LINEARLY CONSTRAINED NONSMOOTH AND NONCONVEX MINIMIZATION*

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Abstract. Motivated by variational models in continuum mechanics, we introduce a novel algorithm for performing nonsmooth and nonconvex minimizations with linear constraints. We show how this algorithm is actually a natural generalization of well-known non-stationary augmented Lagrangian methods for convex optimization. The relevant features of this approach are its applicability to a large variety of nonsmooth and nonconvex objective functions, its guaranteed global convergence to critical points of the objective energy, and its simplicity of implementation. In fact, the algorithm results in a nested double loop iteration, where in the inner loop an augmented Lagrangian algorithm performs an adaptive finite number of iterations on a fixed quadratic and strictly convex perturbation of the objective energy, while the external loop performs an adaptation of the quadratic perturbation. To show the versatility of this new algorithm, we exemplify how it can be easily used for computing critical points in inverse free-discontinuity variational models, such as the Mumford-Shah functional, and, by doing so, we also derive and analyze new iterative thresholding algorithms.

Key words. Variational models in continuum mechanics, linearly constrained nonconvex and nonsmooth optimization, free-discontinuity problems, iterative thresholding algorithms, convergence analysis.

AMS subject classifications. 49M30, 49M25, 90C26, 52A41, 65J22, 65K10, 68U10, 74S30

1. Introduction.

Minimizers of integrals in calculus of variations typically possess singularities, which arise as the result of the nonsmoothness or nonconvexity of the energy. For certain problems in continuum mechanics, such singularities may represent physically interesting instabilities. Singular minimizers exist that model aspects, for instance, of solid phase transformations and certain modes of fracture. More in general, the arising of singular minimizers has to be expected when the energy functionals to be minimized are *nonconvex*. Relevant examples are given for instance by the most realistic models encountered in the literature of elasticity theory (see e.g., [27, 33, 4]). In this context, local minimizers play a pivotal role, as often evolution of physical phenomena proceeds along such energy critical points. Furthermore, usually the given problems have additional conditions, for instance boundary conditions, to be taken into account, which result in linear constraints to be satisfied by the (local) minimizer. Therefore the appropriate solution of constrained genuinely *nonconvex* problems is of the highest interest as well as the accurate numerical treatment of the singularities which are expected to characterize the minimizers.

In the literature one can find efficient algorithmic solutions for linearly constrained *convex and nonsmooth* minimization, e.g., augmented Lagrangian methods [39, 28, 30], and for linearly constrained *nonconvex* minimization, such as sequentially quadratic

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programming (SQP) or (semi-smooth) Newton methods [38]. Unfortunately, in the latter cases only *smooth objective energies*, usually at least C^2 functionals, can be addressed by algorithms, which are then guaranteed to converge *only locally* around the expected critical point.

A typical case which is in fact not falling into the class of *directly* treatable problems through such mentioned algorithms is the so-called Mumford-Shah functional [34], originally introduced as a model of denoising and segmentation for digital images, and more in general widely used as a schematization of many problems in mathematical physics where both volume (bulk) and surface energies are present. However, the nonsmoothness of the problem is usually circumvented by means of smooth approximations, as recalled in the following.

One of the most popular methods to address the numerical minimization of nonconvex functionals, modelling the more general class of *free-discontinuity problems* [1] including the Mumford-Shah functional, is by Γ-approximation with functionals where the surface energy is more easily handled in terms of a smooth indicator function of the discontinuity set of minimal solutions, as proposed by Ambrosio and Tortorelli in [3]. However, very recent adaptive numerical implementations of this regularized model for brittle fracture simulation [9, 8, 11] showed that such relaxation is highly unstable with respect to the approximation parameters, as it very likely creates several spurious local minimizers, resulting in unreliable simulations, unless the resolution of the adaptive finite element discretization is extremely fine [7]. Hence, in this case, the interplay between smoothing and discretization tends to give nonphysical approximations to the problem, urging to find methods where smoothness may play a less relevant role.
On the other hand, there are also efficient semi-heuristic methods, mainly designed for similar image processing models as the Mumford-Shah variational formulation, to seek for *global minimizers* of nonconvex energies. In these cases, global optimization can be addressed using either stochastic algorithms, such as the *simulated annealing* (SA) [31], or continuation-based deterministic relaxations, such as the *graduated nonconvexity* (GNC) pioneered by Blake and Zisserman [5], see also recent developments in [37, 35] and references therein. Being of some conceptual relevance for the scope of this paper, we mention how this latter technique works. For a suitable parameter $\epsilon \in [0, 1]$, one considers a continuous family of smoother objectives $\mathcal{J}^\epsilon$ such that $\lim_{\epsilon \to 1} \mathcal{J}^\epsilon = \mathcal{J}$ (at least pointwise), where $\mathcal{J}$ is the nonconvex energy to be minimized. Then one addresses the global minimization of $\mathcal{J}$ by iterated *local* minimizations along $\mathcal{J}^\epsilon$ when ϵ is increasing from 0 to 1 with a strictly convex initial $\mathcal{J}^0$. More formally, we consider an increasing sequence $(\epsilon_n)_{n\in\mathbb{N}}$, with $\epsilon_0 = 0$ and $\lim_n \epsilon_n = 1$ and the algorithm

$$v^{n+1} = \arg \min_{v \in \mathcal{N}_{\epsilon_n}(v^n)} \mathcal{J}^{\epsilon_n}(v), \tag{1.1}$$

where $\mathcal{N}_{\epsilon_n}(v^n)$ is a suitable neighborhood of the previous iteration v^n of size possibly depending on ϵ_n. While such semi-heuristic algorithms perform very well in practice, usually they do not provide eventually any guarantee for global convergence and their applicability highly depends on the appropriate design of the approximating family $\{\mathcal{J}^\epsilon : \epsilon \in [0, 1]\}$, depending on the particular application and form of $\mathcal{J}$.

Summarizing, both the interest in rigorous and general algorithmic procedures to find constrained critical points of nonsmooth and nonconvex energies, and the relevant applications to free-discontinuity problems, lead us to the motivation of this paper. Its first goal is to propose a *very general* and *simple* iterative algorithm to solve nonsmooth and nonconvex optimization problems with linear constraints. For nonsmoothness we mean that we require our objective function to be in general only a locally Lipschitz function, contrary to the much more restrictive C^2 regularity usually requested by most of the known methods for providing convergence guarantees. Moreover, as one of the most relevant features of our iteration, we will show its *global* convergence, i.e., the initial state does not need to be in a neighborhood of a critical point. Our algorithm is in fact an appropriate combination of the above mentioned techniques, resulting in a nested double loop iteration, where in the inner loop an augmented Lagrangian algorithm or Bregman iteration performs an adaptive finite number of iterations on a fixed *local* quadratic perturbation of the objective energy around the previous iteration, while the external loop performs an adaptation of the quadratic perturbation, similarly to SQP. Moreover, conceptually, our algorithm is reminiscent of the continuation-based deterministic relaxation (1.1), but it will have more general applicability and stronger convergence guarantees, providing also some rigorous justification to those semi-heuristic methods.

In the second part of this paper, we show the versatility of this algorithm by demonstrating how to use it for addressing free-discontinuity problems, in particular the minimization of the Mumford-Shah functional. We show how to reformulate the Mumford-Shah minimization problem into a linearly constrained nonsmooth and nonconvex minimization involving truncated polynomial energy terms, for which the *inner loop*, i.e., the augmented Lagrangian feature of our algorithm, can be realized by means of an *iterative thresholding algorithm*. This technique has been as first proposed in [26] to solve inverse free-discontinuity problems in one dimension, where no approximate smoothing of the energy was used, contrary to other previous approaches, e.g., based on graduated nonconvexity [5, 37, 35]. The extension we provide in this paper allows us now to similarly address problems, which are defined in any dimension, thanks to the appropriate handling of corresponding linear constraints.
Thresholding algorithms have by now a long history of successes, based on their extremely simple implementation, their statistical properties, and, in the iterative case, strong convergence guarantees. We retrace briefly some of the relevant developments, without the intention of providing an exhaustive mention of the many contributions in this area. The terminology "thresholding" comes from image and signal processing, however the associated mathematical concept is the *Moreau proximity map* [14], well-known from convex optimization. The statistical theory of thresholding has been pioneered by Dohono and Johnstone [19] in signal and image denoising and further and extensively explored in other work, e.g., [12]. *Iterative soft-thresholding algorithms* to numerically solve the minimization of convex energies, modelling inverse problems and formed by quadratic fidelity terms and ℓ_p-norm penalties, for $p \geq 1$, have been first proposed in [23]. Their strong convergence has been proven in the seminal work of Daubechies, Defrise, and De Mol [16]. The recent theory of *compressed sensing*, i.e., the universal and nonadaptive compressed acquisition of data [10, 18], stimulated also the research of iterative thresholding algorithms for nonconvex penalty terms, such as the ℓ_p-quasinorms for $0 < p < 1$. Variational and convergence properties of *iterative firm-thresholding* algorithms, in particular the *iterative hard-thresholding*, have been recently studied in [6, 25]. Partially inspired by these latter achievements and the work of Nikolova [36] on the relationships between certain thresholding operators and discrete Mumford-Shah functionals, the results in [26] and in the present paper should be also considered as a contribution to the theory of thresholding algorithms in the

new context of linearly constrained nonsmooth and nonconvex optimization.
While in this paper we are limiting ourself to present the application to the minimization of Mumford-Shah type of functionals, mainly for their typical hard features of nonconvexity and nonsmoothness, in our view the algorithm we study in this work will have significant further numerical applications in several problems involving nonsmooth and nonconvex energies with additional linear (boundary) conditions, as in fracture propagation [7, 27], in elasto-plastic evolutions [33], and in atomic structure computation [4].

The paper is organized as follows. In Section 2 we define an appropriate concept of constrained critical points for certain classes of nonconvex functionals. We introduce then our new algorithm for the solution of nonsmooth and nonconvex minimization with linear constraints and we prove its convergence to critical points. Section 3 is addressed to showing how the general algorithm can be used for the solution of inverse free-discontinuity problems, starting with the reformulation of the classical discrete version of the Mumford-Shah model in terms of a nonsmooth and nonconvex optimization with linear constraints. In Section 4 we show how the core of the algorithm for inverse free-discontinuity problems can actually be realized as an iterative thresholding algorithm. Besides its relevance in terms of simplicity of implementation, the insight provided by certain segmentation properties of such iterations actually allows us to verify all the necessary conditions for the algorithm to converge. For ease of reading, we collect some of the technical results in a concluding Appendix.

2. Linearly Constrained Nonsmooth and Nonconvex Minimization.

2.1. Preliminaries and assumptions. Let $\mathcal{H}$ be a *finite dimensional* Hilbert space and $\mathcal{J}\colon \mathcal{H} \to \mathbb{R}$ a lower semicontinuous functional which we assume to be bounded from below. Since we will be concerned with the search of critical points, without any loss of generality we shall suppose from now on that $\mathcal{J}(v) \geq 0$, for all $v \in \mathcal{H}$. Let $\mathcal{H}_1$ be another finite dimensional Hilbert space and we further consider a linear operator $A\colon \mathcal{H} \to \mathcal{H}_1$. Both the spaces $\mathcal{H}$ and $\mathcal{H}_1$ are endowed with an Hilbertian norm, which we will denote in both cases with $\|\cdot\|$, since it will be always clear from the context in which space we are taking the norm. Dealing with finite dimensional spaces, it remains understood that the only notion of convergence that we will use is the strong convergence in norm, since weak and strong topologies are in this case equivalent.

About the operator A, we shall assume that A has *nontrivial kernel*, and is *surjective*. We shall denote by $A^*\colon \mathcal{H}_1 \to \mathcal{H}$ the adjoint operator of A. By our assumptions, for every $w \in \mathcal{H}_1$ we have that there exists $\delta > 0$ such that

$$\|A^* w\| \geq \delta \|w\| \,. \tag{2.1}$$

We consider $f \in \mathcal{H}_1$ and we are concerned with the problem of finding constrained critical points of $\mathcal{J}$ on the affine space $\mathcal{F}(f) := \{v \in \mathcal{H} : Av = f\}$. As usual in nonsmooth analysis, the notion of critical point is defined via the use of subdifferentiation.

DEFINITION 2.1. *Let $\mathcal{H}$ be an Hilbert space, $\mathcal{J}\colon \mathcal{H} \to \mathbb{R}$ a lower semicontinuous functional, and $v \in \mathcal{H}$. We say that $\xi \in \mathcal{H}' \simeq \mathcal{H}$ belongs to the subdifferential $\partial\mathcal{J}(v)$ of $\mathcal{J}$ at v if and only if*

$$\liminf_{w \to v} \frac{\mathcal{J}(w) - (\mathcal{J}(v) + \langle \xi, w - v\rangle)}{\|w - v\|} \geq 0 \,. \tag{2.2}$$

The subdifferential can be in general empty or multivalued. It is well-known (see, for instance [2, Chapter 1]) that it is a closed convex set. In the case where $\mathcal{J}$ is convex, it is nonempty at every point and it can be shown (see again [2, Proposition 1.4.4]) that

$$\xi \in \partial \mathcal{J}(v) \text{ if and only if } \mathcal{J}(w) - (\mathcal{J}(v) + \langle \xi, w - v \rangle) \geq 0 \tag{2.3}$$

for every $w \in \mathcal{H}$. In the case of a C^1 perturbation of a lower semincontinuous functional, that is $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$ where $\mathcal{J}_1$ is lower semicontinuous, and $\mathcal{J}_2$ is of class C^1, it follows from the definition that if $\partial \mathcal{J}_1(v)$ is nonempty, then $\partial \mathcal{J}(v) \neq \emptyset$ and the decomposition

$$\partial \mathcal{J}(v) = \partial \mathcal{J}_1(v) + D\mathcal{J}_2(v) \,, \tag{2.4}$$

holds true. Here D denotes the Fréchet differential of $\mathcal{J}_2$ at v. In particular, C^1-perturbations of lower semicontinuous convex functionals have nonempty subdifferential at every point. We collect in the following Remark some useful properties of the subdifferential that will be employed in the sequel.

REMARK 2.2. *If $\mathcal{J}$ is a C^1-perturbation of a convex function, one proves that the subdifferential enjoys the following closure property:*

$$\xi_n \in \partial \mathcal{J}(v_n), \quad v_n \to v, \quad \xi_n \to \xi \textit{ implies } \xi \in \partial \mathcal{J}(v) \textit{ and } \mathcal{J}(v_n) \to \mathcal{J}(v) \,. \tag{2.5}$$

The subdifferential of a convex function $\mathcal{J}$ is known to be a monotone operator [22], that is, for every v and $w \in \mathcal{H}$

$$\xi \in \partial \mathcal{J}(v) \textit{ and } \omega \in \partial \mathcal{J}(w) \textit{ implies } \langle \xi - \omega, v - w \rangle \geq 0 \,. \tag{2.6}$$

We shall say that a function is *ν-strongly convex* if a stronger form of (2.6) holds, that is there exists $\nu > 0$ such that

$$\xi \in \partial \mathcal{J}(v) \text{ and } \omega \in \partial \mathcal{J}(w) \text{ implies } \langle \xi - \omega, v - w \rangle \geq \nu \|v - w\|^2 \,. \tag{2.7}$$

It is well-known that this is equivalent to saying that $\mathcal{J}(\cdot) - \frac{\nu}{2}\|\cdot\|^2$ is convex.

We are now ready to recall the definition of critical point and constrained critical point.

DEFINITION 2.3. *Let $\mathcal{H}$ be an Hilbert space, $\mathcal{J} \colon \mathcal{H} \to \mathbb{R}$ a lower semicontinuous functional, and $v \in \mathcal{H}$. We say that v is a critical point of $\mathcal{J}$ if either $\mathcal{J}$ has no subdifferential at v or*

$$0 \in \partial \mathcal{J}(v) \,.$$

Clearly this last condition is the only one of the two to be meaningful whenever $\mathcal{J}$ has nonempty subdifferential at every point. When $\mathcal{J}$ is convex it is sufficient to assure global minimality of v, otherwise it is only a necessary condition for local minimality.

DEFINITION 2.4. *Given a linear operator $A \colon \mathcal{H} \to \mathcal{H}_1$ with nontrivial kernel, and $f \in \mathcal{H}_1$, we say that w is a critical point of $\mathcal{J}$ on the affine space $\mathcal{F}(f) = \{v \in \mathcal{H} : Av = f\}$ if $Aw = f$ and 0 is a critical point for the restriction to $\ker A$ of the functional $\mathcal{J}(w + \cdot)$.*

For $\mathcal{J}$ being a C^1-perturbation of a convex function (in particular, with nonempty subdifferential at every point), the nonsmooth version of Lagrange multiplier Theorem assures that w is a critical point of $\mathcal{J}$ on the affine space $\{v \in \mathcal{H} : Av = f\}$ if and only if $Aw = f$ and

$$\partial \mathcal{J}(w) \cap \operatorname{ran}(A^*) \neq \emptyset , \tag{2.8}$$

where $\operatorname{ran}(A^*)$ is the range of the operator A^* which is known to be the orthogonal complement of $\ker A$ in $\mathcal{H}$.

From now, about the function $\mathcal{J}$, we will make the following more specific assumptions:

(A1) $\mathcal{J}$ is ω-convex, that is there exists $\omega > 0$ such that $\mathcal{J}(\cdot) + \omega\| \cdot \|^2$ is convex;

(A2) the subdifferential of $\mathcal{J}$ satisfies the following growth condition: there exist two nonnegative numbers K, L such that, for every $v \in \mathcal{H}$ and $\xi \in \partial \mathcal{J}(v)$

$$\|\xi\| \leq K\mathcal{J}(v) + L . \tag{2.9}$$

REMARK 2.5. *(a) We observe that condition (A1) is in fact met, for instance, by any C^1 function in finite dimension with piecewise continuous and bounded second derivatives. However, let us stress that, conversely, ω-convexity does not give any information on the smoothness of the function, other than local Lispchitzianity, hence, in finite dimension, its Fréchet-differentiability almost everywhere, by Rademacher's Theorem. We also recall that an ω-convex function is a C^1-perturbation of a convex function, therefore it has nonempty (and locally bounded) subdifferential at every point. If the subdifferential is uniformly bounded, then* (2.9) *is trivially satisfied.*

(b) As just recalled, an ω-convex function in finite dimension has a Fréchet differential almost everywhere, and, if (2.9) *is satisfied only at points of differentiability, then it holds everywhere. This is true since it can be shown that the Fréchet subdifferential is contained in the so-called Clarke subdifferential, which is known to be at every $v \in \mathcal{H}$ the convex hull of limit points of differentials of $\mathcal{J}$ along sequences $v_n \to v$ (for these notions, see for instance [13, Chapter 2]). Therefore one needs not to calculate the subdifferential of $\mathcal{J}$ at non-differentiability points (which is in general quite a hard task) to check if the hypothesis is satisfied everywhere.*

Given $\omega > 0$, and $u \in \mathcal{H}$ we will denote

$$\mathcal{J}_{\omega,u}(v) := \mathcal{J}(v) + \omega\|v - u\|^2 \tag{2.10}$$

Notice that $\mathcal{J}_{\omega,u}$ is coercive whenever $\mathcal{J}$ is bounded from below. We observe that, if $\mathcal{J}$ satisfies (A1) we can always assume that ω is chosen in such a way that $\mathcal{J}_{\omega,u}$ is also ν-strongly convex with ν depending on $\mathcal{J}$ and ω, but not on u. Analogously, if (A1) and (A2) are satisfied, by using (2.4) it is easy to see that $\mathcal{J}_{\omega,u}$ satisfies (2.9) with two constants $\tilde{K}, \tilde{L}$ depending again on $\mathcal{J}$ and ω, but not on u.

2.2. The augmented Lagrangian algorithm in the convex case. We now recall some basic facts about augmented Lagrangian iterations for constrained minimization of convex functionals. Here, we are given a *coercive convex functional* $\tilde{\mathcal{J}}$ and, given two arbitrary $v_0 \in \mathcal{H}$, $q_0 \in \mathcal{H}_1$ such that $A^* q_0 \in \partial \mathcal{J}(v_0)$, for every $k \in \mathbb{N}$, $k \geq 1$, we define:

$$\begin{cases} v_k \in \arg\min_{v \in \mathcal{H}} (\tilde{\mathcal{J}}(v) - \langle q_k, Av \rangle + \lambda \|Av - f\|^2) . \\ q_k = q_{k-1} + 2\lambda(f - Av_k) . \end{cases} \tag{2.11}$$

Convergence of the algorithm has been proved in [39], where it was called *Bregman iteration*, and, since it is equivalent to the *Augmented Lagrangian Method* [30], also in [28]. Precisely it has been shown that $\|Av_k - f\|$ decreases to 0 as k tends to $+\infty$, that the sequence v_k is compact and any limit point is a global minimum of $\tilde{\mathcal{J}}$ under the constraint $Av = f$. Moreover, for every k, $A^*q_k \in \partial\mathcal{J}(v_k)$. When $\tilde{\mathcal{J}}$ is ν-strongly convex for some $\nu > 0$ we have also a quantitative estimate of the convergence of v_k to the unique (due to strict convexity) minimizer of the problem. We give a precise statement and a proof of this additional property, as it will be very useful later in the nonconvex case as well.

PROPOSITION 2.6. *Assume that $\tilde{\mathcal{J}}$ is ν-strongly convex, let v_k and q_k the sequences generated by* (2.11)*, and let $\bar{v}$ the unique global minimizer of $\tilde{\mathcal{J}}$ on the affine space $\{v \in \mathcal{H} : Av = f\}$. Then:*

(i) $(\|Av_k - f\|)_{k\in\mathbb{N}}$ *is a decreasing sequence;*
(ii) $\lim_{k\to+\infty} \|Av_k - f\| = 0$*;*
(iii) $\|v_k - \bar{v}\|^2 \leq \frac{1}{\nu}\|q_0 - \bar{q}\|\,\|Av_k - f\|$*, for all $k \in \mathbb{N}$,*

for every $\bar{q} \in \mathcal{H}_1$ such that $A^*\bar{q} \in \partial\mathcal{J}(\bar{u})$.

Proof. Properties (i) and (ii) are proved in [39]. For the property (iii), we first observe that such a $\bar{q}$ surely exists by (2.8). We define $\Delta q_k := q_k - \bar{q}$ and we prove that $\|\Delta q_k\|$ is decreasing. We actually have, by elementary computations and using (2.11), that

$$\|\Delta q_k\|^2 - \|\Delta q_{k-1}\|^2 \leq 2\langle q_k - q_{k-1}, q_k - \bar{q}\rangle = \\ 4\lambda\langle f - Av_k, q_k - \bar{q}\rangle = 4\lambda\langle \bar{v} - v_k, A^*q_k - A^*\bar{q}\rangle\,.$$

Since $A^*q_k \in \partial\mathcal{J}(v_k)$ and $A^*\bar{q} \in \partial\mathcal{J}(\bar{u})$, the last term in the inequality is nonpositive by (2.6), therefore the claim follows. In particular

$$\|q_k - \bar{q}\| \leq \|q_0 - \bar{q}\|\,. \tag{2.12}$$

Now, by (2.7), we have also

$$\nu\|v_k - \bar{v}\|^2 \leq \langle A^*q_k - A^*\bar{q}, v_k - \bar{v}\rangle = \langle q_k - \bar{q}, Av_k - f\rangle\,,$$

so that we conclude by the Cauchy-Schwarz inequality and (2.12). □

When $\tilde{\mathcal{J}}$ is the function $\mathcal{J}_{\omega,u}$ defined by (2.10), with an appropriate choice of ω, by the previous result, (2.1), and (2.9), we get the following corollary, whose rather immediate proof is therefore omitted.

COROLLARY 2.7. *Consider the function $\mathcal{J}_{\omega,u}$ defined by* (2.10)*, where ω is chosen in such a way that $\mathcal{J}_{\omega,u}$ is ν-strongly convex with ν not depending on u. Let $\bar{v}_u$ the unique global minimizer of $\mathcal{J}_{\omega,u}$ on the affine space $\{v \in \mathcal{H} : Av = f\}$. Then there exist two positive constants C_1 and C_2 depending on A^*, $\mathcal{J}$, and ω, but not on u, such that*

$$\|v_{k,u} - \bar{v}_u\|^2 \leq [C_1(1 + \|q_0\|) + C_2\mathcal{J}_{\omega,u}(\bar{v}_u)]\,\|Av_{k,u} - f\|\,, \tag{2.13}$$

where $v_{k,u} := v_k$ is defined accordingly to (2.11) *for $\tilde{\mathcal{J}} = \mathcal{J}_{\omega,u}$.*

2.3. The algorithm in the nonconvex case. We now present the new algorithm for linearly constrained nonsmooth and nonconvex minimization, and discuss its convergence properties. We pick initial $v_{(0,0)} \in \mathcal{H}$ and $q_{(0,0)} \in \mathcal{H}_1$. Notice that there is no restriction to any specific neighborhood for the choice of the initial iteration. For a fixed scaling parameter $\lambda > 0$, and an *adaptively chosen* sequence of integers $(L_\ell)_{\ell\in\mathbb{N}}$, for every integer $\ell \geq 1$ we set (with the convention $L_0 = 0$):

$$\begin{cases} v_{(\ell,0)} = v_{\ell-1} := v_{(\ell-1,L_{\ell-1})} \quad q_{(\ell,0)} = q_{\ell-1} := q_{(\ell-1,L_{\ell-1})} \\ v_{(\ell,k)} = \arg\min_{v\in\mathcal{H}} \left(\mathcal{J}_{\omega,v_{\ell-1}}(v) - \langle q_{(\ell,k-1)}, Av\rangle + \lambda\|Av - f\|^2\right), \quad k = 1,\dots,L_\ell \\ q_{(\ell,k)} = q_{(\ell,k-1)} + 2\lambda(f - Av_{(\ell,k)})\,. \end{cases} \tag{2.14}$$

Here, thanks to condition (A1), ω is chosen in such a way that $\mathcal{J}_{\omega,v_{\ell-1}}$ is ν-strongly convex, with ν independent of $v_{\ell-1}$, and the *finite* number of inner iterates L_ℓ is defined by the condition

$$(1 + \|q_{\ell-1}\|)\|Av_{(\ell,L_\ell)} - f\| \leq \frac{1}{\ell^\alpha}\,, \quad \alpha > 1\,. \tag{2.15}$$

Since the inner loops are simply the augmented Lagrangian iterations for the functional $\mathcal{J}_{\omega,v_{\ell-1}}$, by Proposition 2.6 (ii) and (2.12) such an integer L_ℓ always exists. We also remark that by construction, for every $\ell \geq 1$ and $k = 0,\dots,L_\ell$ with the only possible exception of $q_{(1,0)}$, we have

$$A^* q_{(\ell,k)} \in \partial\mathcal{J}_{\omega,v_{\ell-1}}(v_{(\ell,k)})\,. \tag{2.16}$$

Moreover, for every $\ell \geq 1$, again by Proposition 2.6, $\|Av_{(\ell,k)} - f\|$ is nonincreasing in k.

Let us also remark that (2.14) is actually a natural generalization of (2.11), as if $\mathcal{J}$ were convex, we could in fact choose $\omega = 0$, and (2.14) would simply reduce to (2.11).

2.4. Analysis of convergence. We now want to analyse the convergence properties of the algorithm defined by (2.14). To do that we will use the following basic calculus lemma.

LEMMA 2.8. *Let $(a_\ell)_{\ell\in\mathbb{N}}$ a sequence of positive numbers, and let $(\delta_\ell)_{\ell\in\mathbb{N}}$ a positive decreasing sequence such that*

$$\sum_{\ell=0}^{\infty} \delta_\ell < +\infty\,.$$

If a_ℓ satisfies for every ℓ the inequality

$$a_\ell \leq (1 + \delta_{\ell-1})a_{\ell-1} + \delta_{\ell-1}\,, \tag{2.17}$$

then $(a_\ell)_{\ell\in\mathbb{N}}$ is a convergent sequence.

Proof. By the recurrence relation (2.17) we deduce

$$a_\ell \leq \left[\prod_{k=0}^{\ell-1}(1+\delta_k)\right] a_0 + \sum_{\ell'=0}^{\ell-1}\left[\prod_{k=\ell'+1}^{\ell-1}(1+\delta_k)\right]\delta_{\ell'}. \tag{2.18}$$

Notice that

$$\begin{aligned} \log\left[\prod_{k=0}^{\infty}(1+\delta_k)\right] &= \sum_{k=0}^{\infty}\log(1+\delta_k) \\ &= \sum_{k=0}^{\infty}\left(\delta_k - \frac{1}{2\xi_k}\delta_k^2\right) < \infty, \end{aligned} \tag{2.19}$$

for suitable $\xi_k \in (1, 1+\delta_k)$, for $k \in \mathbb{N}$, hence

$$\prod_{k=0}^{\infty}(1+\delta_k) < \infty,$$

and, together with (2.18), we deduce that $(a_\ell)_{\ell\in\mathbb{N}}$ is actually uniformly bounded. Now, again by the recurrence relation (2.17), for $k' \le k$, we obtain

$$a_k = a_{k'} + \sum_{\ell=k'+1}^{k}(a_\ell - a_{\ell-1}) \le a_{k'} + \sum_{\ell=k'+1}^{k}\delta_{\ell-1}a_{\ell-1} + \sum_{\ell=k'+1}^{k}\delta_{\ell-1}.$$

Taking first the lim sup as $k \to +\infty$ and then the lim inf as $k' \to +\infty$ in the previous inequality, we conclude from the boundedness of $(a_\ell)_{\ell\in\mathbb{N}}$ and the convergence of the series $\sum_{\ell=0}^{\infty}\delta_\ell$ that $\limsup_{k\to+\infty} a_k \le \liminf_{k'\to+\infty} a_{k'}$, which implies the conclusion. □

In the following theorem we analyse the convergence properties of the proposed algorithm.

THEOREM 2.9. *Assume that $\mathcal{J}$ satisfies (A1) and (A2), and let $(v_\ell)_{\ell\in\mathbb{N}}$ be the sequence generated by* (2.14). *Then,*

(a) $(Av_\ell - f) \to 0$ *as* $\ell \to \infty$;

(b) $(v_\ell - v_{\ell-1}) \to 0$ *as* $\ell \to \infty$.

If in addition $\mathcal{J}$ is coercive then v_ℓ is bounded and $(\mathcal{J}(v_\ell))_{\ell\in\mathbb{N}}$ is a convergent sequence. More in general, if $\mathcal{J}$ only satisfies (A1) and (A2), the implication

$$\text{if } (v_\ell)_{\ell\in\mathbb{N}} \text{ is a bounded sequence, then } (\mathcal{J}(v_\ell))_{\ell\in\mathbb{N}} \text{ is convergent} \tag{2.20}$$

holds.

Proof. Part (a) of the statement is a direct consequence of the construction of v_ℓ and Proposition 2.6 (ii). We now set for every ℓ

$$\bar{v}_\ell := \arg\min_{Av=f}\mathcal{J}_{\omega,v_{\ell-1}}(v)\,. \tag{2.21}$$

By (2.13) with $u = v_{\ell-1}$, $k = L_\ell$, and $q_0 = q_{(\ell,0)}$, and by (2.14) and (2.15), we have that there exist two positive constants C_1 and C_2 independent of ℓ, such that

$$\|v_\ell - \bar{v}_\ell\|^2 \le [C_1 + C_2\mathcal{J}_{\omega,v_{\ell-1}}(\bar{v}_\ell)]\frac{1}{\ell^\alpha}\,. \tag{2.22}$$

By this latter estimate and the minimality of $\bar{v}_{\ell+1}$ we get

$$\begin{aligned}\mathcal{J}_{\omega,v_\ell}(\bar{v}_{\ell+1}) &= \mathcal{J}(\bar{v}_{\ell+1}) + \omega\|v_\ell - \bar{v}_{\ell+1}\|^2\\ &\le \mathcal{J}(\bar{v}_\ell) + \omega\|v_\ell - \bar{v}_\ell\|^2 \le \mathcal{J}(\bar{v}_\ell) + \frac{C_1\omega}{\ell^\alpha} + \frac{C_2\omega}{\ell^\alpha}\mathcal{J}_{\omega,v_{\ell-1}}(\bar{v}_\ell)\\ &\le \frac{C_1\omega}{\ell^\alpha} + \Big(1 + \frac{C_2\omega}{\ell^\alpha}\Big)\mathcal{J}_{\omega,v_{\ell-1}}(\bar{v}_\ell)\,.\end{aligned} \tag{2.23}$$

By Lemma 2.8 we eventually deduce that $(\mathcal{J}_{\omega,v_{\ell-1}}(\bar{v}_\ell))_{\ell\in\mathbb{N}}$ is a convergent sequence, in particular it is bounded. Therefore, there exists a constant C independent of ℓ such that, by (2.22),

$$\|v_{\ell+1} - \bar{v}_{\ell+1}\|^2 \le \frac{C}{(\ell+1)^\alpha}\,, \tag{2.24}$$

and, by (2.23), we have also

$$\mathcal{J}(\bar{v}_{\ell+1}) \leq \mathcal{J}(\bar{v}_{\ell+1}) + \omega\|v_\ell - \bar{v}_{\ell+1}\|^2 \leq \mathcal{J}(\bar{v}_\ell) + \frac{C}{\ell^\alpha}\,. \tag{2.25}$$

Again Lemma 2.8 entails now that

$$\mathcal{J}(\bar{v}_\ell) \text{ is a convergent sequence}\,, \tag{2.26}$$

so that, by (2.25) we get that $(v_\ell - \bar{v}_{\ell+1}) \to 0$ as ℓ goes to $+\infty$, and this vanishing convergence, combined with (2.24), gives part (b) of the statement.

Being $\mathcal{J}$ locally Lipschitz as it is an ω-convex function, if v_ℓ is uniformly bounded, by (2.24) and (2.26) we immediately conclude that $(\mathcal{J}(v_\ell))_{\ell\in\mathbb{N}}$ is a convergent sequence. Moreover, if $\mathcal{J}$ is coercive, then $\bar{v}_\ell$ is bounded by (2.26), and so is also $(v_\ell)_{\ell\in\mathbb{N}}$ by (2.24), as required. □

As a consequence we get our main result of this section. Whenever v_ℓ is bounded, every cluster point is a constrained critical point of $\mathcal{J}$ on the affine space $\{v \in \mathcal{H} : Av = f\}$. We again recall that boundedness of v_ℓ is guaranteed by Theorem 2.9 when $\mathcal{J}$ is assumed to be coercive.

THEOREM 2.10. *Assume that $\mathcal{J}$ satisfies (A1) and (A2), and let $(v_\ell)_{\ell\in\mathbb{N}}$ be the sequence generated by* (2.14). *If $(v_\ell)_{\ell\in\mathbb{N}}$ is bounded, every of its limit points is a constrained critical point of $\mathcal{J}$ on the affine space $\{v \in \mathcal{H} : Av = f\}$.*

Proof. Let $(q_\ell)_{\ell\in\mathbb{N}}$ be the sequence defined by (2.14), and let $p_\ell := A^* q_\ell$, and $\hat{p}_\ell := p_\ell - 2\omega(v_\ell - v_{\ell-1})$. By (2.4) and (2.16), we have

$$\hat{p}_\ell \in \partial\mathcal{J}(v_\ell)\,, \tag{2.27}$$

and by the boundedness of $(v_\ell)_{\ell\in\mathbb{N}}$ and the local Lipschitz continuity of $\mathcal{J}$ we then get that $\hat{p}_\ell$ is bounded too. By Theorem 2.9, part (b), we deduce that $p_\ell - \hat{p}_\ell \to 0$, which in particular gives

$$\lim_{\ell\to+\infty} \operatorname{dist}(\hat{p}_\ell\,, \operatorname{ran}(A^*)) = 0\,. \tag{2.28}$$

Now, if a subsequence $v_{\ell_j} \to v \in \mathcal{H}$, possibly taking a further subsequence we may assume that $\hat{p}_{\ell_j} \to \hat{p} \in \partial\mathcal{J}(v)$, where the last inclusion follows from (2.5) and (2.27). Moreover, since in finite dimension $\operatorname{ran}(A^*)$ is closed, by (2.28) $\hat{p} \in \operatorname{ran}(A^*)$. Since $Av = f$ by part (a) of Theorem 2.9, (2.8) yields now the desired conclusion. □

3. The Application to Free-Discontinuity Problems.

3.1. The Mumford-Shah problem and its reformulation as a finite dimensional linearly constrained nonconvex minimization. The terminology 'free-discontinuity problem' was introduced by De Giorgi [17] to indicate a class of variational problems that consist in the minimization of a functional, involving both volume and surface energies, depending on a closed set $K \subset \mathbb{R}^d$, and a function u on $\mathbb{R}^d$ usually smooth outside of K. In particular,

- K is not fixed a priori and is an unknown of the problem;
- K is not a boundary in general, but a free-surface inside the domain of the problem.

The best-known example of a free-discontinuity problem is the one modelled by the so-called Mumford-Shah functional [34], which is defined by

$$J(u,K) := \int_{\Omega\setminus K} \left[|\nabla u|^2 + \alpha(u-g)^2\right] dx + \beta\mathcal{H}^{d-1}(K\cap\Omega).$$

The set Ω is a bounded open subset of $\mathbb{R}^d$, $\alpha, \beta > 0$ are fixed constants, and $g \in L^\infty(\Omega)$. Here $\mathcal{H}^N$ denotes the N-dimensional Hausdorff measure. Inspired by image processing applications, throughout the rest of this paper, the dimension of the underlying Euclidean space $\mathbb{R}^d$ will always be $d = 2$, although in principle the analysis can be conducted in any dimension. In fact, in the context of visual analysis, g is a given noisy image that we want to approximate by the minimizing function $u \in W^{1,2}(\Omega \setminus K)$; the set K is simultaneously used in order to *segment* the image into connected components. For a broad overview on free-discontinuity problems, their analysis, and applications, we refer the reader to [1].

In fact, the Mumford-Shah functional is the continuous version of a previous discrete formulation of the image segmentation problem proposed by Geman and Geman in [29]; see also the work of Blake and Zisserman in [5]. Let us recall this discrete approach. Let $d = 2$ (as for image processing problems), $\Omega = [0,1]^2$, and let $u_{i,j} = u(hi, hj)$, $(i,j) \in \mathbb{Z}^2$ be a discrete function defined on $\Omega_h := \Omega \cap h\mathbb{Z}^2$, for $h > 0$. Define $W_r^2(t) := \min\left\{t^2, r^2\right\}$, $r > 0$, to be the *truncated quadratic potential*, and

$$\begin{aligned}\mathcal{J}_h(u) := h^2 &\sum_{(hi,hj)\in\Omega_h} W^2_{\sqrt{\frac{\beta}{h}}}\left(\frac{u_{i+1,j}-u_{i,j}}{h}\right)\\ &+ h^2 \sum_{(hi,hj)\in\Omega_h} W^2_{\sqrt{\frac{\beta}{h}}}\left(\frac{u_{i,j+1}-u_{i,j}}{h}\right)\\ &+ \alpha h^2 \sum_{(hi,hj)\in\Omega_h} (u_{i,j}-g_{i,j})^2. \end{aligned} \tag{3.1}$$

We shall now reformulate the minimization of this finite dimensional discrete problem into a linearly constrained minimization of a nonconvex functional of the discrete derivatives. For this purpose, we consider the derivative matrix $D_h : \mathbb{R}^{n^2} \to \mathbb{R}^{2n(n-1)}$ that maps the vector $(u_{j+(i-1)n}) := (u_{i,j})$ to the vector composed of the finite differences in the horizontal and vertical directions u_x and u_y respectively, given by

$$D_h u := \left[\begin{array}{c} u_x \\ u_y \end{array}\right], \quad \left\{\begin{array}{l} (u_x)_{j+n(i-1)} := (u_x)_{i,j} := \frac{u_{i+1,j}-u_{i,j}}{h}, i = 1,\dots,n-1, j = 1,\dots,n \\ (u_y)_{j+(n-1)(i-1)} := (u_y)_{i,j} := \frac{u_{i,j+1}-u_{i,j}}{h}, i = 1,\dots,n, j = 1,\dots,n-1 \end{array}\right. .$$

Note that its range $\operatorname{ran}(D_h) \subset \mathbb{R}^{2n(n-1)}$ is a (n^2-1)-dimensional subspace because $D_h c = 0$ for constant vectors $c \in \mathbb{R}^{n^2}$. It is not difficult to show the representation of any vector $u \in \mathbb{R}^{n^2}$ in terms of the following differentiation-integration formula, given by

$$u = D_h^\dagger D_h u + c,$$

where $D_h^\dagger$ is the pseudo-inverse matrix of D_h (in the Moore-Penrose sense); note that $D_h^\dagger$ maps $\operatorname{ran}(D_h)$ injectively into $\mathbb{R}^{n^2}$. Also, c is a constant vector that depends on u, and the values of its entries coincide with the mean value $h^2 \sum_{(hi,hj)\in\Omega_h} u_{i,j}$ of u.

Therefore, any vector u is uniquely identified by the pair $(D_h u, c)$.

Since constant vectors comprise the null space of D_h, the orthogonality relation

$$\langle D_h^\dagger D_h u, c\rangle = 0 \tag{3.2}$$

holds for any vector u and any constant vector c. Here the scalar product $\langle u, u'\rangle = \sum_{(hi,hj)\in\Omega_h} u_{i,j}u'_{i,j}$ is the standard Euclidean scalar product on $\mathbb{R}^{n^2}$, which induces the Euclidean norm $\|\cdot\|$.

Using the orthogonality property (3.2), we have that

$$\begin{aligned}\|u-g\| &= \|D_h^\dagger D_h u - D_h^\dagger D_h g + (c-c_g)\|\\ &= \|D_h^\dagger D_h - D_h^\dagger D_h g\| + \|c-c_g\|\end{aligned}$$

Hence, with a slight abuse of notation, we can reformulate the original discrete functional (3.1) in terms of derivatives, and mean values, by

$$\mathcal{J}_h(v,c) = h^2\big[\alpha\|D_h^\dagger v - \tilde{g}\|^2 + \alpha\|c-c_g\| + \sum_{i,j}\min\left\{|v_{i,j}|^2, \frac{\beta}{h}\right\}\big].$$

where $v = D_h u \in \mathbb{R}^{2n(n-1)}$, and $\tilde{g} = D_h^\dagger D_h g \in \mathbb{R}^{n^2}$. Of course $c = c_g$ is assumed at the minimizer u, since the corresponding term in $\mathcal{J}_h$ does not depend on z. However, in order to minimize only over vectors in $\mathbb{R}^{2n(n-1)}$ that are derivatives of vectors in $\mathbb{R}^{n^2}$, we must minimize $\mathcal{J}_h(v,c)$ subject to the constraint $(D_h D_h^\dagger - I)v = 0$, and such $2n(n-1)$ linearly independent constraints actually correspond to a discrete curl-free condition on the vector v.
To summarize, we arrive at the following constrained optimization problem:

$$\begin{cases}\text{Minimize} \quad \mathcal{J}_h(v) = h^2\big[\alpha\|Tv-\tilde{g}\|^2 + \sum_{i,j}\min\left\{|v_{i,j}|^2, \frac{\beta}{h}\right\}\big].\\ \\ \text{subject to} \quad Av = 0,\end{cases} \tag{3.3}$$

for $T = D_h^\dagger$ and $A = I - D_h D_h^\dagger$. Once the minimal derivative vector v is computed, we can assemble the minimal u by incorporating the mean value of g as follows:

$$u = D_h^\dagger v + c_g.$$

3.2. Truncated polynomial minimization. We define the *truncated polynomial scalar function* $W_r^p(t) = \min\{|t|^p, r^p\}$, for $r > 0$, $p \geq 1$, and any $t \in \mathbb{R}$. Returning to our more abstract setting of the first part of this paper, we consider again two finite dimensional Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_1$ and a surjective linear constraint map $A : \mathcal{H} \to \mathcal{H}_1$. In addition we consider another finite dimensional Hilbert space $\mathcal{K}$ and a linear operator $T : \mathcal{H} \to \mathcal{K}$. Again, to ease the notation, we indicate with $\|\cdot\|$ the Hilbertian norms on $\mathcal{H}$, $\mathcal{H}_1$, or $\mathcal{K}$ indifferently, as they can be subsumed from the context where they are applied. For fixed $g \in \mathcal{K}$ and $f \in \mathcal{H}_1$, inspired by the reformulation (3.3) of the Mumford-Shah function as a truncated quadratic minimization of the derivative vector, in the following we consider the more general nonsmooth and nonconvex functionals of the type

$$\mathcal{J}_p(v) = \|Tv-g\|^2 + \gamma\sum_{i=1}^m W_r^p(v_i), \tag{3.4}$$

to be minimized, subject to a linear constraint $Av = f$, and $\gamma > 0$ is a positive regularization parameter. Here $(v_i)_{i=1}^m$ are the components of the vector v with respect to a fixed basis in the space $\mathcal{H}$ of dimension $m = \dim(\mathcal{H})$.

First of all, we should mention that, independently of the choice of the linear operators T and A, by [26, Theorem 2.3], the constrained minimization problem

$$\text{Minimize } \mathcal{J}_p(v) = \|Tv - g\|^2 + \gamma \sum_{i=1}^m W_r^p(v_i), \text{ Subject to } Av = f, \tag{3.5}$$

has always minimizers. Notice that this is a remarkable result as the problem is in general not coercive. Concerning uniqueness and stability of minimizers, we refer instead to the work of Durand and Nikolova [20, 21], about cases where T is injective on $\ker A$.

REMARK 3.1. *The proof of existence of solutions of* (3.5) *is based on a special orthogonal decomposition of certain convex sets, see [26, Appendix, Section 8.1]. Let us report the main fact, which it will turn out to be useful to us again later in this paper.*
Define $\bar{\mathcal{J}}_p(v) = \|Tv - g\|^2 + \gamma \sum_{i=1}^m c_i |v_i|^p$ *for* $c_1, \dots c_m$ *scalars; notice that we allow some of them to be negative or zero, as soon as* $\bar{\mathcal{J}}_p(v) \geq C_{\inf} > -\infty$ *for all* $v \in \mathcal{H}$. *Then for any constant* $C > 0$ *and any polyhedral convex set* $X \subset \mathcal{H}$, *there exists a linear subspace* $\mathcal{V} = \mathcal{V}_{X,C} \subset \mathcal{H}$, *such that the orthogonal projection* $X^\perp$ *of* X *onto* $\mathcal{V}^\perp$ *has the properties*

- $X = \{x = x^\perp \oplus tv : x^\perp \in X^\perp, v \in \mathcal{V}, t \in \mathbb{R}^+\}$,
- $M_C = X^\perp \cap \{v \in \mathcal{H} : \bar{\mathcal{J}}_p(v) \leq C\}$ *is compact, and*
- $\bar{\mathcal{J}}_p(\xi_t)$ *is constant along rays* $\xi_t = x^\perp \oplus tv$, *where* $x^\perp \in M_C$, $v \in \mathcal{V}$, *and* $t \in \mathbb{R}^+$.

For $\mathcal{I}_0 \subset \mathcal{I}$ *and* $\mathcal{U}_{\mathcal{I}_0} := \{v \in \mathcal{H} : |v_i| \leq r, i \in \mathcal{I}_0$ *and* $|v_i| > r, i \in \mathcal{I} \setminus \mathcal{I}_0\}$, *in particular this result applies on* $X = \mathcal{F}(f) \cap \overline{\mathcal{U}_{\mathcal{I}_0}}$, *hence*

$$\text{Minimize } \bar{\mathcal{J}}_p(v) = \|Tv - g\|^2 + \gamma \sum_{i=1}^m c_i |t|^p, \text{ Subject to } Av = f \text{ and } v \in \mathcal{U}_{\mathcal{I}_0}, \tag{3.6}$$

has solutions in $\mathcal{H}$, *actually in the compact set* $M_{\bar{\mathcal{J}}_p(v^0)} = X^\perp \cap \{v \in \mathcal{H} : \bar{\mathcal{J}}_p(v) \leq \bar{\mathcal{J}}_p(v^0)\}$, *for any* $v^0 \in \mathcal{H}$.

3.3. Unconstrained minimization and iterative thresholding. Despite the nonsmoothness and the nonconvexity of the functional $\mathcal{J}_p$, in the work [26] a very simple and globally converging iterative algorithm has been studied in order to find local minimizers of $\mathcal{J}_p$ in the case of the unconstrained minimization, i.e., when we are omitting the linear constraint $Av = f$. This case, for instance, solves the minimization of the Mumford-Shah function in one dimension, see [26] for details. Notice that relevant features of this approach are that it does not go through any smooth approximation process of the functional $\mathcal{J}_p$, contrary to other previous approaches, e.g., based on graduated nonconvexity [5, 37, 35], and it works also for inverse problems, where T is not injective. Following a similar approach as in [16], the proposed method is actually a *forward-backward* or *majorization-minimization* algorithm of Douglas-Rachford type [32] for finding minimal solutions to $\mathcal{J}_p$. More precisely, consider the following *surrogate* objective function,

$$\mathcal{J}_p^{surr}(v, a) := \mathcal{J}_p(v) - \|Tv - Ta\| + \|v - a\|, \quad v, a \in \mathcal{H}. \tag{3.7}$$

As $\|T\| \leq 1$ can be assumed without loss of generality possibly rescaling γ and g (see [26, Section 3]), the surrogate functional $\mathcal{J}_p^{surr}$ satisfies $\mathcal{J}_p^{surr}(v,a) \geq \mathcal{J}_p(v)$ everywhere, with equality if and only if $v = a$, and is such that the sequence

$$v^{n+1} = \arg\min_{v \in \mathcal{H}} \mathcal{J}_p^{surr}(v, v^n) \tag{3.8}$$

obtained by successive minimizations of $\mathcal{J}_p^{surr}(v,a)$ in v for fixed a results in a non-increasing sequence of the original functional $(\mathcal{J}_p(v^n))_{n\in\mathbb{N}}$ (see [26, Lemmas 4.1 and 4.2]). Moreover, expanding the squared terms on the right hand side of the expression for $\mathcal{J}_p^{surr}$, we have

$$\begin{aligned}\mathcal{J}_p^{surr}(v,a) &= \|v - (I - T^*T)a + T^*g\| + \gamma \sum_{i=1}^{m} \min\{|v_i|^p, r^p\} + C \\ &= \sum_{i=1}^{m} \Big[(v_i - [a - T^*Ta + T^*g]_i)^2 + \gamma \min\{|v_i|^p, r^p\}\Big] + C,\end{aligned}$$

where the term $C = C(T,a,g)$ depends only on T, a and g. It is now clear that the surrogate functional $\mathcal{J}_p^{surr}$ decouples in the variables v_i, due to the cancellation of terms involving $\|Tv\|$. Because of this decoupling, global v-minimizers of $\mathcal{J}_p^{surr}(v,a)$, for a fixed, can be computed *component-wise* according to

$$\bar{v}_i = \arg\min_{t\in\mathbb{R}} \Big[(t - [a - T^*Ta + T^*g]_i)^2 + \gamma W_r^p(t)\Big], \quad i = 1, \ldots, m. \tag{3.9}$$

The advantage of this strategy is that one can solve (3.9) explicitly when, e.g., $p = 2$, $p = 3/2$, and $p = 1$; in the general case $p \geq 1$, we have the following result:

Proposition 3.2 ([26, Proposition 4.3]).

1. *If $p > 1$, the minimization problem $\bar{v} = \arg\min_{v\in\mathcal{H}} \mathcal{J}_p^{surr}(v,a)$ can be solved component-wise as in* (3.9) *by*

$$\bar{v}_i = H_p([a - T^*Ta + T^*g]_i), \quad i = 1, \ldots, m, \tag{3.10}$$

where $H_p : \mathbb{R} \to \mathbb{R}$ is the 'thresholding function',

$$H_p(\xi) = \begin{cases} F_p^{-1}(\xi), & |\xi| \leq \xi'(r,\gamma,p) \\ \xi, & else \end{cases}$$

Here, $F_p^{-1}(\xi)$ is the inverse of the function $F_p(t) = t + \frac{\gamma p}{2} \operatorname{sign} t |t|^{p-1}$, and $\xi' \in (r, r + \frac{\gamma p}{2} r^{p-1})$ is the unique positive value at which

$$(F_p^{-1}(\xi') - \xi')^2 + \gamma |F_p^{-1}(\xi')|^p = \gamma r^p. \tag{3.11}$$

2. *When $p = 1$, the general form* (3.10) *still holds, but we have to consider two cases:*
   (a) If $r > \gamma/4$, the thresholding function $H_1 : \mathbb{R} \to \mathbb{R}$ satisfies

$$H_1(\xi) = \begin{cases} 0, & |\xi| \leq \gamma/2 \\ (|\xi| - \gamma/2) \operatorname{sign} \xi, & \gamma/2 < |\xi| \leq r + \gamma/4 = \xi' \\ \xi, & |\xi| > r + \gamma/4 \end{cases} \tag{3.12}$$

(b) If, on the other hand, $r \leq \gamma/4$, the function H_1 satisfies

$$H_1(\xi) = \begin{cases} 0, & |\xi| \leq \sqrt{\gamma r} = \xi' \\ \xi, & |\xi| > \sqrt{\gamma r} \end{cases} \tag{3.13}$$

In all cases, the thresholding function is continuous except at $\xi'(r, \gamma, p)$, where it has a jump-discontinuity of size $\delta = |\xi' - H_p(\xi')| > 0$ if $r, \gamma > 0$. In particular, $\xi' > r$ while $H_p(\xi') < r$.

In the previous proposition, we used the notation H_p, neglecting the parameters r and γ, however actually $H_p = H_{r,\gamma,p}$ does depend on them as well and it is characterized by

$$H_{r,\gamma,p}(\xi) := \arg\min_{t \in \mathbb{R}} (t - \xi)^2 + \gamma W_r^p(t). \tag{3.14}$$

To summarize, the iterative algorithm (3.8) can be recast in terms of a component-wise *iterative thresholding algorithm*,

$$v_i^{n+1} = H_p([v^n - T^*Tv^n + T^*g]_i), \tag{3.15}$$

which, for the parameter $p = 2$ of the truncated quadratic functional as in (3.3), reduces simply to

$$u_i^{n+1} = \begin{cases} (1+\gamma)^{-1}([v^n - T^*Tv^n + T^*g]_i), & |[v^n - T^*Tv^n + T^*g]_i| \leq (1+\gamma)^{1/2} r \\ [v^n - T^*Tv^n + T^*g]_i, & \text{else} \end{cases}$$

See [26, Remark 2] for a more detailed account on how to compute the scalar function H_p for a generic $p \geq 1$, see Figure 3.1 for the cases $p = 1$, $p = 3/2$, and $p = 2$. Notice again that H_p is always discontinuous for any $p \geq 1$.

Let us now discuss the convergence properties of such an algorithm. For that we define the operator $\mathbb{H} : \mathcal{H} \to \mathcal{H}$ by its component-wise action,

$$[\mathbb{H}(v)]_i := H_p([v - T^*Tv + T^*g]_i); \tag{3.16}$$

the iteration (3.15) can then be written more concisely in operator notation as

$$v^{n+1} = \mathbb{H}(v^n). \tag{3.17}$$

The following convergence result has been proved in [26, Theorem 4.8 and Theorem 5.1].

THEOREM 3.3. *Suppose $p \geq 1$. Starting from any $v^0 \in \mathcal{H}$, the sequence $(v^n)_{n\in\mathbb{N}}$ defined by $v^{n+1} = \mathbb{H}^n(v^0)$ as in (3.17) will converge to a vector $\bar{v} \in \mathcal{H}$ that satisfies the fixed point condition,*

$$\bar{v} = \mathbb{H}(\bar{v}).$$

Let us further denote Fix($\mathbb{H}$) *the set of fixed points of* $\mathbb{H}$, $\mathscr{G}$ *the set of* global minimizers *of* $\mathcal{J}_p$, *and* $\mathscr{L}$ *the set of* local minimizers *of* $\mathcal{J}_p$. *Then we have the following set inclusions*

$$\mathscr{G} \subset \operatorname{Fix}(\mathbb{H}) \subset \mathscr{L}. \tag{3.18}$$

Let us remark that the proof of convergence of the algorithm to fixed points in Fix($\mathbb{H}$) is strongly based on the *discontinuity* of the thresholding function, see [26, Lemma 4.4].

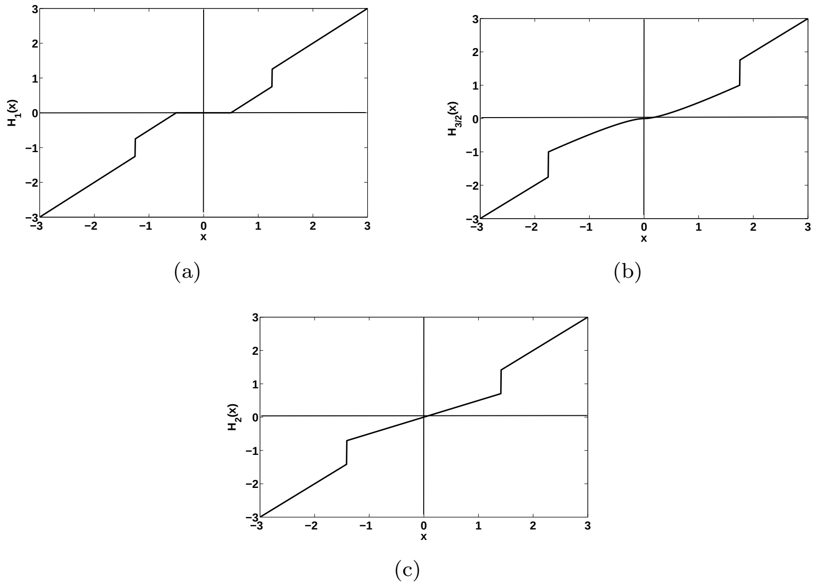

FIG. 3.1. *The discontinuous thresholding functions* H_1, $H_{3/2}$, *and* H_2, *with parameters* $p = 1, 3/2$, *and* 2, *respectively, and* $r = 1$, $\gamma = 1$.

3.4. Constrained minimization. Due to the nonsmoothness and nonconvexity of $\mathcal{J}_p$, the more general linearly constrained minimization (3.5) has been so far an open problem, as standard methods, such as SQP and Newton methods, do not apply, unless one provides a C^2-regularization of the problem. In particular, it would be desirable that an appropriate algorithm performing such an optimization could retain both the simplicity of the thresholding iteration (3.15) and its global convergence properties, as given by Theorem 3.3. Certainly the method (2.14) is a strong candidate, as the iterations of its inner loop actually requires only a unconstrained minimization, which can be again addressed by iterative thresholding, see Section 4 below. However, we encounter two major bottlenecks to the direct application of this algorithm to (3.5). The first problem is that $\mathcal{J}_p$ does not satisfy our main assumption (A1), i.e., it is not ω-convex, as it is not a C^1-perturbation of a convex functional. In fact the term W_r^p is too rough at the kink where the truncation applies. The second trouble comes by the lack of coerciveness of $\mathcal{J}_p$ on the affine space $\mathcal{F}(f)$ in general, for a generic choice of T. In the next sections we address these two issues.

3.5. A smoothing method. In this section we would like to construct an appropriate *slightly smoother* perturbation $\mathcal{J}_p^\varepsilon$ of $\mathcal{J}_p$, which allows eventually for ω-convexity, but does not modify essentially the minimizers over $\mathcal{F}(f)$. Moreover, such modification will not affect the possibility of using *discontinuous* thresholding functions in the numerical setting. We will further clarify in Sections 3.7 and 4 the usefulness of this feature. Actually, let us mention that a similar phenomenon, i.e., the independence of the thresholding under small smooth perturbations, was already observed and used in graduated nonconvexity methods, see [37, Proposition 1]. For the moment, we will only define this smooth perturbation and state its main properties, whose proofs are shifted to the Appendix, for an easier reading.

We start by the following polynomial interpolation result.

LEMMA 3.4. *Let* $0 < s_1 < s_2$ *and assume that*

$$\pi(t) := A(t-s_2)^3 + B(t-s_2)^2 + C,$$

is a third degree polynomial. Given $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ *and by setting*

$$\begin{cases} C = \gamma_3, \\ B = \frac{\gamma_1}{s_2-s_1} - \frac{3(\gamma_3-\gamma_2)}{(s_2-s_1)^2}, \\ A = \frac{\gamma_1}{3(s_2-s_1)^2} + \frac{2B}{3(s_2-s_1)}, \end{cases} \tag{3.19}$$

then we have the following interpolation properties

$$\begin{cases} \pi(s_2) = \gamma_3, & \pi(s_1) = \gamma_2, \\ \pi'(s_2) = 0, & \pi'(s_1) = \gamma_1. \end{cases} \tag{3.20}$$

Proof. The equalities related to s_2 are straightforward, the others related to s_1 follow by simple direct computations:

$$\begin{aligned} \pi(s_1) &= -\frac{\gamma_1}{3}(s_2-s_1) - \frac{2}{3}B(s_2-s_1)^2 + B(s_2-s_1)^2 + \gamma_3 \\ &= -\frac{\gamma_1}{3}(s_2-s_1) + \frac{B}{3}(s_2-s_1)^2 + \gamma_3 \\ &= -\frac{\gamma_1}{3}(s_2-s_1) + \frac{\gamma_1}{3}(s_2-s_1) - (\gamma_3-\gamma_2) + \gamma_3 = \gamma_2, \end{aligned}$$

and

$$\begin{aligned} \pi'(s_1) &= 3A(s_1-s_2)^2 + 2B(s_1-s_2) \\ &= \gamma_1 - 2B(s_1-s_2) + 2B(s_1-s_2) = \gamma_1. \end{aligned}$$

□

Given $0 < \varepsilon < r$ for every $t \in [r-\varepsilon, r+\varepsilon]$ we define $\pi_p(t) = \pi(t)$ as in Lemma 3.4 for $s_1 = (r-\varepsilon)$, $s_2 = (r+\varepsilon)$, $\gamma_1 = p(r-\varepsilon)^{p-1}$, $\gamma_2 = (r-\varepsilon)^p$, and $\gamma_3 = r^p$. For example, for $p = 2$, we have

$$\pi_2(t) = \frac{[t+(r-\varepsilon)][\varepsilon(r+t)-(r-t)^2]}{4\varepsilon}, \quad t \in \mathbb{R}.$$

Let us now set, for all $t \geq 0$,

$$W_r^{p,\varepsilon}(t) = \begin{cases} t^p, & t \leq r-\varepsilon, \\ \pi_p(t), & r-\varepsilon \leq t \leq r+\varepsilon, \\ r^p & t \geq r+\varepsilon, \end{cases} \tag{3.21}$$

whereas for $t \leq 0$, we define $W_r^{p,\varepsilon}(t) = W_r^{p,\varepsilon}(-t)$. Notice that now $W_r^{p,\varepsilon}$ is actually a C^1-function of $\mathbb{R}$, for all $0 < \varepsilon < r$.

As announced before, this modification will not affect the possibility of using discontinuous thresholding techniques in the numerical setting, as we will see in Sections 3.7 and 4. To state this fact in a convenient way, we need to fix some notation. For $\xi \in \mathbb{R}$, $\gamma > 0$, and $r > 0$, we consider the function

$$g_{\gamma,r}^{\xi}(t) = (t-\xi)^2 + \gamma W_r^p(t), \tag{3.22}$$

where again $W_r^p(t) = \min\{|t|^p, r^p\}$.

Remark 3.5. *It follows from the proof of Proposition 3.2 in [26], see also* (3.14)*, that, for fixed* γ, r, ξ *the function* $g^\xi_{\gamma,r}$ *has a unique global minimizer* $\bar{x}$ *in* $\mathbb{R}$*, which is given by*

$$\bar{x} = H_p(\xi),$$

where H_p *is the thresholding function defined in Proposition 3.2. In particular, due to the discontinuity of* H_p*, there exists* $\varepsilon_0 > 0$ *independent of* ξ *such that for all* $0 < \varepsilon < \varepsilon_0$ *such a minimizer does not belong to the set* $[-r-\varepsilon, -r+\varepsilon] \cup [r-\varepsilon, r+\varepsilon]$.

Let us further define the function

$$f^\xi_{\gamma,r}(t) = (t-\xi)^2 + \gamma W_r^{p,\varepsilon}(t). \tag{3.23}$$

Then the following result holds.

Theorem 3.6. *Let* $\varepsilon_0 > 0$ *as in Remark 3.5 and*

$$\varepsilon < \min\left\{\varepsilon_0, \frac{[(5/4)^{1/(p-1)} - 1]}{(5/4)^{1/(p-1)}} r, \frac{\gamma p r^{p-1}}{20}\right\}. \tag{3.24}$$

Then $f^\xi_{\gamma,r}$ *has a unique global minimizer in* $\mathbb{R}$*, which coincides with the one of* $g^\xi_{\gamma,r}$*, that is the thresholding function computes the minimizer*

$$H_p(\xi) = \arg\min_t (t-\xi)^2 + \gamma W_r^{p,\varepsilon}(t),$$

independently of ε *as in* (3.24)*.*

We leave the proof of Theorem 3.6 to the Appendix.

Thanks to the function $W_r^{p,\varepsilon}$, we can define the following perturbation of $\mathcal{J}_p$

$$\mathcal{J}_p^\varepsilon(v) = \|Tv - g\|^2 + \gamma \sum_{i=1}^m W_r^{p,\varepsilon}(v_i). \tag{3.25}$$

About the existence of constrained minimizers of $\mathcal{J}_p^\varepsilon$ we have the following abstract result, whose proof is again shifted to the Appendix.

Theorem 3.7. *For* $0 \le \varepsilon < \varepsilon_0$*, the problem*

$$\text{Minimize } \mathcal{J}_p^\varepsilon(v) = \|Tv - g\|^2 + \gamma \sum_{i=1}^m W_r^{p,\varepsilon}(v_i), \text{ Subject to } Av = f, \tag{3.26}$$

has solutions in $\mathcal{H}$*. Actually, such minimal solutions can be taken in a compact set* $M \subset \mathcal{H}$ *independent of* $0 \le \varepsilon < \varepsilon_0$*.*

Remark 3.8. *The previous result clarifies that, despite the fact that in general* $\mathcal{J}_p^\varepsilon$ *are not coercive functionals, up to restricting them to an appropriate compact set, independent of* ε*, they can be considered equi-coercive.*

Corollary 3.9. *The net of functionals* $(\mathcal{J}_p^\varepsilon)_{0\le\varepsilon<\varepsilon_0}$ Γ*-converges to* $\mathcal{J}_p$ *on* $\mathcal{F}(f)$*. Moreover, if we consider the minimizers* v^*_ε *of* $\mathcal{J}_p^\varepsilon$ *in* M*, as constructed in Theorem 3.7 (which are actually minimizers of* $\mathcal{J}_p^\varepsilon$ *over* $\mathcal{F}(f)$ *as well), then the accumulation points of such a net are minimizers of* $\mathcal{J}_p$*.*

Proof. As $\mathcal{J}_p^\varepsilon$ converges uniformly to $\mathcal{J}_p$ on $\mathcal{F}(f)$, we deduce immediately its Γ-convergence [15]. By Theorem 3.7 and compactness of M we conclude the convergence of minimizers. ∎

PROPOSITION 3.10. *For all* $0 < \varepsilon < \varepsilon_0$, *the functional* $\mathcal{J}_p^\varepsilon$ *satisfies the properties (A1) and (A2), i.e., it is* ω*-convex, and* (2.9) *holds.*

Proof. The ω-convexity follows from the piecewise continuity and boundedness of the second derivatives of $\mathcal{J}_p^\varepsilon$. Since $W_r^{p,\varepsilon}(t) \geq 0$ and $|(W_r^{p,\varepsilon})'(t)| \leq pr^{p-1}$ for every $t \in \mathbb{R}$, by means of the elementary inequality $a \leq \frac{1}{2}(a^2+1)$ we obtain

$$\begin{aligned}\|\nabla \mathcal{J}_p^\varepsilon(v)\| &\leq 2\|T^*(Tv-g)\| + \gamma\|((W_r^{p,\varepsilon})'(v_1), \dots, (W_r^{p,\varepsilon})'(v_m))\| \\ &\leq 2\|T^*\|\|Tv-g\| + \gamma m^{1/2} pr^{p-1} \\ &\leq \|T^*\|\|Tv-g\|^2 + \|T^*\| + \gamma m^{1/2} pr^{p-1} \\ &\leq \|T^*\|\mathcal{J}_p^\varepsilon(v) + \|T^*\| + \gamma m^{1/2} pr^{p-1}.\end{aligned} \tag{3.27}$$

Hence, for $K = \|T^*\|$ and $L = \|T^*\| + \gamma m^{1/2} pr^{p-1}$, we get that (2.9) holds for $\mathcal{J}_p^\varepsilon$. ☐

3.6. The application of the algorithm to the Mumford-Shah case. As we clarified in the previous section, functionals of the type $\mathcal{J}_p^\varepsilon$, for $0 < \varepsilon < \varepsilon_0$, satisfy the assumptions (A1) and (A2) for the applicability of the algorithm (2.14). In particular, when the algorithm is applied for $\mathcal{J} = \mathcal{J}_p^\varepsilon$, then by Theorem 2.9 the sequence $(v_\ell)_{\ell\in\mathbb{N}}$ generated by the algorithm has the properties

(a) $(Av_\ell - f) \to 0$ as $\ell \to \infty$;

(b) $(v_\ell - v_{\ell-1}) \to 0$ as $\ell \to \infty$.

However $\mathcal{J}_p^\varepsilon$ is unfortunately not necessarily coercive on $\mathcal{F}(f) = \{v \in \mathcal{H} : Av = f\}$, although it retains some coerciveness by considering suitable compact subsets M of competitors. Nevertheless, such information does not help when it comes to the application of the algorithm (2.14), as there is no natural or simple way of restricting or projecting the iterations to such compact sets M. Hence, in order to apply Theorem 2.10, we need to explore the mechanism for which the iterations $(v_\ell)_{\ell\in\mathbb{N}}$ generated by the algorithm keep bounded. In this section we treat the case where T is injective on $\ker A$, actually endowing the problem with coerciveness, which is preparatory to the general case where coerciveness is not guaranteed.

Let us first introduce some specific notation for the application of the algorithm (2.14), in particular we denote

$$\mathcal{J}_{\omega,u}(v) := \mathcal{J}_{p,\omega,u}^\varepsilon(v) = \mathcal{J}_p^\varepsilon(v) + \omega\|v-u\|^2. \tag{3.28}$$

LEMMA 3.11. *For all* $0 \leq \varepsilon < \varepsilon_0$, *the sequence* $(\|\nabla \mathcal{J}_p^\varepsilon(v_\ell)\|)_{\ell\in\mathbb{N}}$ *is uniformly bounded, where the iterations* $(v_\ell)_{\ell\in\mathbb{N}}$ *are generated by the algorithm* (2.14).

Proof. As a consequence of (2.26) the sequence $(\|T\bar{v}_\ell\|)_\ell$, where $\bar{v}_\ell$ is defined in (2.21), is uniformly bounded. From (2.24), we have also that $(\|Tv_\ell\|)_\ell$ is uniformly bounded. As pointed out in (3.27) of Proposition 3.10 actually we have $\|\nabla \mathcal{J}_p^\varepsilon(v_\ell)\| \leq (2\|T^*\|)\|Tv_\ell - g\| + \gamma m^{1/2} pr^{p-1}$. Hence the sequence $(\|\nabla \mathcal{J}_p^\varepsilon(v_\ell)\|)_{\ell\in\mathbb{N}}$ is uniformly bounded. ☐

The next lemma is stated in a slightly more general form than the one actually needed in this subsection, in order to allow its application also to the case later discussed in Proposition 3.19. In the statement, given any subset of indexes $\mathcal{I}_1 \subseteq \mathcal{I}$, we denote with P_1 the orthogonal projection onto the subspace $\mathcal{H}^1 := \{v \in \mathcal{H} : v_i = 0, i \in \mathcal{I} \setminus \mathcal{I}_1\}$.

LEMMA 3.12. *Let us consider any* $\mathcal{I}_1 \subseteq \mathcal{I}$ *(hence it may also be* $\mathcal{I}_1 = \mathcal{I}$*) and* $A_1 = AP_1$. *Then, for all* $0 < \varepsilon < \varepsilon_0$, *the sequence* $(A_1^* q_{\ell,L_\ell-1})_{\ell\in\mathbb{N}}$ *generated by the application of the algorithm* (2.14) *for* $\mathcal{J} = \mathcal{J}_p^\varepsilon$ *is uniformly bounded.*

Proof. As $A_1 = AP_1$, then $A_1^* = P_1 A^*$, hence it is sufficient to prove that $(A^* q_{\ell, L_\ell - 1})_{\ell \in \mathbb{N}}$ generated by the application of the algorithm (2.14) is uniformly bounded. By (2.16) we have

$$A^* q_\ell \in \nabla \mathcal{J}_{\omega, v_{\ell-1}}(v_\ell) = \nabla \mathcal{J}_p^\varepsilon(v_\ell) + 2\omega(v_\ell - v_{\ell-1})$$

As, by Lemma 3.11 $\nabla \mathcal{J}_p^\varepsilon(v_\ell)$ is uniformly bounded and $v_\ell - v_{\ell-1} \to 0$, for $\ell \to \infty$, we obtain that also $A^* q_\ell$ is uniformly bounded. By (2.14), we have also

$$A^* q_\ell = A^* q_{\ell, L_\ell - 1} - 2\lambda A^*(A v_\ell - f),$$

from which, together with $(Av_\ell - f) \to 0$ for $\ell \to \infty$, we eventually deduce the uniform boundedness of $A^* q_{\ell, L_\ell - 1}$ as well. □

LEMMA 3.13. *Assume that T is injective on $\ker A$, or $\ker T \cap \ker A = \{0\}$. Then, for all $0 < \varepsilon < \varepsilon_0$, the sequences $(v_\ell)_\ell$ generated by the application of the algorithm (2.14) for $\mathcal{J} = \mathcal{J}_p^\varepsilon$ is uniformly bounded.*

Proof. Notice that, by definition of v_ℓ in (2.14), necessarily it solves the following linear system

$$(T^* T + \frac{1}{2} A^* A) v_\ell = \frac{1}{2} A^*(f + q_{\ell, L_\ell - 1}) + \omega(v_{\ell-1} - v_\ell),$$

where the right-hand-side of this equality if uniformly bounded by Lemma 3.12 and Theorem 2.9 (b). Moreover, as $(Av_\ell - f) \to 0$ for $\ell \to \infty$, we can write that v_ℓ is solution of the system

$$\underbrace{\begin{bmatrix} (T^* T + \frac{1}{2} A^* A) \\ A \end{bmatrix}}_{:=G} v_\ell = w_\ell,$$

where the right-hand-side w_ℓ is actually uniformly bounded with respect to ℓ. Due to our assumption $\ker T \cap \ker A = \{0\}$, we obtain that $\ker G = \{0\}$ and

$$v_\ell = (G^* G)^{-1} G^* w_\ell, \text{ for all } \ell \in \mathbb{N},$$

hence the uniform boundedness of $(v_\ell)_\ell$. □

We summarize this list of technical observations into the following convergence result.

THEOREM 3.14. *Assume that T is injective on $\ker A$, or $\ker T \cap \ker A = \{0\}$. Then, for all $0 < \varepsilon < \varepsilon_0$, the sequences $(v_\ell)_\ell$ generated by the application of the algorithm (2.14) for $\mathcal{J} = \mathcal{J}_p^\varepsilon$ has at least one accumulation point, and every accumulation point is a constrained critical point of $\mathcal{J}_p^\varepsilon$ on the affine space $\mathcal{F}(f) = \{v \in \mathcal{H} : Av = f\}$.*

Proof. The result follows by a direct application of Theorem 2.10, after having recalled the boundedness of $(v_\ell)_\ell$, which results from Lemma 3.13. □

REMARK 3.15. *The previous convergence result actually applies for the case of the Mumford-Shah functional, for which $T = D_h^\dagger$ and $A = I - D_h D_h^\dagger$, since $D_h^\dagger$ is in fact injective on $\operatorname{ran}(D_h)$, see Section 3.1.*

3.7. The application of the algorithm to inverse free-discontinuity problems. When the operator T is actually the composition of $D_h^\dagger$ with another noninvertible operator, say it, S, i.e., $T = S \circ D_h^\dagger$, representing a model of a inverse free-discontinuity problem [24, 26], then the condition $\ker T \cap \ker A = \{0\}$ might not be verified anymore and we wonder in this case under which natural conditions the algorithm can still converge. For this, we will need to make a finer analysis of the behavior of the algorithm (2.14) when applied to $\mathcal{J} = \mathcal{J}_p^\varepsilon$.

For the sake of simplicity and without loss of generality, we consider the application of (2.14) for $\lambda = 1/2$, and we define now the strictly convex functional

$$\mathcal{J}_{\omega,u}(v,q) := \mathcal{J}^\varepsilon_{p,\omega,u}(v,q) = \mathcal{J}_{\omega,u}(v) + \frac{1}{2}\|Av - (f+q)\|^2. \tag{3.29}$$

We further consider the surrogate functional associated to $\mathcal{J}_{\omega,u}(v,q)$, given by

$$\begin{aligned}\mathcal{J}^{surr}_{\omega,u}(v,q,w) := \mathcal{J}^{surr,\varepsilon}_{p,\omega,u}(v,q,w) = \mathcal{J}_{\omega,u}(v,q) &+ (\|v-w\|^2 - \|Tv - Tw\|^2)\\ &+ (\|v-w\|^2 - \frac{1}{2}\|Av - Aw\|^2)\\ &+ (\|v-w\|^2 - \omega\|v-w\|^2).\end{aligned} \tag{3.30}$$

Up to rescaling of g, f, q, γ we can assume here and later and without loss of generality that $\|T\| < 1$, $\frac{1}{\sqrt{2}}\|A\| < 1$, and $\omega < 1$. Hence, we have

$$\mathcal{J}^{surr}_{\omega,u}(v,q,w) \geq \mathcal{J}_{\omega,u}(v,q), \tag{3.31}$$

and

$$\mathcal{J}^{surr}_{\omega,u}(v,q,w) = \mathcal{J}_{\omega,u}(v,q), \tag{3.32}$$

if and only if $w = v$.

Proposition 3.16. *Assume $\|T\| < 1$, $\frac{1}{\sqrt{2}}\|A\| < 1$, and $\omega < 1$. Moreover, let $\varepsilon_0 > 0$ as in Remark 3.5 and*

$$\varepsilon < \min\left\{\varepsilon_0, \frac{[(5/4)^{1/(p-1)} - 1]}{(5/4)^{1/(p-1)}} r, \frac{1}{3}\frac{\gamma p r^{p-1}}{20}\right\}. \tag{3.33}$$

Then

$$v^* = \arg\min_{v\in\mathcal{H}} \mathcal{J}_{\omega,u}(v,q) \tag{3.34}$$

if and only if v^* satisfies the following component-wise fixed-point equation: for $i = 1,\dots,m$,

$$v_i^* = H_p\left(\frac{1}{3}\left\{[(I - T^*T) + (I - \frac{1}{2}A^*A) + (1-\omega)I]v^* + (T^*g + \frac{1}{2}A^*(f+q) + \omega u)\right\}_i\right), \tag{3.35}$$

where H_p is the thresholding function defined in Proposition 3.2 for the parameters r and $\gamma/3$ (notice that an additional factor $\frac{1}{3}$ in fact appears in the last term of (3.33) *with respect to the corresponding one in* (3.24)*).*

Proof. Assume that v^* satisfies (3.34). From (3.31) and (3.32), we have the inequalities

$$\begin{aligned}\mathcal{J}^{surr}_{\omega,u}(v^*,q,v^*) &= \mathcal{J}_{\omega,u}(v^*,q)\\ &\leq \mathcal{J}_{\omega,u}(v,q)\\ &= \mathcal{J}^{surr}_{\omega,u}(v,q,v)\\ &\leq \mathcal{J}^{surr}_{\omega,u}(v,q,v^*).\end{aligned}$$

Hence we obtain also

$$v^* = \arg\min_{v\in\mathcal{H}} \mathcal{J}^{surr}_{\omega,u}(v,q,v^*). \tag{3.36}$$

We notice now by a direct computation that

$$\frac{1}{3}\mathcal{J}^{surr}_{\omega,u}(v,q,v^*) = \left\| v - \left(\frac{b^1+b^2+b^3}{3}\right)\right\|^2 + \frac{\gamma}{3}\sum_{i=1}^m W_r^{p,\varepsilon}(v_i) + C(b^1,b^2,b^3,\gamma), \tag{3.37}$$

where $b^1 = (I-T^*T)v^* + T^*g$, $b^2 = (I-\frac{1}{2}A^*A)v^* + \frac{1}{2}A^*(f+q)$, $b^3 = (I-\omega I)v^* + \omega u$, and $C(b^1,b^2,b^3,\gamma)$ is a term that does not depend on v. One now concludes by an application of Theorem 3.6 and Proposition 3.2 that v^* satisfies (3.35).

Conversely, by (3.37), Proposition 3.2, and Theorem 3.6, if v^* satisfies (3.35), then it also satisfies (3.36). It follows that

$$0 \in \partial\mathcal{J}^{surr}_{\omega,u}(v^*,q,v^*) = \partial\mathcal{J}_{\omega,u}(v^*,q)$$

where the last equality trivially follows from (3.30). By convexity of $\mathcal{J}_{\omega,u}$, this implies (3.34). □

In the rest of the paper, the following notations will be useful. For $r > 0$ and $v \in \mathcal{H}$ fixed, we denote $\mathcal{I}_0 := \{i \in \mathcal{I} = \{1,\dots,m\} : |v_i| \leq r+\varepsilon\}$ and $\mathcal{I}_1 = \mathcal{I}\setminus\mathcal{I}_0$. We also define $\mathcal{H}^0 := \{v \in \mathcal{H} : v_i = 0, i \in \mathcal{I}_1\}$ and $v_1 \in \mathcal{H}^1 := \{v \in \mathcal{H} : v_i = 0, i \in \mathcal{I}_0\}$. Let P_0 and P_1 denote the orthogonal projections onto the subspaces $\mathcal{H}^0$ and $\mathcal{H}^1$, respectively. We fix the notation for the operator $T_i = TP_i$ and $A_i = AP_i$ for $i = 0,1$.

REMARK 3.17. *Notice that the fixed-point condition* (3.35) *in particular implies*

$$\begin{aligned}\mathcal{I}_0 &= \{i \in \mathcal{I} : |v_i| \leq r+\varepsilon\} = \{i \in \mathcal{I} : |v_i| \leq \xi'(r,\gamma/3,p) - \delta\}\\ \mathcal{I}_1 &= \{i \in \mathcal{I} : |v_i| > r+\varepsilon\} = \{i \in \mathcal{I} : |v_i| > \xi'(r,\gamma/3,p)\},\end{aligned}$$

In particular, for $\varepsilon > 0$ sufficiently small

$$\xi'(r,\gamma/3,p) - \delta < r-\varepsilon < r < r+\varepsilon < \xi'(r,\gamma/3,p),$$

where $\xi' = \xi'(r,\gamma/3,p)$ is the position of the jump-discontinuity of the thresholding function H_p and δ is the size of its jump-discontinuity, as defined in Proposition 3.2. We call this phenomenon, the separation of the components.

Let us now resume the algorithm (2.14) and see how the separation of the components affects its iteration.

LEMMA 3.18 (Fixation of the index set $\mathcal{I}_1$). *Assume $\|T\| < 1$, $\frac{1}{\sqrt{2}}\|A\| < 1$, and $\omega < 1$. Moreover, let $\varepsilon_0 > 0$ as in Remark 3.5 and*

$$\varepsilon < \min\left\{\varepsilon_0, \frac{[(5/4)^{1/(p-1)}-1]}{(5/4)^{1/(p-1)}}r, \frac{1}{3}\frac{\gamma p r^{p-1}}{20}\right\}. \tag{3.38}$$

Then, for the sequence $(v_\ell)_{\ell\in\mathbb{N}}$ generated by the algorithm (2.14), we consider the time-dependent partition of the index set $\mathcal{I}=\{1,\dots,m\}$ into "small" components

$$\mathcal{I}_0^\ell := \{i \in \mathcal{I} : |(v_\ell)_i| \le \xi' - \delta\}, \tag{3.39}$$

and "large" components

$$\mathcal{I}_1^\ell := \{i \in \mathcal{I} : |(v_\ell)_i| > \xi'\}. \tag{3.40}$$

Then for $N \in \mathbb{N}$ sufficiently large, this partition fixes during the iteration for $\ell \ge N$; that is, there exists $\mathcal{I}_0$ such that for all $\ell \ge N$, $\mathcal{I}_0^\ell = \mathcal{I}_0$ and $\mathcal{I}_1^\ell = \mathcal{I}\setminus\mathcal{I}_0$.

Proof. Notice that, by definition in (2.14) and (3.29), we have actually

$$v_\ell = \arg\min_{v\in\mathcal{H}} \mathcal{J}^{\varepsilon}_{p,\omega,v_{\ell-1}}(v, q_{\ell,L_\ell-1}).$$

Hence, by Proposition 3.16 and Remark 3.17 we have

(a) $|(v_\ell)_i| \le \xi' - \delta < \xi'$, if $i \in \mathcal{I}_0^\ell$, or

(b) $|(v_\ell)_i| > \xi'$, if $i \in \mathcal{I}_1^\ell$,

Thus, $|(v_{\ell+1})_i - (v_\ell)_i| \ge \delta$ if $i \in \mathcal{I}_0^{\ell+1}\cap\mathcal{I}_1^\ell$, or if $i \in \mathcal{I}_0^{\ell}\cap\mathcal{I}_1^{\ell+1}$. At the same time, Theorem 2.9 (b), together with the results of the previous sections, imply

$$|(v_{\ell+1})_i - (v_\ell)_i| \le \|v_{\ell+1} - v_\ell\| \le \epsilon, \tag{3.41}$$

once $\ell \ge N(\epsilon)$, and $\epsilon > 0$ can be taken arbitrarily small. In particular, (3.41) implies that $\mathcal{I}_0$ and $\mathcal{I}_1$ must be fixed once $\ell \ge N(\epsilon)$ and $\epsilon < \delta$. □

If we assumed that $\ker T \cap \ker A \neq \{0\}$, then we could not use anymore Lemma 3.13 to infer the uniform boundedness of $(v_\ell)_{\ell\in\mathbb{N}}$ which is the key ingredient for again obtaining a convergence result as in Theorem 3.14. However, we notice that, due to the separation of the components and after fixation of the associated index sets as in Lemma 3.18, actually the component $P_0(v_\ell)$, where P_0 is the orthogonal projection onto vectors supported on $\mathcal{I}_0$, is uniformly bounded, by definition. If we could obtain an affine dependence of the component $P_1(v_\ell)$ supported on $\mathcal{I}_1$ on the component $P_0(v_\ell)$, then we could infer the boundedness of $P_1(v_\ell)$ as well and hence of $v_\ell = P_0(v_\ell) + P_1(v_\ell)$. This is the scope of the following result.

Proposition 3.19. *Assume $\|T\| < 1$, $\frac{1}{\sqrt{2}}\|A\| < 1$, and $\omega < 1$. Moreover, let $\varepsilon_0 > 0$ as in Remark 3.5 and*

$$\varepsilon < \min\left\{\varepsilon_0, \frac{[(5/4)^{1/(p-1)} - 1]}{(5/4)^{1/(p-1)}} r, \frac{1}{3}\frac{\gamma p r^{p-1}}{20}\right\}.$$

Assume that $\mathcal{I}_0$ and $\mathcal{I}_1$ are the index sets produced by the algorithm (2.14) after fixation as in Lemma 3.18. Let us consider the restricted operators $T_j = TP_j$ and $A_j = AP_j$, associated to the partition $\mathcal{I}_j$, for $j = 0, 1$, respectively. If

$$\ker(T_1^* T_1 + \frac{1}{2}A_1^* A_1) = \{0\}, \tag{3.42}$$

then the sequence $(v_\ell)_{\ell\in\mathbb{N}}$ generated by the algorithm is uniformly bounded.

Proof. By minimality of $v_\ell = \arg\min_{v\in\mathcal{H}} \mathcal{J}^{\varepsilon}_{p,\omega,v_{\ell-1}}(v, q_{\ell,L_\ell-1})$, and if we fix $v_\ell^0 := P_0(v_\ell)$, the vector $v_\ell^1 := P_1(v_\ell)$, satisfies

$$v_\ell^1 = \arg\min_{z\in\mathcal{H}^1} \mathcal{J}^1_{\omega,v_{\ell-1}}(z),$$

where

$$\mathcal{J}^1_{\omega,v_{\ell-1}}(z) := \|T_1 z-(g-T_0 v_\ell^0)\|^2+\frac{1}{2}\|A_1 z-(f+q_{\ell,L_\ell-1}-A_0 v_\ell^0)\|^2+\omega\|z-P_1 v_{\ell-1}\|^2+\sum_{i\in\mathcal{I}_1} W_r^{p,\varepsilon}(z_i).$$

As $|(v_\ell^1)_i| > \xi' > r+\varepsilon$, $i \in \mathcal{I}_1$, then v_ℓ^1 is also the unique minimizer of

$$\|T_1 z - (g - T_0 v_\ell^0)\|^2 + \frac{1}{2}\|A_1 z - (f + q_{\ell,L_\ell-1} - A_0 v_\ell^0)\|^2 + \omega\|z - P_1 v_{\ell-1}\|^2, \quad (3.43)$$

or, else, the vector z^* minimizing (3.43) would satisfy $\mathcal{J}^1_{\omega,v_{\ell-1}}(z^*) < \mathcal{J}^1_{\omega,v_{\ell-1}}(v_\ell^1)$, in contradiction to the minimality of v_ℓ^1. The uniqueness of such minimization comes from the strict convexity of (3.43). Hence, v_ℓ^1 is also characterized as the solution of the Euler-Lagrange equations

$$2T_1^*(T_1 z - (g - T_0 v_\ell^0)) + A_1^*(A_1 z - (f + q_{\ell,L_\ell-1} - A_0 v_\ell^0)) + 2\omega(z - P_1 v_{\ell-1}) = 0,$$

or

$$[T_1^* T_1 + \frac{1}{2}A_1^* A_1]v_\ell^1 = T_1^*(g - T_0 v_\ell^0) + \frac{1}{2}A_1^*(f + q_{\ell,L_\ell-1} - A_0 v_\ell^0) + \omega P_1(v_\ell - v_{\ell-1}).$$

Since $\ker(T_1^*T_1 + \frac{1}{2}A_1^*A_1) = \{0\}$ the operator $T_1^*T_1 + \frac{1}{2}A_1^*A_1$ is actually invertible and hence

$$v_\ell^1 = [T_1^*T_1+\frac{1}{2}A_1^*A_1]^{-1}\underbrace{\left[T_1^*(g - T_0 v_\ell^0) + \frac{1}{2}A_1^*(f + q_{\ell,L_\ell-1} - A_0 v_\ell^0) + \omega P_1(v_\ell - v_{\ell-1})\right]}_{:=w_\ell}.$$

We observe now that w_ℓ is uniformly bounded, because, for construction v_ℓ^0 is bounded, by Lemma 3.12 $A_1^* q_{\ell,L_\ell-1}$ is bounded, and by Theorem 2.9 (b) $(v_\ell - v_{\ell-1})$ is also bounded. We conclude that v_ℓ^1 and therefore $v_\ell = v_\ell^0 + v_\ell^1$ are also uniformly bounded. ∎

Having been able to recover the boundedness of the sequence $(v_\ell)_{\ell\in\mathbb{N}}$ thanks to condition (3.42), we are now able to conclude this section with a corresponding convergence result.

THEOREM 3.20. *Assume $\|T\| < 1$, $\frac{1}{\sqrt{2}}\|A\| < 1$, and $\omega < 1$. Moreover, let $\varepsilon_0 > 0$ as in Remark 3.5 and*

$$\varepsilon < \min\left\{\varepsilon_0, \frac{[(5/4)^{1/(p-1)} - 1]}{(5/4)^{1/(p-1)}} r, \frac{1}{3}\frac{\gamma p r^{p-1}}{20}\right\}.$$

Assume that $\mathcal{I}_0$ and $\mathcal{I}_1$ are the index sets produced by the algorithm (2.14) *after fixation as in Lemma 3.18. Let us consider the restricted operators $T_j = TP_j$ and $A_j = AP_j$, associated to the partition $\mathcal{I}_j$, for $j = 0, 1$, respectively. If*

$$\ker(T_1^*T_1 + \frac{1}{2}A_1^*A_1) = \{0\},$$

then the sequence $(v_\ell)_{\ell\in\mathbb{N}}$ generated by the algorithm (2.14) *for $\mathcal{J} = \mathcal{J}_p^\varepsilon$ has at least one accumulation point, and every accumulation point is a constrained critical point of $\mathcal{J}_p^\varepsilon$ on the affine space $\mathcal{F}(f) = \{v \in \mathcal{H} : Av = f\}$.*

Proof. The result follows by a direct application of Theorem 2.10, after having recalled the boundedness of $(v_\ell)_{\ell\in\mathbb{N}}$, which results from Proposition 3.19. □

REMARK 3.21. *Although the condition* $\ker(T_1^*T_1 + \frac{1}{2}A_1^*A_1) = \{0\}$ *depends on the actual realization of the algorithm, in principle the null space of minors of the matrix* $T^*T + \frac{1}{2}A^*A$ *can be checked a priori, identifying the possible admissible configurations of* $\mathcal{I}_1$ *for which the algorithm will be guaranteed to perform convergence. It is also known that for certain random matrices, restricted injectivity properties are actually ensured as soon as* $\mathcal{I}_1$ *is not too large, see for instance [40].*

4. Iterative Thresholding Algorithms Revisited. As already mentioned in Section 3.3 an iterative thresholding algorithm can be used for identifying local minimizers of the Mumford-Shah functional in one dimension [26]. This algorithm is actually very attractive for its exceptional simplicity, and its ability of performing a *separation of components* at a finite number of iterations, as we have showed in Lemma 3.18. This property highlights the identification of possible discontinuities after only a finite number of iterations, in principle allowing then for refinement strategies, when it comes to adaptive discretizations, see [9] for a recent approach to adaptive discretization in fracture simulation. Hence, we would like to see whether an iterative thresholding algorithm can play a profitable role also for free-discontinuity problems in higher dimension.

4.1. Formulation of an iterative thresholding algorithm. In the inner loop of algorithm (2.14), one has to recursively solve a smooth and unconstrained convex optimization of the type

$$v^* = \arg\min_{v\in\mathcal{H}} \mathcal{J}_{\omega,u}(v,q). \tag{4.1}$$

In principle, as $\mathcal{J}_{\omega,u}(v,q)$ is smooth and strictly convex, one can use relatively simple gradient descent methods and this task does not present any particular difficulty. Actually, it does constitute one of the strong features of algorithm (2.14), making it very easily implementable. Nevertheless, as we pointed out in Proposition 3.16, the vector v^* can be point-wise characterized by the following fixed-point equation, for $i = 1,\dots,m$,

$$v_i^* = H_p\left(\frac{1}{3}\left\{[(I - T^*T) + (I - \frac{1}{2}A^*A) + (1-\omega)I]v^* + (T^*g + \frac{1}{2}A^*(f+q) + \omega u)\right\}_i\right). \tag{4.2}$$

Hence, it is natural to wonder whether the corresponding fixed-point iteration

$$v_i^{n+1} = H_p\left(\frac{1}{3}\left\{[(I - T^*T) + (I - \frac{1}{2}A^*A) + (1-\omega)I]v^n + (T^*g + \frac{1}{2}A^*(f+q) + \omega u)\right\}_i\right), \tag{4.3}$$

generates a sequence $(v^n)_{n\in\mathbb{N}}$ which converges to v^*. The next Theorem, which is again based on the separation of components, gives a positive answer to this question.

THEOREM 4.1. *Assume* $\|T\| < 1$, $\frac{1}{\sqrt{2}}\|A\| < 1$, *and* $\omega < 1$. *Moreover, let* $\varepsilon_0 > 0$ *as in Remark 3.5 and*

$$\varepsilon < \min\left\{\varepsilon_0, \frac{[(5/4)^{1/(p-1)} - 1]}{(5/4)^{1/(p-1)}} r, \frac{1}{3}\frac{\gamma p r^{p-1}}{20}\right\}. \tag{4.4}$$

Let $v^* = \arg\min_{v\in\mathcal{H}} \mathcal{J}_{\omega,u}(v,q)$, *and consider the sequence* v^n *defined by the iteration* (4.3) *and its time-dependent partition of the index set* $\mathcal{I} = \{1,\dots,m\}$ *into "small"* *components*

$$\mathcal{I}_0^n := \{i \in \mathcal{I} : |(v_\ell)_i| \leq \xi' - \delta\},$$

and "large" components

$$\mathcal{I}_1^n := \{i \in \mathcal{I} : |(v_\ell)_i| > \xi'\}.$$

Then for $N \in \mathbb{N}$ sufficiently large, this partition fixes during the iteration for $n \geq N$; that is, there exists $\mathcal{I}_0$ such that for all $n \geq N$, $\mathcal{I}_0^n = \mathcal{I}_0$ and $\mathcal{I}_1^n = \mathcal{I} \setminus \mathcal{I}_0$. Moreover for every $m \in \mathbb{N}$ one has

$$\|v^{N+m} - v^*\| \leq (1 - \tfrac{\omega}{3})^m \|v^N - v^*\| \tag{4.5}$$

so that in particular $v^n \to v^*$ as n tends to $+\infty$.

Proof. The proof follows closely similar arguments in [26, Section 4]. First of all, by Proposition 3.2 and Theorem 3.6, if v^n satisfies (4.3), then for every $n \in \mathbb{N}$

$$v^n = \arg\min_{v \in \mathcal{H}} \mathcal{J}^{surr}_{\omega,u}(v, q, v^{n-1}).$$

Now the same argument as in [26, Lemma 4.2] gives that $(v^{n+1} - v^n) \to 0$ as n tends to $+\infty$, whence, arguing exactly as in Lemma 3.18, we deduce the property of fixation of the indexes.

After fixation of the index set $\mathcal{I}_0$, for every $n \geq N$ one has $v^{n+1} = \mathbb{U}_{\mathcal{I}_0}(v^n)$, where $\mathbb{U}_{\mathcal{I}_0}$ is an operator having component-wise action defined by

$$[\mathbb{U}_{\mathcal{I}_0}(v)]_i = F_p^{-1}\left(\frac{1}{3}\left\{[(I - T^*T) + (I - \frac{1}{2}A^*A) + (1-\omega)I]v + (T^*g + \frac{1}{2}A^*(f+q) + \omega u)\right\}_i\right)$$

if $i \in \mathcal{I}_0$, and

$$[\mathbb{U}_{\mathcal{I}_0}(v)]_i = \frac{1}{3}\left\{[(I - T^*T) + (I - \frac{1}{2}A^*A) + (1-\omega)I]v + (T^*g + \frac{1}{2}A^*(f+q) + \omega u)\right\}_i$$

if $i \in \mathcal{I}_1$. The function F_p is here defined as in Proposition 3.2. Using the hypotheses, it is easy to show that $\|\frac{1}{3}[(I - T^*T) + (I - \frac{1}{2}A^*A) + (1-\omega)I]\| \leq 1 - \frac{\omega}{3}$, therefore, since the mapping F_p^{-1} is nonexpansive, we get that

$$\mathrm{Lip}(\mathbb{U}_{\mathcal{I}_0}) \leq 1 - \tfrac{\omega}{3}\,;$$

in particular, $\mathbb{U}_{\mathcal{I}_0}$ is a contraction mapping. By Banach fixed point Theorem, we infer that (4.5) holds with v^* the unique fixed point of $\mathbb{U}_{\mathcal{I}_0}$. It follows that v^* satisfies (4.2), so that the proof is concluded by using Proposition 3.16. □

REMARK 4.2. *In view of Theorem 4.1, the algorithm* (2.14) *applied for $\mathcal{J} = \mathcal{J}_p^\varepsilon$ depends on ε through the choice of $\omega = \omega(\varepsilon)$ to make $\mathcal{J}^\varepsilon_{p,\omega,u}$ convex. Actually $\omega = \omega(\varepsilon)$ grows with $\varepsilon \to 0$. In view of the necessary rescaling, this determines a deterioration of the convergence quality of the algorithm. In particular the iteration N in* (4.5)*, from which the algorithm fixes the index sets $\mathcal{I}_0$ and $\mathcal{I}_1$ and starts to converge exponentially fast, might get delayed. Hence, while $\varepsilon > 0$ cannot be chosen too large according to* (4.4)*, it should also not be chosen too small. A more detailed analysis of the dependencies on ε of the convergence properties of the algorithm will be explored in a successive numerical analysis work.*

5. Appendix.

5.1. Proof of Theorem 3.6. In order to achieve the proof, we need two preliminary lemmas. Let us start with a simple technical result.

LEMMA 5.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be of class C^1 and $g : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Let $0 < \varepsilon < r$ and we assume*

1) g has a unique minimizer $\bar{x}$ in $\mathbb{R}$;
2) the minimizer $\bar{x} \notin [-r-\varepsilon, -r+\varepsilon] \cup [r-\varepsilon, r+\varepsilon]$;
3) $f = g$ in $\mathbb{R} \setminus \{(-r-\varepsilon, -r+\varepsilon) \cup (r-\varepsilon, r+\varepsilon)\}$;
4) f' is decreasing in $(-r-\varepsilon, -r+\varepsilon) \cup (r-\varepsilon, r+\varepsilon)$.

Then $\bar{x}$ is also the unique global minimizer for f.

Proof. If $x \in \mathbb{R} \setminus \{(-r-\varepsilon, -r+\varepsilon) \cup (r-\varepsilon, r+\varepsilon)\}$ then we have

$$f(x) = g(x) \geq g(\bar{x}) = f(\bar{x}),$$

with equality only if $x = \bar{x}$. If $x \in (r-\varepsilon, r+\varepsilon)$ and $f'(x) = 0$, by 4) we would immediately obtain by the Fundamental Theorem of Calculus that

$$f(x) \geq \min\{f(r-\varepsilon), f(r+\varepsilon)\}.$$

Actually the same holds if $f' \neq 0$ in $(r-\varepsilon, r+\varepsilon)$, as it follows directly by the Weierstrass Theorem. Hence, it follows

$$f(x) \geq \min\{f(r-\varepsilon), f(r+\varepsilon)\} = \min\{g(r-\varepsilon), g(r+\varepsilon)\} > g(\bar{x}) = f(\bar{x}).$$

Analogously in the case $x \in (-r-\varepsilon, -r+\varepsilon)$. □

In the following Lemma we address the concavity of polynomials π of third degree as in Lemma 3.4.

LEMMA 5.2. *Referring to the notations of Lemma 3.4, let $s_1 = (r-\varepsilon)$, $s_2 = (r+\varepsilon)$, $\gamma_1 = p(r-\varepsilon)^{p-1}$, $\gamma_2 = (r-\varepsilon)^p$, and $\gamma_3 = r^p$. Let us set $\delta = (\frac{5}{4})^{\frac{1}{p-1}} - 1$, and suppose $\varepsilon \leq \frac{\delta}{1+\delta} r$. Then for all $t \in (r-\varepsilon, r+\varepsilon)$, we have:*

$$\pi''(t) \leq -\frac{p}{10\varepsilon} r^{p-1}.$$

Proof. In this case we have

$$B = \frac{p(r-\varepsilon)^{p-1}}{2\varepsilon} + \frac{3}{4\varepsilon^2}[(r-\varepsilon)^p - r^p], \tag{5.1}$$

$$A = \frac{p(r-\varepsilon)^{p-1}}{12\varepsilon^2} + \frac{B}{3\varepsilon}. \tag{5.2}$$

Since $(r-\varepsilon)^p - r^p \leq -\varepsilon p(r-\varepsilon)^{p-1}$ by convexity, we deduce from (5.1) that

$$B \leq -\frac{p(r-\varepsilon)^{p-1}}{4\varepsilon},$$

and therefore, from (5.2) we have also $A \leq 0$. But then for all $t \in (r-\varepsilon, r+\varepsilon)$, we deduce from these negativity relationships and again (5.2) that

$$\pi''(t) = 6A(t-(r+\varepsilon)) + 2B \leq -12\varepsilon A + 2B \leq -\frac{p(r-\varepsilon)^{p-1}}{\varepsilon} - 2B. \tag{5.3}$$

By the mean value theorem there exists $\xi \in (r-\varepsilon, r)$ such that

$$r^p - (r-\varepsilon)^p = \varepsilon p \xi^{p-1} \le \varepsilon p r^{p-1},$$

and therefore, from (5.1)

$$B \ge \frac{p(r-\varepsilon)^{p-1}}{2\varepsilon} - \frac{3}{4\varepsilon} p r^{p-1}.$$

By substituting this latter estimate in (5.3) we obtain

$$\pi''(t) \le -\frac{2p(r-\varepsilon)^{p-1}}{\varepsilon} + \frac{3}{2\varepsilon} p r^{p-1} \tag{5.4}$$

Now, $\varepsilon \le \frac{\delta}{1+\delta} r$ if and only if $0 \le \delta(r-\varepsilon) - \varepsilon$ if and only if $r \le (1+\delta)(r-\varepsilon)$ or $r^{p-1} \le (1+\delta)^{p-1}(r-\varepsilon)^{p-1}$. For the chosen value of δ, we obtain $r^{p-1} \le \frac{5}{4}(r-\varepsilon)^{p-1}$ or $-(r-\varepsilon)^{p-1} \le -\frac{4}{5} r^{p-1}$. Substituting the latter estimate in (5.4), we eventually have

$$\pi''(t) \le -\frac{8p}{5\varepsilon} r^{p-1} + \frac{3p}{2\varepsilon} r^{p-1} = -\frac{p}{10\varepsilon} r^{p-1}.$$

□

We can now give the proof of Theorem 3.6. We intend to apply Lemma 5.1. From Remark 3.5, by construction and assumption $\varepsilon < \varepsilon_0$, both $f = f^{\xi}_{\gamma,r}$ and $g = g^{\xi}_{\gamma,r}$ verify the assumptions 1), 2), and 3) of Lemma 2.2. In order to show that also assumption 4) holds, it is sufficient to show that $(f^{\xi}_{\gamma,r})''(t) < 0$ for all $t \in (-r-\varepsilon, -r+\varepsilon) \cup (r-\varepsilon, r+\varepsilon)$. As $\frac{d^2}{dt^2}(t-\xi)^2 \equiv 2$ and by parity of $W^{p,\varepsilon}_r$, it is sufficient to check the negativity of $(f^{\xi}_{\gamma,r})''(t)$ for all $t \in (r-\varepsilon, r+\varepsilon)$. In this interval, by Lemma 5.2 and construction of $f^{\xi}_{\gamma,r}$ we have

$$(f^{\xi}_{\gamma,r})''(t) = 2 + \gamma \pi''(t) \le 2 - \frac{\gamma p}{10\varepsilon} r^{p-1} < 0,$$

where we used the assumption (3.24).

5.2. Proof of Theorem 3.7. This proof uses a similar approach as for [26, Theorem 2.3]. Let us first consider a partition $\mathscr{P} = \{\mathcal{U}_{\mathcal{I}_0^j}\}_{j=1}^{2^m}$ of $\mathcal{H}$ indexed by all subsets $\mathcal{I}_0^j \subset \mathcal{I}$, as follows

$$\mathcal{U}_{\mathcal{I}_0^j} = \{v \in \mathcal{H} : |v_i| \le r+\varepsilon, i \in \mathcal{I}_0^j, |v_i| > r+\varepsilon, i \in \mathcal{I} \setminus \mathcal{I}_0^j\}.$$

The minimization of $\mathcal{J}_p^{\varepsilon}$ over $\mathcal{F}(f) \cap \mathcal{U}_{\mathcal{I}_0^j}$ can be reformulated as

$$\begin{cases} \text{Minimize } \bar{\mathcal{J}}_p^{\varepsilon}(v) = \|Tv - g\|^2 + \gamma \sum_{i=1}^m c_i \bar{W}^{p,\varepsilon}_r(v_i), \text{ Subject to } v \in \mathcal{F}(f) \cap \mathcal{U}_{\mathcal{I}_0^j}, \\ c_i = 0 \text{ if } i \in \mathcal{I} \setminus \mathcal{I}_0^j \text{ and } c_i = 1 \text{ if } i \in \mathcal{I}_0^j, \end{cases} \tag{5.5}$$

where

$$\bar{W}^{p,\varepsilon}_r(t) = \begin{cases} t^p, & t \le r-\varepsilon, \\ \pi_p(t), & r-\varepsilon \le t \le r+\varepsilon, \\ |t-\varepsilon|^p & t \ge r+\varepsilon. \end{cases}$$

If we prove that the minimization (5.5) has always a solution $v(\mathcal{I}_0^j)$ for all $j = 1,\dots,2^m$, and such a minimizer belongs to a compact set M^j, independent of $\varepsilon \geq 0$, then

$$v^* = \arg \min_{j=1,\dots,2^m} \mathcal{J}_p^\varepsilon(v(\mathcal{I}_0^j)),$$

is actually a solution for (3.26) and it belongs to the compact set $M = \cup_{j=1}^{2^m} M^j$, independent of $\varepsilon \geq 0$. Hence, it is sufficient now to address (5.5). For that, we first show the following technical observation:

If $x, v \in \mathcal{H}$ are fixed and $\bar{\mathcal{J}}_p^\varepsilon$ is bounded above and below on the ray $R_{x,v} = \{x + tv, t \geq 0\}$, then $\bar{\mathcal{J}}_p^\varepsilon$ is actually constant on $R_{x,v}$. In fact, let us consider the function $\mu(t) = \bar{\mathcal{J}}_p^\varepsilon(x + tv)$. By the boundedness of $\bar{\mathcal{J}}_p^\varepsilon(x + tv)$, without loss of generality, we can assume that $0 \leq \mu(t) \leq 1$. Hence there exists a sequence $(t_n)_n \subset \mathbb{R}^+$ of points $t_n \to +\infty$ for $n \to \infty$ such that $\mu(t_n) \to \eta \in [0, 1]$ for $n \to \infty$. Moreover, by definition of $\bar{W}_r^{p,\varepsilon}$, for $t > 0$ sufficiently large we have actually the general expression $\mu(t) = P(t) + \gamma \sum_{i=1}^m c_i|x_i - \varepsilon + tv_i|^p$, where P is a polynomial of degree at most 2. Assume now, for instance, that $1 \leq p \leq 2$. As $\lim_n \frac{\mu(t_n)}{t_n^2} = 0$ we deduce that all the coefficients in P of second degree are actually vanishing. In turn, then $0 = \lim_n \frac{\mu(t_n)}{|t_n|^p}$ has the implication that for each i one of the coefficients c_i or d_i must vanish as well. Following in the same manner, we conclude that all linear coefficients in $\mu(t)$ also vanish, leaving only the possibility that $\mu(t)$ is a constant function. A similar approach can be conducted to prove the observation also for $p > 2$.

Notice now that $\bar{\mathcal{J}}_p^\varepsilon$ converges uniformly to $\bar{\mathcal{J}}_p$ on $\mathcal{U}_{\mathcal{I}_0^j}$ for $\varepsilon \to 0$, as defined in (3.6), or

$$|\bar{\mathcal{J}}_p^\varepsilon(v) - \bar{\mathcal{J}}_p(v)| \leq \Gamma(\varepsilon), \quad \text{for all } v \in \mathcal{U}_{\mathcal{I}_0^j}, \tag{5.6}$$

for a continuous function $\Gamma(\varepsilon) = o(\varepsilon)$, $\varepsilon \to 0$. By Remark 3.1, for $X = \mathcal{F}(f) \cap \overline{\mathcal{U}_{\mathcal{I}_0^j}}$ and any $v^0 \in X$, there exists a linear subspace $\mathcal{V} \subset \mathcal{H}$, such that the orthogonal projection $X^\perp$ of X onto $\mathcal{V}^\perp$ has the properties

- $X = \{x = x^\perp \oplus tv : x^\perp \in X^\perp, v \in \mathcal{V}, t \in \mathbb{R}^+\}$,
- $M_C^j = X^\perp \cap \{v \in \mathcal{H} : \bar{\mathcal{J}}_p(v) \leq C\}$, for $C \geq \bar{\mathcal{J}}_p^\varepsilon(v^0) + \Gamma(\varepsilon)$ is compact, and
- $\bar{\mathcal{J}}_p(\xi_t)$ is constant along rays $\xi_t = x^\perp \oplus tv$, where $x^\perp \in M_C^j$, $v \in \mathcal{V}$, and $t \in \mathbb{R}^+$.

By the uniform estimate (5.6) and the last property, we deduce that $\bar{\mathcal{J}}_p^\varepsilon(\xi_t)$ is bounded from above and below by $\bar{\mathcal{J}}_p(x^\perp) \pm \Gamma(\varepsilon)$ on rays $\xi_t = x^\perp \oplus tv$, where $x^\perp \in M_C$, $v \in \mathcal{V}$, and $t \in \mathbb{R}^+$. Hence, we conclude that $\bar{\mathcal{J}}_p^\varepsilon(\xi_t)$ is also constant for $t \geq 0$. From (5.6), the set

$$X^\perp \cap \{v \in \mathcal{H} : \bar{\mathcal{J}}_p^\varepsilon(v) \leq \bar{\mathcal{J}}_p^\varepsilon(v^0)\},$$

is included in M_C^j, and

$$\inf_{v \in \mathcal{F}(f) \cap \mathcal{U}_{\mathcal{I}_0^j}} \bar{\mathcal{J}}_p^\varepsilon(v) = \inf_{v \in M_C^j} \bar{\mathcal{J}}_p^\varepsilon(v).$$

By compactness of $M^j = M_C^j$ and continuity of $\bar{\mathcal{J}}_p^\varepsilon$ we conclude the existence of minimizers in M^j. As pointed out above, this further implies the existence of minimal solutions in $M = \cup_{j=1}^{2^m} M^j$ of the original problem (3.26). Notice further that, by continuity of $\bar{\mathcal{J}}_p^\varepsilon(v^0) + \Gamma(\varepsilon)$ with respect to ε, the sets $M^j = M_C^j$ actually do not depend on $0 \leq \varepsilon < \varepsilon_0$ as soon as $C \geq \max_{0<\varepsilon<\varepsilon_0} \bar{\mathcal{J}}_p^\varepsilon(v^0) + \Gamma(\varepsilon)$ is large enough.

REFERENCES

[1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free-Discontinuity Problems.* Oxford Mathematical Monographs. Oxford: Clarendon Press. xviii, 2000.

[2] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures. 2nd ed.* Basel: Birkhäuser, 2008.

[3] L. Ambrosio and V. M. Tortorelli. Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence. *Commun. Pure Appl. Math.*, 43(8):999–1036, 1990.

[4] Y. Au-Yeung, G. Friesecke, and B. Schmidt. Minimizing atomic configurations of short range pair potentials in two dimensions: crystallization in the Wulff shape. *Calc. Var. PDE*, to appear.

[5] A. Blake and A. Zisserman. *Visual Reconstruction.* MIT Press, 1987.

[6] T Blumensath and M. E. Davies. Iterative hard thresholding for compressed sensing. *Appl. Comput. Harmon. Anal.*, 27(3):265–274, 2009.

[7] B. Bourdin. Numerical implementation of the variational formulation for quasi-static brittle fracture. *Interfaces Free Bound.*, 9(3):411–430, 2007.

[8] B. Bourdin and A. Chambolle. Implementation of an adaptive finite-element approximation of the Mumford-Shah functional. *Numer. Math.*, 85(4):609–646, 2000.

[9] S. Burke, C. Ortner, and E. Süli. An adaptive finite element approximation of a variational model of brittle fracture. *SIAM J. Numer. Anal.*, 48(3):980–1012, 2010.

[10] E. J. Candès, J. K. Romberg, and T. Tao. Stable signal recovery from incomplete and inaccurate measurements. *Commun. Pure Appl. Math.*, 59(8):1207 – 1223, 2006.

[11] A. Chambolle and G. Dal Maso. Discrete approximation of the Mumford-Shah functional in dimension two. *M2AN, Math. Model. Numer. Anal.*, 33(4):651–672, 1999.

[12] A. Chambolle, R. A. DeVore, N. Lee, and B. J. Lucier. Nonlinear wavelet image processing: variational problems, compression, and noise removal through wavelet shrinkage. *IEEE Trans. Image Process.*, 7(3):319–335, 1998.

[13] F. H. Clarke. *Optimization and nonsmooth analysis.* New York: John Wiley, and sons, 1983.

[14] P. L. Combettes and V. R. Wajs. Signal recovery by proximal forward-backward splitting. *Multiscale Model. Simul.*, 4(4):1168–1200, 2005.

[15] G. Dal Maso. *An Introduction to Γ-Convergence.* Birkhäuser, Boston, 1993.

[16] I. Daubechies, M. Defrise, and C. De Mol. An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. *Commun. Pure Appl. Math.*, 57(11):1413–1457, 2004.

[17] E. De Giorgi. Free-discontinuity problems in calculus of variations. In R. Dautray, editor, *Frontiers in pure and applied mathematics, a collection of papers dedicated to J.-L. Lions on the occasion of his* 60^{th} *birthday*, pages 55–62. North Holland, 1991.

[18] D. L . Donoho. Compressed sensing. *IEEE Transactions on Information Theory*, 52(4):1289–1306, 2006.

[19] D. L. Donoho and I. M. Johnstone. Ideal spatial adaptation by wavelet shrinkage. *Biometrika*, 81(3):425–455, 1994.

[20] S. Durand and M. Nikolova. Stability of the minimizers of least squares with a non-convex regularization. I: Local behavior. *Appl. Math. Optimization*, 53(2):185–208, 2006.

[21] S. Durand and M. Nikolova. Stability of the minimizers of least squares with a non-convex regularization. II: Global behavior. *Appl. Math. Optimization*, 53(3):259–277, 2006.

[22] I. Ekeland and R. Temam. *Convex Analysis and Variational Problems. Translated by Minerva Translations, Ltd., London.* Studies in Mathematics and its Applications. Vol. 1. Amsterdam - Oxford: North-Holland Publishing Company; New York: American Elsevier Publishing Company, Inc., 1976.

[23] M. A. T. Figueiredo and R. D. Nowak. Wavelet-based image estimation: An empirical Bayes approach using Jeffrey's noninformative prior. *IEEE Trans. Image Process.*, 10(9):1322–1331, 2001.

[24] M. Fornasier, R. March, and F. Solombrino. Existence of minimizers of the mumford-shah functional with singular operators and unbounded data. *Annali Mat. Pura App.*, doi:10.1007/s10231-011-0228-8, 2012.

[25] M. Fornasier and H. Rauhut. Iterative thresholding algorithms. *Appl. Comput. Harmon. Anal.*, 25(2):187–208, 2008.

[26] M. Fornasier and R. Ward. Iterative thresholding meets free-discontinuity problems. *Found. Comput. Math.*, 10(5):527–567, 2010.

[27] G.A. Francfort and J.-J. Marigo. Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids*, 46(8):1319–1342, 1998.

[28] K. Frick and O. Scherzer. Regularization of ill-posed linear equations by the non-stationary augmented Lagrangian method. *J. Integral Equations Appl.*, 22(2):217–257, 2010.

[29] S. Geman and D. Geman. Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images. *IEEE Trans. Pattern Anal. Mach. Intell*, 6:721–741, 1984.

[30] K. Ito and K. Kunisch. *Lagrange multiplier approach to variational problems and applications.* Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM), 2008.

[31] S. Kirkpatrick, C.D. jun. Gelatt, and M.P. Vecchi. Optimization by simulated annealing. *Science*, 220(4598):671–680, 1983.

[32] P.-L. Lions and B. Mercier. Splitting algorithms for the sum of two nonlinear operators. *SIAM J. Numer. Anal.*, 16:964–979, 1979.

[33] A. Mielke. Evolution of rate-independent inelasticity with microstructure using relaxation and Young measures. Dordrecht: Kluwer Academic Publishers, 2003.

[34] D. Mumford and J. Shah. Optimal approximation by piecewise smooth functions and associated variational problems. *Commun. Pure Appl. Math.*, 42:577–684, 1989.

[35] M. Nikolova. Markovian reconstruction using a GNC approach. *IEEE Transactions on Image Processing*, 8(9):1204–1220, 1999.

[36] M. Nikolova. Thresholding implied by truncated quadratic regularization. *IEEE Trans. Signal Process.*, 48:3437–3450, 2000.

[37] M. Nikolova, J. Idier, and A. Mohammad-Djafari. Inversion of large-support ill-posed linear operators using a piecewise Gaussian MRF. *IEEE Trans. Image Process.*, 7(4):571–585, 1998.

[38] J. Nocedal and S. J. Wright. *Numerical optimization. 2nd ed.* New York, NY: Springer, 2006.

[39] S. Osher, M. Burger, D. Goldfarb, J. Xu, and W. Yin. An iterative regularization method for total variation-based image restoration. *Multiscale Model. Simul.*, 4(2):460–489, 2005.

[40] H. Rauhut. Compressive sensing and structured random matrices. In Massimo Fornasier, editor, *Theoretical Foundations and Numerical Methods for Sparse Recovery.* Radon Series on Computational and Applied Mathematics 9, De Gruyter, 2010.