n_i(\nu) = \mu_1, \ldots, \mu_i \oplus \nu, \ldots, \mu_n$.

In the sequel, when there is no danger of confusion we identify the element $x \in S$ with the multiset $\mu$ with support \{ $x$ \} such that $\mu(x) = 1$. Moreover, given (not necessarily distinct) elements $a_1, \ldots, a_n$, $a_1 \oplus \cdots \oplus a_n$ (written without parentheses) denotes the multiset $\mu$ such that $S(\mu) = \{ x : x = a_i \text{ for some } i \text{ with } 1 \leq i \leq n \}$, and for $x \in S(\mu)$, $\mu(x)$ is the cardinality of the set $\{ i : x = a_i \}$.

As in the case of projective and semi-projective logics [8,18], a hyperprojective logic $L$ has an associated semantic first-order theory $T_L$ whose intended range of discourse are sets of truth values. We will make some assumptions on $T_L$. The first is on its language and we assume that:

(TL0) The language of $T_L$ consists of: (a) an $n$-ary operation for each $^2$ $n$-ary connective of $L$ and a constant for each propositional constant of $L$; (b) possibly, finitely many constants and finitely many unary function symbols, denoted in the sequel by $c_1, \ldots, c_s$ and $f_1, \ldots, f_h$ respectively, not in the language of $L$; (c) a constant $\varepsilon$ for the empty multiset and a binary operation $\oplus$ for the union of multisets; (d) some relation symbols, not in the language of $L$; (e) for each $n$-ary relation symbol $P$, we assume that either the negation of $P(x_1, \ldots, x_n)$ is equivalent in $T_L$ to a disjunction of atomic formulas without function symbols, or that the language of $T_L$ has a relation symbol $P^*$, along with the axiom $P^*(x_1, \ldots, x_n) \Leftrightarrow \neg P(x_1, \ldots, x_n)$. In any case, the negation of an atomic formula $P$ is equivalent to a formula which is either atomic

\[ ^2 \text{We assume that } L \text{ has finitely many constants and connectives, and we identify each connective with its corresponding operation and each propositional constant with its corresponding constant symbol. } \]
or the disjunction of atomic formulas, and which will be denoted by $P^\ast$. Moreover, we set $P^{\ast\ast} = P$.

Before introducing the other conditions, we briefly comment on (TL0). Condition (a) allow us to treat formulas of $L$ as terms of $T_L$. As in the definition of semi-projective logics [18], the unary function symbols and constant symbols in condition (b) may help to “reduce” the connectives to the right format, (see condition (TL4) below); when present in $T_L$, we assume that the extension $L'$ of $L$ by the unary connectives is a conservative extension of $L$. Condition (c) allows us to represent not only formulas, but also multisets of formulas, while condition (d) will permit to represent the proof theory of $L$ inside $T_L$. Finally, condition (e) allows us to eliminate negations and to keep the language finite.

**Example 1.** The semantic theory $T_L$ for product logic is the theory of product chains with large order $\leq$ and strict order $<\,\oplus$ is interpreted as product, and $\varepsilon$ is interpreted as $1$. Moreover $x \leq^* y$ is equivalent to $y < x$, and hence we may eliminate negations.

To introduce the next conditions on $T_L$, we need some auxiliary definitions.

**Definition 2.** A formula of $T_L$ that does not contain any quantifier, negation, implication and function symbol with the exception of $\oplus$ is called simple. It is called weakly simple if it has no quantifiers, negation or implication and its terms are either variables, or constants, or of the form $f_i(x)$, where $f_i$ is a unary function symbol not in $L$, or of the form $t_1 \oplus \cdots \oplus t_k$, where $t_1, \ldots, t_k$ are terms of the form shown above.

The next three assumptions on $T_L$ are:

(TL1) There is a weakly simple formula $Des(x)$ of $T_L$ such that for every formula $\phi$, $T_L \vDash Des(\phi)$ iff $L \vDash \phi$.

(TL2) The set of valid formulas of $T_L$ that are universal closures of weakly simple formulas is decidable.

(TL3) If $T_L$ has unary function symbols $f_1, \ldots, f_h$ not in $L$, then for all $i, j = 1, \ldots, h$, $T_L$ has an axiom of the form $f_i(f_j(x)) = x$ or an axiom of the form $f_i(f_j(x)) = f_r(x)$ for some $r$.

The next (and last) condition on $T_L$ describes the connectives of $L$ and will determines the logical rules of the proof systems defined in the next section.

We introduce some notation and terminology first: an overlined letter will denote the sequence consisting of the same letter with subscripts. Thus for instance, $\overline{x}$ will denote $x_1, \ldots, x_n$ for some $n > 0$ and $\overline{\delta}$ will denote $\delta_1, \ldots, \delta_m$, for some $m > 0$. As usual whenever writing $P(\overline{x})$ or $f(\overline{x})$ we implicitly assume that the arities of $P$ and $f$ coincide with the number of $x_i$.

**Definition 3.** The formulas $Q_1, \ldots, Q_s$ of $T_L$ form a partition of the unit if $T_L \vDash Q_i \rightarrow \neg Q_j$ for $i \neq j$ and $T_L \vDash \bigvee_{j=1}^s Q_j$.

We denote by $L'$ the logic $L$ extended by the symbols of constant and of unary
function in $T_L$ but not in $L$, to be interpreted as propositional constants and as unary connectives, respectively.

The connectives of hyperprojective logics have a case-reduction of the following form: for every connective $\Box$ in the language of $L$, for every unary connective $u$ in the language of $T_L$ but not in the language of $L$, for every predicate symbol $P$ of $T_L$ with arity $k$ and every $1 \leq i \leq k$

\[
P(\overline{\mu} \oplus \Theta_i^P(\Box(\overline{\pi}))) = \begin{cases} 
P(\overline{\mu} \oplus \varphi_1^{(\Box P_i)}) & \text{if } Q_1^{(\Box P_i)}(\overline{\pi}) \\
\ldots & \ldots \\
P(\overline{\mu} \oplus \varphi_{\ell}^{(\Box P_i)}) & \text{if } Q_{\ell}^{(\Box P_i)}(\overline{\pi}) \
\end{cases}
\]

\[
P(\overline{\mu} \oplus \Theta_i^n(u(\Box(\overline{\pi})))) = \begin{cases} 
P(\overline{\mu} \oplus \varphi_1^{(u(\Box) P_i)}) & \text{if } Q_1^{(u(\Box) P_i)}(\overline{\pi}) \\
\ldots & \ldots \\
P(\overline{\mu} \oplus \varphi_{\ell'}^{(u(\Box) P_i)}) & \text{if } Q_{\ell'}^{(u(\Box) P_i)}(\overline{\pi}) 
\end{cases}
\]

where:

- $\overline{\mu}$ is a sequence of multisets of formulas of $L'$;
- $\varphi_a^{(\Box P_i)}$, $a = 1, \ldots, \ell$, and $\varphi_b^{(u(\Box) P_i)}$, $b = 1, \ldots, \ell'$, are sequences of length $k$ of multisets whose support is contained in the set

\[Z(\overline{\pi}) = \{x, c_v, f_w(x) \mid x \in \{\overline{\pi}\} \cup \{c_1, \ldots, c_s\}, v = 1, \ldots, s, w = 1, \ldots, h\};\]

- $Q_a^{(\Box P_i)}(\overline{\pi})$, $a = 1, \ldots, \ell$ and $Q_b^{(u(\Box) P_i)}(\overline{\pi})$, $b = 1, \ldots, \ell'$, are partitions of the unit consisting of weakly simple formulas.

The dependancy of $\varphi_a^{(\Box P_i)}$, $\varphi_b^{(u(\Box) P_i)}$, $Q_a^{(\Box P_i)}(\overline{\pi})$ and $Q_b^{(u(\Box) P_i)}(\overline{\pi})$ on $\Box$, $P$, $i$ will be omitted in what follows. Thus, with the notation just introduced, (TL4) reads:

(TL4) For each $k$-ary predicate $P$, for each position $i$ with $1 \leq i \leq k$, for each $n$-ary connective $\Box$ and for each unary function symbol $u$ in $T_L$ and not in $L$, there are two partitions of the unit $Q_1(\overline{\pi}), \ldots, Q_\ell(\overline{\pi})$ and $Q_1^n(\overline{\pi}), \ldots, Q_{\ell'}^n(\overline{\pi})$, consisting of weakly simple formulas, and sequences of multisets $\varphi_a : a = 1, \ldots, \ell$, and $\varphi_b^n : b = 1, \ldots, \ell'$, with support $Z(\overline{\pi})$, such that, for $a = 1, \ldots, \ell$, for $b = 1, \ldots, \ell'$, for every sequence $\overline{\mu}$ of multisets of formulas of $L'$ and for every substitution $\sigma$ of variables with formulas of $L'$, the following conditions hold:

(TL4/1) $T_L \models Q_a(\sigma(\overline{\pi})) \Rightarrow (P(\overline{\mu} \oplus \sigma(\Theta_i^P(\Box(\overline{\pi})))) \Leftrightarrow P(\overline{\mu} \oplus \sigma(\varphi_a)))$,

(TL4/2) $T_L \models Q_b^n(\sigma(\overline{\pi})) \Rightarrow (P(\overline{\mu} \oplus \sigma(\Theta_i^n(u(\Box(\overline{\pi})))) \Leftrightarrow P(\overline{\mu} \oplus \sigma(\varphi_b^n)))$,

where $\Rightarrow$ and $\cap$ indicate the implication and the conjunction in $T_L$, and $Q \Leftrightarrow R$ stands for $(Q \Rightarrow R) \cap (R \Rightarrow Q)$. 

7
Definition 4. A logic $L$ is hyperprojective if it has a semantic theory $T_L$ satisfying conditions (TL0), ..., (TL4) above. $L$ is said to be regular if all unary function symbols or constants in $T_L$ are already in $L$.

Remark 2.1. Projective and semi-projective logics are particular cases of hyperprojective logics (see Example 11). Indeed, every connective of a projective logic has the following form

$$\square(\overline{x}) = \begin{cases} 
    t_1 & \text{if } Q_1(\overline{x}) \\
    \ldots \\
    t_k & \text{if } Q_k(\overline{x}) 
\end{cases}$$

where $Q_1(\overline{x}), \ldots, Q_k(\overline{x})$ are simple formulas which form a partition of the unit and $t_1, \ldots, t_k$ are either variables among $\overline{x}$ or constants. In the case of a semi-projective logic $L$, we have instead that for every $n$-ary connective $\square$ of $L$ and every unary connective $u$ in the language of the semantic theory $T_L$:

$$\square(\overline{x}) = \begin{cases} 
    t_1 & \text{if } Q_1(\overline{x}) \\
    \ldots \\
    t_k & \text{if } Q_k(\overline{x}) 
\end{cases} \quad \text{and} \quad u(\square(\overline{x})) = \begin{cases} 
    t'_1 & \text{if } Q_1^u(\overline{x}) \\
    \ldots \\
    t'_k & \text{if } Q_k^u(\overline{x}) 
\end{cases}$$

where $Q_1(\overline{x}), \ldots, Q_k(\overline{x})$, and $Q_1^u(\overline{x}), \ldots, Q_k^u(\overline{x})$ respectively, still form a partition of the unit, but are now weakly simple (i.e., with unary function symbols in $T_L$), and $t_i, t'_i$ are either variables in $\overline{x}$, constants or terms of the form $f_i(x)$, $x$ a variable in $\overline{x}$ and $f_i$ a unary function symbols in $T_L$.

After identifying a multiset consisting of just one formula with the formula itself, we see that projective or semiprojective logics are a special case of hyperprojective logics.

In regular hyperprojective logics weakly simple formulas are also simple, condition (TL3) becomes irrelevant, and (TL4/2) does not apply.

A nonregular hyperprojective logic is a logic that admits a conservative extension which is a regular hyperprojective logic. Regular hyperprojective logics are the three most famous Fuzzy Logics [22]: Gödel, Łukasiewicz and Product logic (see Example 11), while a hyperprojective logic that is not regular is Hajek’s Basic Fuzzy Logic [22], see the example below.

Example 5. Recall that $BL$ is complete with respect to the ordinal sum $A$ of $\omega$ copies of $[0,1]_{MV}$. This algebra is defined as follows: fix an increasing sequence $0 = a_0 < a_1 < \cdots < a_n < \cdots < 1$ such that $\lim_n a_n = 1$. The order is the order of $[0,1]$, and conjunction $\&$ is defined as follows: if $x, y$ are in the same interval $[a_i, a_{i+1}]$, then $x \& y = \max\{a_i, x + y - a_{i+1}\}$, otherwise, $x \& y = \min\{x, y\}$. Implication in $A$ is the residual of conjunction. Note that
each component \([a_i, a_{i+1}]\) is isomorphic to \([0, 1]_{MV}\) via the map \(x \mapsto \frac{x-a_i}{a_{i+1}-a_i}\).

Hence we will start from \(A\), in the construction of the semantic theory for \(BL\). Besides 0 and 1, the bottom and the top elements of \(A\), we will use two new constant symbols 0\(^+\) and 1\(^+\) which stand for the bottom and for the top of the component we are dealing with. Since elements of different components are treated independently, we can use the same symbols 0\(^+\) and 1\(^+\) for each component. We expand \(A\) to a structure \(A^*\) by adding the multiset operation \(\oplus\) along with the following relations:

(a) For \(x, y \in A\), we set \(x \ll y\) iff \(x < y\) and either \(y = 1\) or \(x, y\) belong to different components. We will only use sequents of the form \(\mu \ll \nu\) when \(\mu\) and \(\nu\) have cardinality 1, that is, they both consist of a single formula.

(b) For \(x, y \in A\), we set \(x \vartriangleleft y\) if \(x < y < 1\) and \(x, y\) are in the same component, and \(x \preceq y\) if either \(x \vartriangleleft y\) or \(x = y\). Moreover if \(\sigma, \nu\) are multisets with support included in \(A\), then we set \(\sigma \vartriangleleft \nu\) if the elements of the support of \(\sigma\) and \(\nu\) are all in the same component and different from 1, and \(\sum_{x \in S(\sigma)} \mu(x) \cdot x < \sum_{y \in S(\nu)} \mu(y) \cdot y\). We set \(\sigma \preceq \nu\) if either the elements of \(\sigma\) and \(\nu\) are all equal to 1, or they are all in the same component and all different from 1, and \(\sum_{x \in S(\sigma)} \mu(x) \cdot x \leq \sum_{y \in S(\nu)} \mu(y) \cdot y\).

When \(x, y \in A\), we write \(x \llleq y\) to mean that either \(x \ll y\) or \(x \preceq y\) or \(x \llleq y\). The negation, \(x \llleq^* y\), of \(x \llleq y\), also denoted by \(y \llleq x\), may be represented as \(y \llleq x|y \preceq x < y\). Moreover \(x \vartriangleleft y\) and \(x \llleq^* y\) may be represented as \(y \llleq x|y \preceq x < y\) and as \(x \llleq y|y < x < y\), respectively (where \(|\) is interpreted as disjunction). Finally, negations of sequents are equivalent to disjunctions of sequents. For instance, if \(\sigma, \nu\) are multisets, then the negation of \(\sigma \preceq \nu\) is equivalent to the disjunction of \(\nu \vartriangleleft \sigma\) and of all sequents \(x \ll y\) where \(x, y \in S(\sigma) \cup S(\nu)\).

Lattice operations \(\wedge\) and \(\vee\) are defined in terms of \(\leq\) as usual.

The theory \(T_{BL}\) will be basically the first-order theory of \(A^*\). A multiset with cardinality greater than 1 will only appear in sequents of the form \(\leq\) or \(\vartriangleleft\) and in a context where all formulas in the multiset are supposed to be in the same component. The intended interpretation of \(\oplus\) will be the isomorphic copy of the sum, namely, \(x \oplus y = \frac{x-a_i}{a_{i+1}-a_i} + \frac{y-a_i}{a_{i+1}-a_i}\), where \([a_i, a_{i+1}]\) is the component which \(x\) and \(y\) belong to. In this context, \(0^+\) and \(1^+\) will represent \(a_i\) and \(a_{i+1}\), respectively.

We immediately see that \(T_{BL}\) satisfies condition (TL0), and we put emphasis on the fact that with the exception of \(1^+\) and \(0^+\), all function symbols and constant symbols of \(T_{BL}\) are already in the language of \(BL\). Regarding condition
We may take $\text{Des}(x) := 1 \leq x$, while condition (TL2) is easily proved to be satisfied (see [15, Lemma 4.5]). Notice that condition (TL3) is empty since we do not add any extra unary connective to $T_{BL}$.

The connectives of $BL$ can be represented in the theory $T_{BL}$ in the following way:

$$x \land y = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } y < x \end{cases} \quad x \lor y = \begin{cases} y & \text{if } x \leq y \\ x & \text{if } y < x \end{cases}$$

$$x \& y = \begin{cases} x & \text{if } x \ll y \\ y & \text{if } y \ll x \\ 0^+ & \text{if } x \equiv y \text{ and } x \oplus y < 1^+ \end{cases} \quad x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } y \ll x \end{cases}$$

The symbol $-$ is not in the language, but we may avoid it replacing $x - y \triangleleft z$ by $x \triangleleft y \oplus z$ and $z \triangleleft x - y$ by $z \oplus y \ll x$, where $\triangleleft$ is any of $\ll$ or $\leq$. Moreover, under the assumption $\phi \equiv \psi$, sequents of the form $\phi \ll \gamma$ or $\gamma \ll \phi \& \psi$ reduce to $\phi \ll \gamma$ and to $\gamma \ll \phi$, respectively, while under the assumption $\psi \ll \phi$, sequents of the form $\phi \rightarrow \psi \ll \gamma$ or $\gamma \ll \phi \rightarrow \psi$ reduce to $\psi \ll \gamma$, or to $\gamma \ll \psi$, respectively. Hence, the symbol $-$ is never needed in our reductions.

Notice that this representation is neither projective nor semi-projective, since there are terms appearing in the decomposition which involve $\ominus$. Let us now see how to obtain the conditions in (TL4) from this format of the connectives, in the case of the binary predicate $\ll$, the position 1 (thus at the left hand side of $\ll$) and the connective $\&$.

Consider a substitution of variables with formulas which associates the (generic) formulas $\phi$ and $\psi$ in $L'$ to $x$ and $y$, respectively. Taking into account all we have said, for every pair of multisets of formulas $\Gamma$ and $\Delta$ we have the following conditions for (TL4/1):

- $(\phi \ll \psi) \Rightarrow ((\Gamma, \phi \& \psi \ll \Delta) \Leftrightarrow (\Gamma, \phi \ll \Delta))$
- $(\psi \ll \phi) \Rightarrow ((\Gamma, \phi \& \psi \ll \Delta) \Leftrightarrow (\Gamma, \psi \ll \Delta))$
- $((\phi \equiv \psi) \cap (\psi, \phi \ll 1^+)) \Rightarrow ((\Gamma, \phi \& \psi \ll \Delta) \Leftrightarrow (\Gamma, 0^+ \ll \Delta))$
- $((\phi \equiv \psi) \cap (1^+ \ll \phi, \psi)) \Rightarrow ((\Gamma, \phi \& \psi \ll \Delta) \Leftrightarrow (\Gamma, \phi, \psi \ll \Delta, 1^+))$

The remaining reductions are obtained analogously. A complete presentation of a proof system for $BL$ will be given in the last section.
3 Proof search oriented calculi for hyperprojective logics

Proof systems for various many-valued logics have been defined using hypersequents [3], that are finite "disjunctions" of standard sequents; these include Gödel, Lukasiewicz and Product logic, as well as Monoidal T-norm based logic MTL [20]; see [26] for an overview. A hypersequent is a multiset of the form

\[ \Gamma_1 \Rightarrow \Delta_1 \vert \ldots \vert \Gamma_n \Rightarrow \Delta_n, \]

where each component \( \Gamma_i \Rightarrow \Delta_i \) is an ordinary sequent, i.e. of the form \( \phi_1^i, \ldots, \phi_n^i \Rightarrow \psi_1^i, \ldots, \psi_m^i \). In contrast with the above mentioned logics, Hajek’s basic fuzzy logic BL (cf. Example 5) seems to escape an analytic formalization using hypersequents, that is a hypersequent calculus whose proofs proceed by stepwise decomposition of the formulas to be proved. Even when analytic, hypersequent calculi are in general not suitable for proof search. The main reason being that their rules are usually not invertible (exceptions are [28,27]). For instance, termination is still an open problem for the hypersequent calculus of MTL, that also does not help characterizing the computational complexity of the logic.

Relational hypersequents are a generalization of hypersequents introduced in [17] to define proof search oriented calculi for Gödel, Lukasiewicz and Product logic. A relational hypersequent, or, for short, r-hypersequent, is a multiset of two different types of sequents, where Gentzen’s sequent arrow \( \Rightarrow \) is replaced in one by \( < \) and in the other by \( \leq \). Relational hypersequents were also used in [29] to define analytic calculi for (a conservative extension of) BL. All these calculi are however defined on a logic by logic basis and their discovery has required some ingenuity.

In this section we introduce a methodology to define relational hypersequent calculi for all hyperprojective logics. To this purpose along the line of the sequents of relations [8,18] we generalize r-hypersequents to objects understood as a disjunction of arbitrary predicates (not only the binary ones \( < \) and \( \leq \)) belonging to a chosen semantic theory.

3.1 From hyperprojective logics to r-hypersequent calculi

Let \( L \) be a hyperprojective logic with semantic theory \( T_L \). A relational hypersequent for \( T_L \), has the form

\[ P_1(\overline{p}_1) \vert \ldots \vert P_\ell(\overline{p}_\ell) \]

(each \( P_i(\overline{p}_i) \) is called a component of the r-hypersequent) where \( P_1, \ldots, P_\ell \) are predicate symbols of \( T_L \) and \( \overline{p}_1, \ldots, \overline{p}_\ell \) are sequences of multisets of formulas
of the conservative extension $L'$ of $L$. In what follow, according to a long standing tradition in proof theory we will often write “,” for $\oplus$ and multisets of formulas $\phi_1 \oplus \cdots \oplus \phi_n$ will be represented as $\phi_1, \ldots, \phi_n$. In a relational hypersequent, $|$ is interpreted as a disjunction, that is, for any sequence of multisets $\mu = \mu_1, \ldots, \mu_r$,

$$M, v \models P_1(\mu_1) | \ldots | P_t(\mu_t) \quad \text{iff} \quad M, v \models P_1(\mu_1) \cup \ldots \cup P_t(\mu_t).$$

The $T_L$ formula $P_1(\mu_1) \cup \ldots \cup P_t(\mu_t)$ will be called the formula associated to the $r$-hypersequent $P_1(\mu_1) | \ldots | P_t(\mu_t)$.

Conditions (TL4/1) and (TL4/2) can be translated into logical rules as follows: first of all, we can assume that $Q_a(\pi)$ and $Q_b^n(\pi)$ are conjunctions of multisets of atomic formulas. Since the negation of a conjunction of atomic formulas is equivalent to a disjunction of negations of atomic formulas (and since our semantic theory satisfies condition (TL0), each negation of an atomic formula is equivalent to an atomic formula), the negations of $Q_a(\pi)$ and $Q_b^n(\pi)$ can be written as $r$-hypersequents. We denote such $r$-hypersequents by $(Q_a)^*(\pi)$ and $(Q_b^n)^*(\pi)$. Hence we have

**Definition 6** (Logical rules). For any $n$-ary connective $\Box$ of $L$ (and for any unary function symbol $u$ in the language of $T_L$ and not in that of $L$), any predicate symbol $P$ of $T_L$ with arity $r$, and for any position $i$ with $1 \leq i \leq r$ we have the rule $(P, \Box, i)$ (resp. $(P, u(\Box), i)$) for introducing $\Box(\pi)$ (resp. $u(\Box(\pi))$) at position $i$ into a component of an $r$-hypersequent containing the symbol $P$.

Let $Q_1(\pi), \ldots, Q_{\ell}(\pi)$ and $Q_1^n(\pi), \ldots, Q_{\ell}^n(\pi)$ be two partitions of the unit, consisting of weakly simple formulas, and $\overline{\pi}_a, \overline{\pi}_b$ be sequences of multisets with support in $Z(\pi)$, such that, for $a = 1, \ldots, \ell$, for $b = 1, \ldots, \ell'$, for any sequence $\pi$ of multisets of formulas of $L'$ and for every substitution $\sigma$ of variables with formulas of $L'$, conditions (TL4/1) and (TL4/2) hold. Then we have the rules

$$H[(Q_1)^*(\sigma(\pi))|P(\overline{\pi} \oplus \sigma(\overline{\pi}_1))] \ldots H[(Q_{\ell})^*(\sigma(\pi))|P(\overline{\pi} \oplus \sigma(\overline{\pi}_t))]/H[P(\overline{\pi} \oplus \Theta^n(\Box(\sigma(\pi))))] \quad (P, \Box, i)$$

$$H[(Q_1^n)^*(\sigma(\pi))|P(\overline{\pi} \oplus \sigma(\overline{\pi}_1^n))] \ldots H[(Q_{\ell}^n)^*(\sigma(\pi))|P(\overline{\pi} \oplus \sigma(\overline{\pi}_t^n))]/H[P(\overline{\pi} \oplus \Theta^n(\Box(\sigma(\pi))))] \quad (P, u(\Box), i)$$

where $H$ is any side $r$-hypersequent.

In the above rules, the formula $P(\overline{\pi} \oplus \Theta^n(\Box(\sigma(\pi))))$ (resp. $P(\overline{\pi} \oplus \Theta^n(\Box(\sigma(\pi))))$) is called main formula of the rule, while $(Q_a)^*(\sigma(\pi))$ (resp., $(Q_b^n)^*(\sigma(\pi))$) are called contexts of the rule. For each $a = 1, \ldots, \ell$ (resp., for $b = 1, \ldots, \ell'$), $P(\overline{\pi} \oplus \sigma(\overline{\pi}_a))$ (resp. $P(\overline{\pi} \oplus \sigma(\overline{\pi}_b))$) is called reduced formula of the main formula.

Given a hyperprojective logic $L$, we denote by $\mathbb{H}L$ the $r$-hypersequent calculus whose rules are defined as indicated above and whose axioms are
Definition 7 (Axioms). Suppose that \( P_1(\mu_1), \ldots, P_\ell(\mu_\ell) \) are weakly simple atomic formulas. Then the r-hypersequent \( P_1(\mu_1) \mid \ldots \mid P_\ell(\mu_\ell) \) is an axiom of \( \mathbb{H}L \) iff the universal closure of the formula associated to it is valid in \( T_L \).

From condition (TL4) of \( T_L \) immediately follows that each rule constructed as in Definition 6 is sound and invertible for \( L \), i.e. for any rule

\[
\begin{array}{c}
H_1 \quad \ldots \quad H_n \\
\hline \\
\hspace{1cm} H
\end{array}
\]

in \( \mathbb{H}L \) and for any model \( \mathcal{M} \) of \( T_L \), \( \mathcal{M} \models H_i \) for any \( i = 1, \ldots, n \) iff \( \mathcal{M} \models H \).

3.2 Soundness, completeness and decidability

Let \( L \) be a hyperprojective logic. We show that the calculus \( \mathbb{H}L \) is sound and complete for \( L \) and use it to show that \( L \) is decidable. Towards this section we fix a hyperprojective logic \( L \) with semantic theory \( T_L \) and r-hypersequent calculus \( \mathbb{H}L \). An r-hypersequent \( H \) of \( T_L \) is said to be provable in \( \mathbb{H}L \) if there is an upward tree of r-hypersequents rooted in \( H \), such that every leaf is an axiom of \( \mathbb{H}L \) and every other r-hypersequent is obtained from the ones standing immediately above it by application of one of the rules of \( \mathbb{H}L \). Such a tree is called a derivation of \( H \); we define the length of a derivation as the number of inferences in a maximal branch of that derivation.

Being \( Des \) a simple formula (cf. condition (TL1)), it can be written as a conjunctions of disjunctions of atomic simple formulas, indicated in what follows as \( Des_1, \ldots, Des_k \). The two results below establish the soundness and completeness of \( \mathbb{H}L \) with respect to \( L \).

Theorem 8 (Soundness). Let \( \phi \) be any formula in the language of \( L \). If \( Des_1(\phi), \ldots, Des_k(\phi) \) are provable in \( \mathbb{H}L \) then \( \phi \) is valid in \( L \).

Proof. We will prove, arguing by induction on the length \( l \) of the derivation of \( H \), the more general statement: if \( H \) is provable in \( \mathbb{H}L \) then for every model \( \mathcal{M} \) of \( T_L \) and for every valuation \( v \) of \( \mathcal{M} \), \( \mathcal{M}, v \models H \).

Base step: if \( l = 0 \) then \( H \) is an axiom of \( \mathbb{H}L \), and the associated formula is valid in \( T_L \).

Inductive step: assume that the claim holds for derivations with length \( n \) and let \( l = n + 1 \). If the r-hypersequents above \( H \) are \( H_1, \ldots, H_n \), then by the inductive hypothesis, \( \mathcal{M}, v \models H_i \) for \( i = 1, \ldots, n \). The claim follows by condition (TL4).

Thus, if \( Des_1(\phi), \ldots, Des_k(\phi) \) are provable in \( \mathbb{H}L \) then \( T_L \models Des(\phi) \), and
Theorem 9 (Completeness). Let $\phi$ be any formula in the language of $L$. If $\phi$ is valid in $L$ then $\text{Des}_1(\phi), \ldots, \text{Des}_k(\phi)$ are provable in $\mathbb{H}L$.

Proof. If $\phi$ is valid in $L$ then, by (TL1), $\text{Des}_1(\phi), \ldots, \text{Des}_k(\phi)$ are provable in $T_L$, and hence for every model $\mathcal{M}$ of $T_L$ and for every valuation $v$ of $\mathcal{M}$, $\mathcal{M}, v \models \text{Des}_i(\phi)$ for $i = 1, \ldots, k$. Applying the rules of $\mathbb{H}L$ backwards to every $\text{Des}_i(\phi)$ we can build a tree, called the reduction tree of $\text{Des}_i(\phi)$. The leaves of the reduction tree are r-hypersequents $P_1(\overline{m}_1) \mid \ldots \mid P_t(\overline{m}_t)$ such that $P_1(\overline{m}_1) \cup \ldots \cup P_t(\overline{m}_t)$ is a weakly simple formula of $T_L$, and by the invertibility of the rules of $\mathbb{H}L$, their universal closure is valid in $T_L$. Hence the reduction tree of $\text{Des}_i(\phi)$ is actually a derivation of $\text{Des}_i(\phi)$, and thus $\text{Des}_1(\phi), \ldots, \text{Des}_k(\phi)$ are provable in $\mathbb{H}L$.

A first, easy but important property of hyperprojective logics is given by the following result.

Theorem 10. Any hyperprojective logic $L$ is decidable.

Proof. Given a formula $\phi$ of $L$, we apply the rules of $\mathbb{H}L$ backwards, starting from $\text{Des}(\phi)$. We can assume that no consecutive occurrences of unary functions not in $L$ occur in $\phi$ (otherwise, we eliminate them using (TL3)). Let $a(\phi)$ denote the number of occurrences of function symbols in $\phi$ in the language of $L$, and $b(\phi)$ be the number of occurrences of unary function symbols in $\phi$ and not in $L$, and set $c(\phi) = 2a(\phi) + b(\phi)$ (so, function symbols in $L$ count more than unary function symbols not in $L$ in the computation of $c(\phi)$). We call $c(\phi)$ the complexity of $\phi$. Now let for each r-hypersequent $H$, $c(H)$ denote the maximum complexity of the formulas in $H$ and $k(H)$ denote the number of occurrences of formulas of maximum complexity. We reduce first the formulas with maximal complexity. It is easily seen that if $H'$ is any premise of a rule acting on a formula of maximal complexity and $H$ is its conclusion, then $(c(H'), k(H')) < (c(H), k(H))$. It follows that every path of the reduction tree terminates with a r-hypersequent containing formulas are either atomic or of the form $f_j(p)$ with $p$ a propositional variable or a constant and $f_j$ a unary operation not in $L$. These r-hypersequents are weakly simple formulas and hence they are decidable by our assumptions on $T_L$. This shows that $L$ is decidable.

Remark 3.1. Being invertible, the rules of our calculi decompose the main formula in a set of reduced formulas which have strictly lower complexity (in the sense of the above proof) than the main formula. This justifies our use of the word reduction in place of rule, to underline the fact that whenever we read a reduction tree starting from the root, each rule is actually reducing the complexity of the starting r-hypersequent.
3.3 How to build the semantic theory for a hyperprojective logic

To show that a logic $L$ is hyperprojective we need to provide a semantic theory $T_L$ satisfying conditions (TL0), ..., (TL4) (cf. Definition 4). Though this could be tricky in general, for many interesting logics this process can be automated; this is the case when the logic is algebraizable and its equivalent algebraic semantics is a variety $\mathcal{V}$ generated by a single algebra $A$, and whose first-order theory satisfies suitable conditions, which will be described below.

If $L$ is algebraizable, then the connectives of $L$ may be regarded as operation symbols and formulas of $L$ as terms in the language of $\mathcal{V}$. Let $A$ be an algebra which generates $\mathcal{V}$ as a variety, with universe $A$. Let $A^m$ be the family of all multisets with support included in $A$ (we may assume, without loss of generality, that $A \cap A^m = \emptyset$). We equip $A \cup A^m$ with the operations of $A$, extended by the clause $f(a_1, \ldots, a_n) = \varepsilon$ if some $a_i$ is in $A^m$, with the operation $\oplus$ for the union of multisets, extended by the condition $a \oplus b = \varepsilon$ if either $a \notin A^m$ or $b \notin A^m$, and, possibly, with finitely many additional unary operations and constants. Moreover we equip $A \cup A^m$ with some relations, thus getting a structure $A^*$. We assume that:

(A1) There is a weakly simple formula $\text{Des}(x)$ such that for each formula $\phi$ of $L$ we have: $L \models \phi$ iff $A^* \models \text{Des}(\phi)$.

(A2) The set of weakly simple formulas valid in $A^*$ is decidable.

(A3) If $f, g$ are unary operations not in the language of $L$, there is a unary operation $h$ not in the language of $L$ such that $A^* \models \forall x(f(g(x)) = h(x))$.

(A4) With the notation used in the definition of hyperprojective logic, for each $k$-ary predicate $P$, for each position $i$ with $1 \leq i \leq k$, for each $n$-ary operation $\square$ and for each unary function symbol $u$ in $T_L$ and not in $L$, there are two partitions of the unit $Q^u_1(\pi), \ldots, Q^u_\ell(\pi)$ and $Q^n_1(\pi), \ldots, Q^n_\ell'(\pi)$, consisting of weakly simple formulas, and sequences of multisets $v_a : a = 1, \ldots, \ell, v_b^u : b = 1, \ldots, \ell'$, whose support is included in $A$, such that, for $a = 1, \ldots, \ell$, for $b = 1, \ldots, \ell'$, for every sequence $\pi$ of multisets of elements of $A$ and for every sequence $\pi$ of elements of $A$, the following conditions hold:

(A4/1) $A^* \models Q_a(\pi) \Rightarrow (P(\pi \oplus \Theta^u_i(\square(\pi))) \Leftrightarrow P(\pi \oplus v_a))$, and

(A4/2) $A^* \models Q^u_b(\pi) \Rightarrow (P(\mu \oplus \Theta^u_i(u(\square(\pi)))) \Leftrightarrow P(\pi \oplus v^u_b))$.

When conditions (A1), ..., (A4) are satisfied, we take $T_L$ to be the first-order theory of the model $A^*$.

**Example 11.**

(1) As said before, every projective [8] or semi-projective logic [18] is hyperprojective, provided that we identify any formula $\phi$ with the multiset with cardinality 1 whose support is $\{\phi\}$.
(2) Though Product Logic is neither projective nor semi-projective, it is hyperprojective. To see this, let \( A \) be the product algebra on \([0,1]\), (see [22]), and let us extend \( A \) to a structure \( A^* \) on \( A \cup A^m \) along the lines indicated at the beginning of this section. We introduce the relations \( \leq \) and \( < \) on multisets, defined as follows:

\[
\sigma \leq \nu \quad \text{iff} \quad \prod_{x \in S(\sigma)} x^\mu(x) \leq \prod_{y \in S(\nu)} y^\mu(y)
\]

\[
\sigma < \nu \quad \text{iff} \quad \prod_{x \in S(\sigma)} x^\mu(x) < \prod_{y \in S(\nu)} y^\mu(y)
\]

(\text{an empty product is 1 by definition}).

Let \( T_L \) be the first-order theory of \( A^* \), where we write \( < \) instead of \( \prec \) and \( \leq \) instead of \( \preceq \). Then \( T_L \), along with the decomposition rules for the connectives of Product Logic presented in the introduction, witnesses the fact that product logic is hyperprojective. The predicate \( \text{Des} \) can be defined by \( \text{Des}(x) := 1 \leq x \). Note that the set of weakly simple formulas of \( T_L \) is not only decidable, but also in \( P \) (the complexity of linear programming [30]). The rules of the \( r \)-hypersequent calculus for Product Logic are contained in Section 1.

(3) Lukasiewicz Logic [19] is hyperprojective. To build its semantic theory, let \( A \) be the standard MV-algebra on \([0,1]\), and construct \( A^* \) instead of \( A \) on multisets, defined as follows:

\[
\sigma < \nu \quad \text{iff} \quad \sum_{x \in S(\sigma)} \mu(x) \cdot x < \sum_{y \in S(\nu)} \mu(y) \cdot y
\]

\[
\sigma \leq \nu \quad \text{iff} \quad \sum_{x \in S(\sigma)} \mu(x) \cdot x \leq \sum_{y \in S(\nu)} \mu(y) \cdot y
\]

(where an empty sum is 0 by definition).

The predicate \( \text{Des} \) can be defined by \( \text{Des}(x) := 1 \leq x \). Moreover, recalling that \( x \cdot y = \max\{x+y-1,0\} \) and \( x \cdot y = \min\{1-x+y,1\} \), and writing “,” instead of \( \oplus \), \( < \) instead of \( \prec \) and \( \leq \) instead of \( \preceq \), we have the following rules (we omit the side hypersequent \( H \) for space reasons):

\[
1 \leq \phi, \psi | \Gamma \preceq \Delta \quad \phi, \psi < 1 | \Gamma, \phi, \psi \preceq \Delta, 1 \\
\Gamma, \phi \& \psi \preceq \Delta
\]

\[
1 \leq \phi, \psi | \Gamma \preceq \Delta \quad \phi, \psi < 1 | \Gamma, 1 \preceq \Delta, \phi, \psi \\
\Gamma \preceq \Delta, \phi \& \psi
\]

\[
\phi < \psi | \Gamma, \psi \preceq \Delta \quad \psi \leq \phi | \Gamma, \phi \preceq \Delta \\
\Gamma, \phi \& \psi \preceq \Delta
\]

\[
\phi < \psi | \Gamma, \phi \preceq \Delta \quad \psi \leq \phi | \Gamma, \psi \preceq \Delta \\
\Gamma, \phi \& \psi \preceq \Delta
\]

\[
\psi < \phi | 1 \preceq \Delta \quad \phi \leq \psi | \Gamma, \psi, 1 \preceq \Delta, \phi \\
\Gamma, \phi \rightarrow \psi \preceq \Delta
\]

\[
\phi < \psi | \Gamma \preceq \Delta \quad \psi \leq \phi | \Gamma, \phi \preceq \Delta, \psi \\
\Gamma \preceq \Delta, \phi \rightarrow \psi
\]

where \( \preceq \) stands for \( < \) or \( \leq \) uniformly in each rule.
4 Co-NP completeness

We identify sufficient conditions that make a hyperprojective logic \( L \) Co-NP complete. The conditions are on \( T_L \) and on the r-hypersequent calculus \( \mathbb{HL} \). As we only treat logics that are Co-NP hard, by [24], we only have to worry about Co-NP containment.

4.1 Uniform sets of rules

We start discussing uniformity of contexts, a useful property of r-hypersequent rules, and show that all calculi for hyperprojective logics can be modified in order to fulfill it. Intuitively, having a uniform set of rules means that the rules for introducing the same connective have all the same form, i.e. they do not depend on the particular predicate symbol or the position inside it.

**Definition 12.** Let \( L \) be an hyperprojective logic with semantic theory \( T_L \) and r-hypersequent calculus \( \mathbb{HL} \). We say that \( \mathbb{HL} \) has a uniform set of rules if for each rule \((P, □, i)\) (resp., \((P, u(□), i)\)), the contexts \((Q_a)^*(\sigma(\overline{x}))\) (resp., \((Q_b^u)^*(\sigma(\overline{x}))\)) and the number of premises in the rule \((P, □, i)\) (resp., \((P, u(□), i)\)) only depend on □ (resp., on \( u \) and on □) but not on \( P \) or on \( i \).

Uniform rules allow us to reduce several occurrences of the same formula simultaneously and make the proof search algorithm more efficient, as the following example shows.

**Example 13.** In Product Logic we can simultaneously reduce all occurrences of \( A \rightarrow B \) in the sequent \( H := A \rightarrow B \leq C \mid D < A \rightarrow B \), where \( A, B, C \) and \( D \) are propositional variables, taking advantage of the fact that the contexts are the same in each rule. As already outlined in Section 1, the rules for → are (where \( \triangleleft \) is either \( \leq \) or \( < \) and \( H' \) is any side-sequent)

\[
\begin{align*}
\frac{H'|\phi < \psi|\Gamma, 1 \triangleleft \Delta}{H'|\Gamma, \phi \rightarrow \psi \triangleleft \Delta} & \quad (\triangleleft, \rightarrow, \text{left}) \\
\frac{H'|\phi < \psi|\Gamma, 1 \triangleleft \Delta}{H'|\Gamma \triangleleft \Delta, \phi \rightarrow \psi} & \quad (\triangleleft, \rightarrow, \text{right})
\end{align*}
\]

Thus, by repeatedly applying them to the sequent \( H \) we have the following (compact) reduction:

\[
\begin{array}{c}
B < A | 1 \leq C | D < 1 \\
A \leq B | B \leq A, C | D, A < B \\
A \rightarrow B \leq C | D < A \rightarrow B
\end{array}
\]

Our first result is that in any hyperprojective logic we can always get uniformity for free. This is based on the following facts:
(1) Given two partitions of the unit $Q_1, \ldots, Q_n$ and $R_1, \ldots, R_k$, the partition consisting of all formulas $W_{i,j} = Q_i \cap R_j$, $i = 1, \ldots, n, j = 1, \ldots, k$ is a common refinement of the original partitions.

(2) Hence, for every connective $\Box$ of $L$ and every unary function symbol $u$ in $T_L$ and not in $L$, we can find a common refinement $W(x)$ of all partitions used for the rules of all sequents of the form $P(\pi \Theta^n_i (\Box(x)))$ for any $i = 1, \ldots, n$, and, respectively, a common refinement $W^u(x)$ of all partitions of unit used in the rules of all sequents of the form $P(\pi \Theta^n_i (u(\Box(x))))$ for any $i = 1, \ldots, n$. Now we may suppose that each element $W_{a}(x)$ of $W_a$ and each element $W_{a}^u(x)$ of $W_a^u$ is a conjunction of atomic formulas, and hence using our assumptions on negations we may write their negations as r-hypersequents, which will be denoted by $(W_a)^*(x)$ and $(W_a^u)^*(x)$, respectively. It follows that the original rules can be replaced by the uniform rules

\[
\begin{align*}
H | ((W_1(\sigma(x)))^* & P(\pi \Theta^n_1 (\Box(x)))) \ldots H | ((W_e(\sigma(x)))^* P(\pi \Theta^n_e (\Box(x)))) \quad (P, \Box, i) \\
H | ((W_1^u(\sigma(x)))^* P(\pi \Theta^n_1 (u(\Box(x)))) \ldots H | ((W_e^u(\sigma(x)))^* P(\pi \Theta^n_e (u(\Box(x))))) \quad (P, u(\Box), i)
\end{align*}
\]

Hence it follows

**Proposition 14.** Every hyperprojective logic $L$ has a uniform set of rules.

In the sequel, we tacitly assume to apply rules for (different occurrences of) the same formula simultaneously, whenever possible.

### 4.2 Resource-boundedness

We define the size of an r-hypersequent as the number of symbols occurring in each formula contained in it. To guarantee that the size of each leaf of the reduction tree is polynomial in the size of the end r-hypersequent we need a further assumption on the calculus’ rules.

**Example 15.** Assume to have a hyperprojective logic $L$ with a binary connective $\Box$ such that $P(x)$ is unary predicate symbol of $T_L$ and that its r-hypersequent calculus $\mathbb{H}_L$ has the rule

\[
P(\Gamma, \phi, \psi, \psi) \quad (P, \Box, 1)
\]

where $\Gamma$ is an arbitrary multiset of formulas and $\phi$ and $\psi$ are metavariables for formulas in $L'$. Clearly the rule is uniform (it has no context). Now let $\Phi_0 = \phi$ and $\Phi_{n+1} = \Box(\phi, \Phi_n)$. Then the size of $\Phi_n$ is linear in $n$, but, writing $h\phi$ for $\phi, \ldots, \phi$ $h$ times, the nodes in the unique branch of the reduction tree
with root $P(\Phi_n)$ are
\[
P(\phi, 2\Phi_{n-1}), P(3\phi, 4\Phi_{n-2}), P(7\phi, 8\Phi_{n-3}), \ldots, P((2^{n+1} - 1)\phi).
\]

Hence, the size of the leaf of the tree is exponential in the size of the root.

To avoid such situations, we introduce the following constraint.

**Definition 16.** A rule of the form $(P, \Box, i)$ (resp., $(P, u(\Box), i)$), cf. Definition 6, is said to be resource-bounded if for all $P, i$ (resp., $P, i, u$, where $u$ is a unary function in the language of $T_L$ but not in $L$), every element of the union of the multisets in $\nu_a$, $a = 1, \ldots, \ell$ (resp. $\nu_b^u$, $b = 1, \ldots, \ell'$) in condition (TL4) has multiplicity at most 1.

The definition says that although in each rule’s context there may be repetitions of (proper) subformulas of the main formula of the rule, any reduced formula of the main formula can be increased by one occurrence at most of each of such proper subformulas.

A steady example of a family of logics with a resource-bounded $r$-hypersequent calculi is given by semi-projective logics (and thus also projective logics). We have seen that, for the calculi of these logics, multisets of formulas can be replaced simply by formulas, and hence Definition 16 is trivially true for their rules.

A trivial corollary of Proposition 14 states that resource-boundedness is stable when replacing a set of rules with a uniform set of rules.

**Corollary 17.** If $L$ has a resource bounded proof system, then it has a proof system which is both uniform and resource bounded.

### 4.3 Main Theorem

A final condition for the Co-NP containment of a hyperprojective logic is that the set of valid weakly simple formulas of its semantic theory is in Co-NP. The main result of this section reads:

**Theorem 18.** Let $L$ be a hyperprojective logic with semantic theory $T_L$ and $r$-hypersequent calculus $H_L$ with uniform and resource bounded rules. Suppose further that the set of weakly simple formulas which are valid in $T_L$ is in Co-NP. Then the set of theorems of $L$ is in Co-NP.

**Proof.** We show that the set of formulas of $L$ that are not valid is in NP. Let $\phi$ be any such formula. Again, write $\text{Des}(\phi)$ as a conjunction of disjunctions of atomic formulas $\text{Des}_1(\phi), \ldots, \text{Des}_k(\phi)$; then, for some $1 \leq i \leq k$, the reduction tree starting from $\text{Des}_i(\phi)$ has a leaf which is not an axiom. Let $J$ be the maximum cardinality of all multisets occurring in some context of a rule of
L (remember that a context is a disjunction of atomic formulas of the form \(Q(\mu_1, \ldots, \mu_n)\), where each \(\mu_i\) is a multiset of the form \(\phi_1 \oplus \cdots \oplus \phi_h\), with \(\phi_1, \ldots, \phi_h\) formulas of \(L'\)). Let \(K\) be the sum of the arities of all predicates occurring in a context of some rule (if a predicate occurs \(n\) times, its arity is multiplied by \(n\)). Note that, due to the fact that each rule is resource-bounded, the total size of a reduced formula does not exceed the size of the main formula. Hence, denoting, for each expression \(E\), the size of \(E\) by \(s(E)\), if \(H\) is the conclusion of a rule and \(H'\) is one of its premises, we have \(s(H') \leq J \cdot K \cdot s(H)\). If we reduce all occurrences of the main formula simultaneously, the situation does not change, because the contexts are the same and hence do not change, and the size of reduced formulas does not exceed the size of the main formula after performing all possible reductions. Hence, if the length of a branch starting from \(\text{Des}_i(\phi)\) is \(I\), the maximal size of a node in the branch is bounded by \(s(\text{Des}_i(\phi)) + I \cdot J \cdot K \cdot s(\phi)\). Proving the following claim will yield the desired result.

**Proposition 19.** Each branch of the reduction tree starting from \(\text{Des}_i(\phi)\) has length linear in the size of \(\phi\).

**Proof.** Since we have uniform rules, we reduce all occurrence of a formula together, starting from the formula of highest complexity. So, if a subformula of \(\phi\) is reduced in one node, it does not appear in the nodes above it. Hence, the length of the branch does not exceed the number of subformulas of \(\phi\), and thus it is linear in the size of \(\phi\). \(\square\)

Continuing with the proof of Theorem 18, the total size of a branch starting from \(\text{Des}_i(\phi)\) is bounded by \(s(\text{Des}_i(\phi)) + J \cdot K \cdot M \cdot (s(\phi))^2\), where \(M \cdot s(\phi)\) is a bound for the length of the branch \(I\), by the claim above. Hence, it is possible to guess a branch of the reduction tree and to reach its leaf in polynomial time (since the length of the branch is linear in there size of \(\phi\)). It follows that a non-deterministic polynomial-time algorithm for non-provability in \(L\) is the following:

(1) Guess an \(i\) with \(1 \leq i \leq k\).

(2) Guess a maximal branch in the reduction tree of \(\text{Des}_i(\phi)\) and reach its leaf (in polynomial time). Note that this leaf is a weakly simple formula.

(3) Apply a non-deterministic polynomial algorithm to check if the leaf is not valid in \(T_L\) (by assumption, this algorithm does exist). \(\square\)

**Corollary 20.** Each hyperprojective logic \(L\) having a resource bounded proof system and whose axioms are in Co-NP is Co-NP complete.
Proof. Proposition 14 and Theorem 18 ensure that \( L \) is in co-NP. The claim follows from [24].

Remark 4.1. Although assuming that the axioms are in Co-NP is a sufficient condition for a hyperprojective resource-bounded logic to be in Co-NP, we believe that in a reasonable proof system the axiom set should be in P. R-hypersequent calculi whose axiom set is in Co-NP (but possibly not in P) are the calculus for Weak Nilpotent Minimum WNM of [18] and that for Hajek's BL in [29].

4.3.1 Examples

We discuss some known logics that fall into our framework and prove that they are Co-NP complete.

Projective and semi-projective logics: From the projective definition of connectives it follows that projective and hyperprojective logics have a resource bounded and uniform set of rules. Hence when their axioms are in Co-NP, the logics are Co-NP complete. We discuss below three of them: Gödel and classical logic (both projective logics) and the logic \( RM \) (semi-projective logic).

Uniform and resource-bounded rules for Gödel Logic are (\( \lessdot \) stands for either \(<\) or \(\leq\), uniformly in each rule)

\[
\frac{H|\phi < \psi | \psi \lessdot \chi}{H|\phi \land \psi \lessdot \chi}
\frac{H|\psi \leq \phi | \phi \lessdot \chi}{H|\chi < \phi \land \psi}
\frac{H|\phi < \psi | \phi \lessdot \chi}{H|\phi \lor \psi \lessdot \chi}
\frac{H|\psi \leq \phi | \psi \lessdot \chi}{H|\chi < \phi \lor \psi}
\frac{H|\phi \leq \psi | \psi \lessdot \chi}{H|\phi \lessdot \psi \land \chi}
\frac{H|\psi < \phi | \phi \lessdot 1}{H|\psi \land \phi \lessdot 1}
\frac{H|\phi \lessdot \psi \land \chi}{H|\chi \land \phi \lessdot \psi}
\]

The above rules differ from those of the calculus in [8] and from the \( r \)-hypersequent rules in [17] that are not uniform.

Axioms for Gödel Logic are all \( r \)-hypersequents that either contain \( 0 \leq \phi \) or \( \phi \leq 1 \) or a cycle \( \phi_1 \lessdot_1 \phi_2 \lessdot_2 \phi_3 \ldots \lessdot_n \phi_1 \) where for all \( i \lessdot_i \) is either \(<\) or \(\leq\), and at least one \( \lessdot_i \) is \(\leq\), see [7].

Remark 4.2. Being a two-valued logic, Classical Logic CL is a regular hyperprojective logic with an \( r \)-hypersequent calculus which has a uniform, resource-bounded set of rules. An alternative sequent-style calculus for CL is obtained by adding to the above calculus for Gödel logic axioms of the form \( \phi \leq \psi | \psi \leq \chi \). Note that, in contrast with Gentzen sequent calculus LK [21] for CL, the logical rules above are uniform in the sense of Definition 12 (i.e., left and right rules for the same connective have the same structure).
The next example of an hyperprojective logic is the relevant logic R-Mingle [2], indicated in the following by $RM$. This logic has binary connectives $\&$, $\rightarrow$, $\land$, $\lor$ and unary connective $\neg$, axiomatized Hilbert-style by the following set of formulas:

(B) $(\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))$

(C) $(\phi \rightarrow (\psi \rightarrow \chi) \rightarrow (\psi \rightarrow (\phi \rightarrow \chi)))$

(I) $\phi \rightarrow \phi$

($\& 1$) $\phi \rightarrow (\psi \rightarrow (\phi \& \psi))$

($\& 2$) $\psi \rightarrow (\phi \rightarrow (\phi \& \psi))$

(DIS) $(\phi \land (\psi \lor \chi)) \rightarrow ((\phi \land \psi) \lor (\phi \land \chi))$

(\neg 1) $(\phi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \neg \phi)$

(\neg 2) $\neg \neg \phi \rightarrow \phi$

(\land 1) $(\phi \land \psi) \rightarrow \phi$

(\land 2) $(\phi \land \psi) \rightarrow \psi$

(\land 3) $((\phi \rightarrow \psi) \land (\phi \rightarrow \chi)) \rightarrow (\phi \rightarrow (\psi \land \chi))$

($\lor 1$) $\phi \rightarrow (\phi \lor \psi)$

($\lor 2$) $\psi \rightarrow (\phi \lor \psi)$

($\lor 3$) $((\phi \rightarrow \chi) \land (\psi \rightarrow \chi)) \rightarrow ((\phi \lor \psi) \rightarrow \chi)$

($C$) $\phi \rightarrow (\phi \& \phi)$

($M$) $(\phi \& \phi) \rightarrow \phi$

The rules are modus ponens and adjunction:

$\frac{\phi}{\phi \rightarrow \psi} \quad \frac{\phi \land \psi}{\phi \land \psi}$

A first analytic calculus for this logic was defined in [4] using hypersequents. The rules of this calculus are however not invertible and do not help proving that the validity problem of $RM$ is Co-NP complete (see, e.g., [25] for a proof).

It is a well-known fact that the logic $RM$ has the variety of Sugihara monoids as equivalent algebraic semantics. This variety is generated by the algebra

$\mathbb{Z}_0^- = \langle \mathbb{Z} \setminus \{0\}, \&\,, \rightarrow\,, \lor\,, \land\,, \neg\,, 1 \rangle$

where $\mathbb{Z}$ is the ordered set of integers, $\land$ and $\lor$ are min and max respectively, and the other connectives are defined as follows:

$x \& y = \begin{cases} x \land y & \text{if } |x| = |y| \\
 x & \text{if } |x| < |y| \\
 y & \text{if } |y| < |x| \end{cases}$

$x \rightarrow y = \begin{cases} (\neg x) \lor y & \text{if } x \leq y \\
 (\neg x) \land y & \text{if } y < x \end{cases}$

where $|\cdot|$ is the absolute value function. The following format of the above connectives makes evident the semi-projective nature of $RM$:

$x \& y = \begin{cases} x & \text{if } x < -y \text{ and } y \leq x \\
 y & \text{if } x < -y \text{ and } x < y \\
 y & \text{if } -y \leq x \text{ and } y \leq x \\
 x & \text{if } -y \leq x \text{ and } x < y \end{cases}$

$x \rightarrow y = \begin{cases} -x & \text{if } x \leq y \text{ and } y \leq -x \\
 y & \text{if } x \leq y \text{ and } -x < y \\
 y & \text{if } y < x \text{ and } y \leq -x \\
 -x & \text{if } y < x \text{ and } -x < y \end{cases}$
Hence, $RM$ is semi-projective, and multisets are not needed. The rules for $\lor$, $\land$, $-\lor$ and $-\land$ are as in Nilpotent Minimum logic $NM$ (with $-$ in place of $\neg$, see [18]), and the rules for the connectives $\&$, $-\&$, $\to$ and $-(\to)$ are as follows (having used “,” in place of $\oplus$):

The premises of the rule ($\triangleright$, $\&$, left) are

\[
H | -\psi \leq \phi | \phi < \psi | \phi \triangleright \chi \\
H | \phi < -\psi | \phi < \psi | \phi \triangleright \chi
\]

and the conclusion is $H | \phi \& \psi \triangleright \chi$. The premises of the ($\triangleright$, $\&$, left) rule are:

\[
H | -\psi \leq \phi | \phi < \psi | \chi \triangleright \phi \\
H | \phi < -\psi | \phi < \psi | \chi \triangleright \phi
\]

and the conclusion is $H | \chi \triangleright \phi \& \psi$. The premises of the ($\triangleright$, $-\&$, left) rule are:

\[
H | -\psi \leq \phi | \phi < \psi | \chi \triangleleft -\phi \\
H | \phi < -\psi | \phi < \psi | \chi \triangleleft -\phi
\]

and the conclusion is $H | \chi \triangleleft \phi \& \psi$. The premises of the ($\triangleright$, $\to$, left) rule are:

\[
H | \psi < \phi | -\phi < \psi | -\phi \triangleright \chi \\
H | \phi \leq \psi | -\phi < \psi | -\phi \triangleright \chi
\]

and the conclusion is $H | \phi \rightarrow \psi \triangleright \chi$. The premises of the ($\triangleright$, $\to$, right) rule are:

\[
H | \psi < \phi | -\phi < \psi | \chi \triangleright -\phi \\
H | \phi \leq \psi | -\phi < \psi | \chi \triangleright -\phi
\]

and the conclusion is $H | \chi \triangleright \phi \rightarrow \psi$. The premises of the ($\triangleright$, $-(\to)$, left) rule are:

\[
H | \psi < \phi | -\phi < \psi | \phi \triangleright \chi \\
H | \phi \leq \psi | -\phi < \psi | \phi \triangleright \chi
\]

and the conclusion is $H | -(\phi \to \psi) \triangleright \chi$. The premises of the ($\triangleright$, $-(\to)$, right) rule are:

\[
H | \psi < \phi | -\phi < \psi | \chi \triangleright \phi \\
H | \phi \leq \psi | -\phi < \psi | \chi \triangleright \phi
\]
and the conclusion is $H \models \chi \Longleftrightarrow (\phi \rightarrow \psi)$.

The semantic theory, $T_{RM}$, for $RM$ is just the first-order theory of the structure $\mathbb{Z}_0$ and the designated truth predicate $Des(x) := 1 \leq x$. Clearly, for every formula $\phi$ of $RM$, we have $RM \models \phi$ iff $T_{RM} \models Des(\phi)$.

**Theorem 21.** There is a P-time procedure for deciding whether a weakly simple formula of $T_{RM}$ is valid.

**Proof.** Any weakly simple formula $Q$ of $T_{RM}$ is equivalent to a formula of the form $Q_1 \cap \ldots \cap Q_n$, such that for every $i = 1, \ldots, n$ there is an $m > 0$ such that

$$Q_i = (u_{i1} \triangleleft v_{i1}) \cup \ldots \cup (u_{im} \triangleleft v_{im})$$

where $u_{ij}, v_{ij}$ are either variables or constants or terms of the form $-x$, where $x$ is a variable, and $\triangleleft \in \{\leq, <\}$. Then the negation of $Q$ is the disjunction of conjunctions of atomic formulas of the form $u \triangleleft v$ (since the negation of an atomic formula is equivalent to another atomic formula in $T_{RM}$). Denote by $\Sigma_i$ the set of atomic formulas in $Q_i$, and let $\Sigma_i^\sharp$ be the set of inequalities obtained from $\Sigma_i$ by adding, for each inequality $t \triangleleft s$, the inequality $-s \triangleleft -t$, where $\triangleleft$ is $\leq$ if $\triangleleft$ and is $<$ vice versa, and where we identify $-(s)$ with $s$. It is readily seen that $\Sigma_i^\sharp$ is satisfiable if and only if $\Sigma_i$ is. Hence, $Q$ is valid in $T_{RM}$ if and only if all sets $\Sigma_i^\sharp$ are satisfiable.

To check whether $\Sigma_i^\sharp$ is satisfiable, set $SL(t, t')$ iff $t < t'$ is in $\Sigma_i^\sharp$ and $LE(t, t')$ iff $t \leq t'$ is in $\Sigma_i^\sharp$. Set $s \leq^t t$ if there is a finite sequence $s = u_1, \ldots, u_n = t$ such that for $i = 1, \ldots, n - 1$, either $LE(u_i, u_{i+1})$ or $SL(u_i, u_{i+1})$ and $s \leq^t t$ if there is a sequence $s = u_1, \ldots, u_n = t$ as above, such that in addition for at least one $i$, $SL(u_i, u_{i+1})$. Then $\Sigma_i^\sharp$ is satisfiable in $T_{RM}$ if and only if for every term $t$ in $\Sigma_i^\sharp$, we do not have $t \leq^t t$ or $t \leq^t -t$ and $-t \leq^t t$ (remember that in $\mathbb{Z}_0$ the function $-$ has no fixed point). If these conditions are satisfied, then we may map the set of terms in $\Sigma_i^\sharp$ into $\mathbb{Z}_0$ so that the relations $\leq^t$ and $<^t$, as well as the function $-$, are preserved.

Since the procedure outlined above is polynomial in the size of $Q$, the theorem is proved. \hfill $\square$

**Product, Lukasiewicz and Hajek’s Basic Logic:**

As shown before, Product Logic and Lukasiewicz Logic are examples of regular hyperprojective logics having uniform rules. These rules are easily proved to be resource-bounded. Moreover the set of weakly simple formulas of the corresponding semantic theory is in $P$ (which is the complexity of linear programming [30]), and hence these logics are Co-NP complete.

One of the most important Co-NP complete many-valued logics is Hayek’s
Basic Logic $BL$. We are going to introduce a uniform and resource-bounded proof system for this logic whose axiom set is in P. Note that Vetterlein introduced in [29] a simpler system for $BL$, which is uniform and resource-bounded. However, it is not clear whether his axiom set is in P (it is clearly in Co-NP, because the whole logic $BL$ is in Co-NP).

With reference to the semantic theory for $BL$ described in Example 5, for $\preceq \in \{\ll, \prec, \preceq\}$, we have the rules (we omit $H$ and use “,” in place of $\oplus$):

$\phi \ll \psi | \Gamma, \psi \ll \Delta \quad \psi \leq \phi | \Gamma, \phi \ll \Delta$

$\Gamma, \phi \land \psi \ll \Delta$

$\phi \ll \psi | \Gamma, \phi \ll \Delta \quad \psi \leq \phi | \Gamma \ll \Delta, \phi$

$\Gamma \ll \Delta, \phi \land \psi$

$\phi \ll \psi | \Gamma, \phi \ll \Delta \quad \psi \leq \phi | \Gamma, \psi \ll \Delta$

$\Gamma \ll \Delta, \phi \lor \psi$

$\phi \ll \psi | \Gamma, \phi \ll \Delta \quad \psi \leq \phi | \Gamma \ll \Delta, \phi$

$\Gamma \ll \Delta, \phi \lor \psi$

For the $\&$-rules and for the $\rightarrow$-rules we must distinguish between the case where $\preceq$ is $\ll$ and the case where $\preceq$ is $\preceq$ or $\prec$. If $\preceq$ is $\preceq$ or $\prec$, the rules are as follows: The premises of the ($\ll, \&$, left) rule are

$H | \psi \ll \phi | \Gamma, \phi \ll \Delta$

$H | \phi \ll \psi | \Gamma, \psi \ll \Delta$

$H | \phi \ll \psi | 1^+ \ll \phi, \psi | \Gamma \ll \Delta$

$H | \phi \ll \psi | 1^+ \ll \phi, \psi | 1^+ \ll \Delta, \phi, \psi$

and the conclusion is $H | \Gamma, \phi \& \psi \ll \Delta$.

The premises of the ($\ll, \&$, right) rule are

$H | \psi \ll \phi | \Gamma, 1 \ll \Delta$

$H | \phi \ll \psi | \Gamma, 1 \ll \Delta$

$H | \phi \ll \psi | 1^+ \ll \phi, \psi | \Gamma \ll \Delta$

$H | \phi \ll \psi | 1^+ \ll \phi, \psi | 1^+ \ll \Delta, \phi, \psi$

and the conclusion is $H | \Gamma \ll \phi \& \psi, \Delta$. The rules for $\rightarrow$ are:

$H | \psi \ll \phi | \Gamma, 1 \ll \Delta$

$H | \phi \ll \psi | \Gamma, \psi \ll \Delta$

$H | \phi \ll \psi | \phi \ll \psi | \Gamma, 1^+, \psi \ll \Delta, \phi$

$H | \Gamma, \phi \rightarrow \psi \ll \Delta$

$H | \psi \ll \phi | \Gamma, \phi \ll \Delta, \psi$

$H | \phi \ll \psi | \phi \ll \psi | \Gamma \ll \Delta, \psi$

$H | \phi \ll \psi | \phi \ll \psi | \phi \ll \Delta, 1^+, \psi$

$H | \Gamma \ll \Delta, \phi \rightarrow \psi$

The premises of the rule ($\ll, \&$, left) are

$H | \psi \ll \phi | \phi \ll \chi$

$H | \phi \ll \psi | \psi \ll \chi$

$H | \phi \ll \psi | 1^+ \ll \phi, \psi \ll \chi$

$H | \phi \ll \psi | \phi \ll \psi \ll 1^+ | \phi \ll \chi$

and the conclusion is $H | \phi \& \psi \ll \chi$. The premises of the rule ($\ll, \&$, right) are

$H | \psi \ll \phi | \chi \ll \phi$

$H | \phi \ll \psi | \chi \ll \psi$

$H | \phi \ll \psi | 1^+ \ll \phi, \psi \ll \chi$

$H | \phi \ll \psi | \phi \ll \psi \ll 1^+ | \chi \ll \phi$
and the conclusion is $H \mid \chi \ll \phi \& \psi$. Finally,

$$
\frac{H \mid \psi < \phi \mid 1 \ll \chi \quad H \mid \phi \ll \psi \mid \chi \quad H \mid \phi \leq \psi \mid \psi \ll \phi \mid \psi \ll \chi}{H \mid \phi \rightarrow \psi \ll \chi}
$$

$$
\frac{H \mid \psi < \phi \mid \chi \ll 1 \quad H \mid \phi \ll \psi \mid \chi \ll \psi \quad H \mid \phi \leq \psi \mid \psi \ll \phi \mid \chi \ll \psi}{H \mid \chi \ll \phi \rightarrow \psi}
$$

**Remark 4.3.**

1. The symbol $0^+$ used in Example 5 is omitted in our rules for BL because it is interpreted as the neutral element for sum, and hence, for any multiset $\Gamma$ of formulas, $\Gamma$ and $\Gamma, 0^+$ are interpreted in the same way. Moreover in the $\ll$ rules $0^+$ is not needed: for instance, $\phi \& \psi \ll \alpha$ reduces either to $\phi \ll \alpha$ or to $\psi \ll \alpha$.

2. The rules might be simplified considerably: for instance, in the $\ll$-rules the same reduction corresponds to different contexts, and in the $\ll$ rule for $\rightarrow$, the condition $\Gamma, 1 \ll \Delta$ is impossible and may be deleted. However, we have chosen this more complicate formalization in order to make the rules of each connective uniform.

3. An important advantage of our system w.r.t. that in [29] is that its axiom set is in $P$ (see [15, Lemma 4.5] for a proof).

**Acknowledgments:** M. Bongini is supported by the ERC starting Grant HD-SPCONTR - 306274 and by the International Research Training Group IGDK 1754 of the German Science Found. A. Ciabattoni is supported by the FWF Austrian Science Fund project START Y544-N23.

**References**


