Modeling and Simulation with ODE for MSE

The exercises should be handed in during the week 06.11. – 10.11.!

Exercise 1 (An IVP)
In dependence of $y_0 \in \mathbb{R}$ determine the solution of the Initial value problem

$$xy'(x) + y^2(x) + 1 = 0, \quad y(1) = y_0,$$

and its maximal domain of definition.

Solution:
Even before starting to try to solve the ODE, we can immediately conclude that the point $x = 0$ will be troublesome: Any point where one or more of the (non-constant) coefficients vanish will necessarily be of special interest, and in our case we can directly see that no real-valued solution $y(x)$ can fulfill the equation for $x = 0$, since here the equation reduces to $y^2(0) + 1 = 0$. Since we have real coefficients and a real initial condition, we are only interested in real-valued solutions. Altogether this means that $x = 0$ necessarily lies outside the domain of definition of our solution.

Now let’s come to solving the ODE. Though perhaps not obvious from a first glance, it is not hard to rewrite the equation so that we can identify it as an ODE with separable variables: We find

$$xy' + y^2 + 1 = 0 \iff xy' = -y^2 - 1 \iff \frac{y'}{y^2 + 1} = -\frac{1}{x}.$$

Integrating both sides leads to

$$\arctan(y(x)) = -\ln |x| + c, \quad c \in \mathbb{R},$$

hence our solution is

$$y(x) = \tan(c - \ln |x|) = \tan \ln \frac{C}{|x|}, \quad C > 0.$$

Inserting the initial condition $y(1) = y_0$, we obtain

$$y_0 = y(1) = \tan \ln C \iff C = e^{\arctan y_0}.$$

Altogether we have found

$$y(x) = \tan(\arctan y_0 - \ln |x|) = \frac{y_0 - \tan \ln |x|}{1 + y_0 \tan \ln |x|}.$$

The last step follows from an addition formula for the tangent,

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}.$$
What is the maximal domain (interval) of definition for this solution? First note that any such interval must contain the point \( x = 1 \) from the initial condition (otherwise it may still be a solution of the ODE on that interval, but there is no longer any connection to the IVP). Then any interval of definition already can only be a subinterval of the positive real numbers (as we already know the solution can’t be defined in 0, and of course \( \ln |0| \) fails to be defined). Next, we must look at the original version of the solution, not the last one, since simplifications may introduce new problems (e.g. the addition formula is not valid for arbitrary choices of \( \alpha \) and \( \beta \); generally one needs to be careful with “simplifications” in case we have a restricted domain of definition).

To get the domain of definition we recall that the tangent has definition gaps at \( \frac{\pi}{2} + k\pi \), hence \( \arctan y_0 - \ln |x| \) must belong to some interval \( I_k = (-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi) \). Since 1 must belong to our domain, this fixes \( k = 0 \), since \( I_0 \) is the natural range of values for \( \arctan y_0 \). Hence the domain of definition for our solution turns out to be

\[
D = \left( \exp(\arctan y_0 - \frac{\pi}{2}), \exp(\arctan y_0 + \frac{\pi}{2}) \right)
\]

(this can be seen by backward verification: we indeed have \( \arctan y_0 - \ln x \in I_0 \), and in view of the range of \( \arctan \) we have \( \arctan y_0 - \frac{\pi}{2} < 0 < \arctan y_0 + \frac{\pi}{2} \), thus \( e^0 = 1 \in D \)).

Exercise 2 (Substitution)

(a) Solve the initial value problem

\[
(3x^2y^2(x) - 2y(x)) - xy'(x) = 0, \quad y(1) = 1
\]

via the substitution \( u(x) = x^2y(x) \).

**Solution:**

With the suggested substitution we obtain

\[
\left( 3x^2 \frac{u^2(x)}{x^4} - 2 \frac{u(x)}{x^2} \right) - x \frac{x^2u'(x) - 2xu(x)}{x^4} = 0, \quad u(1) = 1.
\]

Simplifying gives

\[
u'(x) = \frac{3}{x} u^2(x), \quad u(1) = 1.
\]

We continue with separation of variables and integrate:

\[
\int_{\xi=1}^{x} \frac{u'\xi d\xi}{u^2\xi} = 3 \int_{\xi=1}^{x} \frac{d\xi}{\xi} \implies -\frac{1}{u(x)} + 1 = 3 \ln(x).
\]

As a solution of the original initial value problem we obtain

\[
y(x) = \frac{u(x)}{x^2} = \frac{1}{x^2(1 - 3 \ln(x))}, \quad 0 < x < \exp\left(\frac{1}{3}\right).
\]

(b) Find a suitable substitution to solve the ODE

\[
y'(x) = \sin(x + y(x)), \quad y(0) = 0.
\]

**Solution:**
If we want to have a chance to separate variables, our only choice is the substitution
\[ v(x) = x + y(x) \]
(we could try addition formulas, but they would not be of help here). Then we immediately obtain
\[ v'(x) = 1 + y'(x), \]
hence the original ODE is equivalent to
\[ v'(x) - 1 = \sin v(x) \]
with initial value \( v(0) = 0 + y(0) = 0 \). This is indeed an equation with separable variables,
\[ \frac{v'(x)}{1 + \sin v(x)} = 1. \]
We first need to discuss (and exclude) the roots of the denominator, which lead to the special solutions
\[ y_k(x) = -\frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}, \]
of the differential equation. However, none of those fulfills the initial condition. We now want to integrate both sides of the equation. For this we shall use the standard substitution \( u = \tan \frac{v}{2} \). We insert
\[ \sin v = 2 \sin \frac{v}{2} \cos \frac{v}{2} = 2 \tan \frac{v}{2} \cos \frac{v}{2} = 2u \frac{1}{\sin^2 \frac{v}{2} + \cos^2 \frac{v}{2}} = \frac{2u}{1 + u^2}. \]
as well as \( dv = \frac{2}{1+u^2}du \), which results in
\[
\int \frac{v'(x)}{1 + \sin v(x)} dx = \int \frac{1}{1 + \sin v} dv = \int \frac{2}{1 + u^2} du = 2 \int \frac{1}{(1+u)^2} du = -\frac{2}{1+u} + c = \frac{-2}{1 + \tan \frac{v(x)}{2}} + c.
\]
With addition formulas this simplifies to \( \tan \left( \frac{v}{2} - \frac{\pi}{4} \right) + c \).
We thus have found the equation
\[ x = \tan \left( \frac{v(x)}{2} - \frac{\pi}{4} \right) + c. \]
Before we solve for \( v(x) \) we determine \( c \) from the initial condition. Inserting \( x = 0 \) and \( v(0) = 0 \) gives
\[ 0 = \tan \left( -\frac{\pi}{4} \right) + c \iff c = 1. \]
Now solving the previous equation for \( v(x) \) yields
\[ v(x) = 2 \arctan(x - 1) + \frac{\pi}{2}, \]
and hence
\[ y(x) = v(x) - x = 2 \arctan(x - 1) + \frac{\pi}{2} - x. \]
Exercise 3 Solve the problem
\[2x\sqrt{ax} - x^2 y'(x) + a^2 + y^2(x) = 0, \quad y(a) = 0,\]
with a parameter \(a \in \mathbb{R}\).

Solution:
We first note that, depending on \(a\), we again immediately get a restriction of the domain of definition of any potential solution, without having to do any calculations towards solving the equation. Since also here we have an equation with real coefficients and real initial condition, we are primarily interested in real-valued solutions. But for this purpose, particularly the square root needs to be well-defined in \(\mathbb{R}\). This leads to the restriction \(x \in I = (0, a)\) for \(a > 0\) and \(x \in I = (a, 0)\) for \(a < 0\) (recall that we need open intervals to define derivatives in the usual way). In case \(a = 0\) this means there can’t exist any solution, even though \(y = 0\) would formally solve the ODE and satisfy the initial condition.

Particularly note that in this case formally the initial point does NOT belong to the domain of definition of the solution of the ODE. In such a case the initial condition is to be understood in a limiting sense (with one-sided limits), i.e. for \(a > 0\) it should hold
\[\lim_{x \to a} y(x) = 0,\]
accordingly for \(a < 0\).

Now let \(x \in I\). Then the ODE can be re-written as
\[y'(x) = -(a^2 + y^2(x)) \frac{1}{2x\sqrt{ax} - x^2},\]
and thus can be identified as an equation with separable variables. The integrations now yield
\[
\int \frac{y'(x)}{a^2 + y^2(x)} dx = \frac{1}{a} \int \frac{1}{1 + v^2} dv = \frac{1}{a} \arctan v = \frac{1}{a} \arctan \frac{y(x)}{a}
\]
with a substitution \(v = \frac{y(x)}{a}\), as well as
\[
\int \frac{1}{2\sqrt{\xi} \sqrt{a^2 - \xi^2}} d\xi = -\int \frac{1}{2a\sqrt{a^2 - u^2}} (-\frac{a}{u^2}) du = \frac{1}{|a|} \int \frac{1}{2\sqrt{u - 1}} du
\]
\[= \frac{1}{|a|} \sqrt{u - 1} + c = \frac{1}{|a|} \sqrt{\frac{a^2}{x} - 1} + c,
\]
where we used the substitution \(u = \frac{a^2}{x}\). For the solution we hence ultimately find
\[y(x) = a \tan \left(\frac{a}{|a|} \sqrt{\frac{a^2}{x} - 1} + C\right) = |a| \tan \left(\sqrt{\frac{a^2}{x} - 1} + C'\right).
\]
It now remains to insert the initial condition. We find
\[0 = y(a) = |a| \tan(C') \iff C' = 0,
\]
and hence the solution of the IVP is given by
\[y(x) = |a| \tan \left(\sqrt{\frac{a^2}{x} - 1}\right), \quad x \in I.
\]
Exercise 4 (Another initial value problem, but a little tricky)
Determine the general solution of the ODE
\[ xy' = y^2 + y. \]
Further solve the initial value problem with initial value \( y(0) = 1. \)

**Solution:**
The equation can be written in explicit form as
\[ y' = \frac{y^2 + y}{x}, \]
but this only works for \( x \neq 0. \) This is once more a separable ODE. As before we have to take care of potential problems with denominators first. In particular, we have to exclude the constant solutions \( y_1(x) = 0 \) and \( y_2(x) = -1, \) where 0 and -1 are the roots of \( y^2 + y. \) Again as before, we therefore may expect to have to consider cases \( y < -1, -1 < y < 0 \) and \( y > 0, \) as well as potential limits to the range of definition at \( x = 0. \)
The solution of the ODE begins straightforward: We have
\[
\int \frac{dy}{y^2 + y} = \int \frac{dx}{x},
\]
and with \( \frac{1}{y^2 + y} = \frac{1}{y} - \frac{1}{y+1} \) we immediately obtain
\[
\ln \left| \frac{y}{y+1} \right| = \ln |x| + c,
\]
and after exponentiation
\[
\left| \frac{y}{y+1} \right| = C |x|
\]
with \( C = e^c > 0. \) At this point, the mentioned cases come into play. Let’s consider first the case \( y < -1 \) or \( y > 0, \) in which case on the left-hand side both nominator and denominator are of the same sign. Then we obtain
\[
C|x| = \frac{y}{y+1} = 1 - \frac{1}{y+1} \iff y_I = -1 + \frac{1}{1 - C|x|} = \frac{C|x|}{1 - C|x|}.
\]
This function solves the ODE for \( 0 < x < \frac{1}{C} \) (this corresponds to \( y > 0) \) or \( |x| > \frac{1}{C} \) (then we have \( y < -1) \).
Similarly, in case \(-1 < y < 0\) we find
\[
\frac{y}{y+1} = -C|x| \iff y_{II} = -1 + \frac{1}{1 + C|x|} = \frac{C|x|}{1 + C|x|},
\]
which solves the ODE for all \( x \neq 0. \)
Now what about the initial condition? Since we had to exclude the point \( x = 0 \) before due to our method of solving the ODE, we now have to investigate whether any of the solutions can be extended to a continuously differentiable function also in \( x = 0, \) and which values are possible at all. Clearly, both functions admit a continuous extension to \( x = 0, \) as the respective limit exists
\[
\lim_{x \to 0} y_I(x) = \lim_{x \to 0} y_{II}(x) = 0,
\]
independent of \( C > 0. \) Since obviously also the constant solution \( y_1 \) and \( y_2 \) do not match the required initial value \( y(0) = 1, \) there exists no solution to the posed initial valued problem.
Information and material related to the lecture can be found at the lecture webpage

http://www-m15.ma.tum.de/Allgemeines/ModelingSimulation