Modeling and Simulation with ODE for MSE

The exercises can be handed in during the week 15.01. – 19.01. 2018

Exercise 1 (Parameter multistep method)
Let \( a \in \mathbb{R} \) and consider the following family of multistep methods:

\[
y_{n+2} + \alpha_1 y_{n+1} + a y_n = h (\beta_2 f(t_{n+2}, y_{n+2}) + \beta_1 f(t_{n+1}, y_{n+1}) + \beta_0 f(t_n, y_n)).
\]

Determine \( \alpha_1, \beta_0, \beta_1, \beta_2 \in \mathbb{R} \) such that the family of multistep methods has order of consistency 4, i.e.,

\[
|y_{n+2} - y(t_{n+2})| = \mathcal{O}(h^4)
\]

provided all data from previous iteration steps are exact, i.e. \( y(t_k) = y_k \) for all \( k < n + 2 \). For simplicity, one may also assume \( f(t_{n+2}, y_{n+2}) = f(t_{n+2}, y(t_{n+2})) \).

Solution: For multistep methods there are basically three strategies to determine their order of consistency: first via Taylor expansion, second via the characteristic polynomials \( \rho \) and \( \sigma \), and third via order conditions. In each of the following exercises we shall demonstrate one of those methods.

Here we shall estimate the consistency error directly, with the help of Taylor expansions (note that using suitable Taylor expansions is a method which works for every type of numerical scheme – be it basic single-step method like in the last exercise, multistep methods or Runge-Kutta methods, whereas order conditions or other approaches are tailored towards the specific type of method).

Recall that to determine the consistency error of a numerical scheme (the systematic error which remains even upon replacing the iterations steps \( y_n \) by the corresponding data from the exact solution \( y(t_n) \); and thus the error incurred in one iteration step even upon the assumption that all previous data had been exact), we consider the difference \( y(t_{n+2}) - y_{n+2} \), and then replace all data from previous iterations by their counterpart from the exact solution. Explicitly, this means we have to estimate

\[
y(t_{n+2}) + \alpha_1 y(t_{n+1}) + a y(t_n) - h \beta_2 f(t_{n+2}, y(t_{n+2})) - h \beta_1 f(t_{n+1}, y(t_{n+1})) - h \beta_0 f(t_n, y(t_n)).
\]

In the second line we now can invoke the ODE, to find \( f(t_{n+2}, y(t_{n+2})) = y'(t_{n+2}) \), so that we need to consider

\[
y(t_{n+2}) + \alpha_1 y(t_{n+1}) + a y(t_n) - h \beta_2 y'(t_{n+2}) - h \beta_1 y'(t_{n+1}) - h \beta_0 y'(t_n). \quad (1)
\]

The main idea now is to use Taylor expansions to link all appearing terms of \( y \) and \( y' \) to only one point, e.g. to \( t_n \). In that regards we first obtain

\[
y(t_{n+2}) = y(t_n + 2h) = y(t_n) + 2h y'(t_n) + \frac{(2h)^2}{2} y''(t_n) + \frac{(2h)^3}{6} y'''(t_n) + \mathcal{O}(h^4).
\]
Therein we stop after the third derivative since we are interested in ultimately deriving an error estimate by $ch^4$, so we only need a Taylor expansion of $y$ up to the same accuracy. Moreover, we also need Taylor expansions of $y'$,

$$y'(t_{n+2}) = y'(t_n + 2h) = y'(t_n) + 2hy''(t_n) + \frac{(2h)^2}{2}y'''(t_n) + \mathcal{O}(h^3),$$

note $(y')'' = y'''$ etc. Moreover, we only need a Taylor expansion up to the second derivative, since all terms in (1) involving a first derivative then get multiplied by $h$, thus a Taylor-expansion of accuracy $\mathcal{O}(h^3)$ is sufficient.

Everything put together, we now obtain

$$y(t_{n+2}) + \alpha_1 y(t_{n+1}) + ay(t_n) - h\beta_2 y'(t_{n+2}) - h\beta_1 y'(t_{n+1}) - h\beta_0 y'(t_n)$$

$$= \left[ y(t_n) + 2hy'(t_n) + \frac{(2h)^2}{2}y''(t_n) + \frac{(2h)^3}{6}y'''(t_n) + \mathcal{O}(h^4) \right]$$

$$+ \alpha_1 \left[ y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(t_n) + \frac{h^3}{6}y'''(t_n) + \mathcal{O}(h^4) \right] + ay(t_n)$$

$$- h\beta_2 \left[ y'(t_n) + 2hy''(t_n) + \frac{(2h)^2}{2}y'''(t_n) + \mathcal{O}(h^3) \right]$$

$$- h\beta_1 \left[ y'(t_n) + hy''(t_n) + \frac{h^2}{2}y'''(t_n) + \mathcal{O}(h^3) \right] - h\beta_0 y'(t_n)$$

$$= y(t_n) \left[ 1 + \alpha_1 + a \right] + y'(t_n) \left[ 2 + \alpha_1 - \beta_2 - \beta_1 - \beta_0 \right] h$$

$$+ y''(t_n) \left[ 2 + \frac{\alpha_1}{2} - 2\beta_2 - \beta_1 \right] h^2 + y'''(t_n) \left[ \frac{4}{3} + \frac{\alpha_1}{6} - 2\beta_2 - \frac{\beta_1}{2} \right] h^3 + \mathcal{O}(h^4).$$

If we want to bound the consistency error by $ch^4$ we thus have to choose the parameters in such a way that all terms of lower $h$-power vanish, i.e. that the terms in brackets become 0. This results in a linear system of four equations for the four unknown $\alpha_1$, $\beta_2$, $\beta_1$ and $\beta_0$ (another reason not to expand beyond $h^4$ – otherwise the linear system would be overdetermined)

$$1 + \alpha_1 + a = 0,$$

$$2 + \alpha_1 - \beta_2 - \beta_1 - \beta_0 = 0,$$

$$2 - \frac{\alpha_1}{2} - 2\beta_2 - \beta_1 = 0,$$

$$\frac{4}{3} + \frac{\alpha_1}{6} - 2\beta_2 - \frac{\beta_1}{2} = 0,$$

with solution

$$\alpha_1 = -a - 1,$$

$$\beta_0 = -\frac{1}{12}(9a - 1),$$

$$\beta_1 = \frac{2}{3}(1 - a),$$

$$\beta_2 = \frac{1}{12}(5 - a).$$

To summarize: The (implicit) scheme

$$y_{n+2} - (a + 1)y_{n+1} + ay_n$$

$$= h \left( \frac{1}{12}(5 + a)f(t_{n+2}, y_{n+2}) + \frac{2}{3}(1 - a)f(t_{n+1}, y_{n+1}) - \frac{1}{12}(9a - 1)f(t_n, y_n) \right)$$
is of consistency order 3 for every choice of the parameter \( a \in \mathbb{R} \). Of course, two choices of that parameter seem favorable: For the choice \( a = -5 \) the scheme turns out to be an explicit scheme (since \( \beta_2 = 0 \) the only term involving \( y_{n+2} \) on the right-hand side disappears), and for some other specific choice for \( a \) the order increases up to 4 (we do not wish to determine that choice of \( a \) at this point, but it should be clear how to approach such a task). Finally, we note that the choice \( a = 0 \) yields the 2-step Adams-Moulton method from Exercise 3 (recall that the Adams-Moulton methods are \( s \)-step methods with consistency order \( s + 1 \)).

**Exercise 2** (Characterization of order)
Consider the multistep scheme

\[
y_{n+2} - y_n = \frac{1}{3} h (f(t_{n+2}, y_{n+2}) + 4 f(t_{n+1}, y_{n+1}) + f(t_n, y_n))
\]

for approximating the solution to

\[
y'(t) = f(t, y(t)), y(t_0) = y_0.
\]

Calculate the order of the scheme.

**Solution:** Let us here use the theorem presented in the lecture, which uses polynomials characteristic for the multistep method, derived via the various coefficients. Recall: For a multistep method written in the form

\[
\sum_{j=0}^{s} a_j y_{n+j} = h \sum_{j=0}^{s} b_j f(t_{n+j}, y_{n+j})
\]

(where usually one assumes a normalization such that \( a_n = 1 \)) we define polynomials

\[
\rho(w) = \sum_{j=0}^{s} a_j w^j \quad \text{and} \quad \sigma(w) = \sum_{j=0}^{s} b_j w^j.
\]

Then the multistep method has order (at least) \( p \) if, and only if

\[
\rho(w) - \ln(w) \sigma(w) = \mathcal{O}(w^{p+1}) \quad \text{as} \quad w \to 1.
\]

In other words, it has order precisely \( p \) if, and only if

\[
\rho(\xi + 1) - \ln(\xi + 1) \sigma(\xi + 1) = c_\xi^{p+1} + \mathcal{O}(\xi^{p+2}) \quad \text{as} \quad \xi \to 0
\]

for some constant \( c \neq 0 \).

In our situation, we have

\[
\rho(w) = w^2 - 1 \quad \text{and} \quad \sigma(w) = \frac{1}{3} w^2 + \frac{4}{3} w + \frac{1}{3},
\]

hence for \( w = \xi + 1 \) we find

\[
\tilde{\rho}(\xi) = \rho(\xi + 1) = (\xi + 1)^2 - 1 = \xi^2 + 2\xi
\]

as well as

\[
\tilde{\sigma}(\xi) = \sigma(\xi + 1) = \frac{1}{3} (\xi + 1)^2 + \frac{4}{3} (\xi + 1) + \frac{1}{3} = \frac{1}{3} \xi^2 + 2\xi + 2.
\]
Furthermore, we have the well-known Taylor expansion
\[
\ln(w) = \ln(\xi + 1) = \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{j} \xi^j = \xi - \frac{1}{2} \xi^2 + \frac{1}{3} \xi^3 - \frac{1}{4} \xi^4 \pm \ldots.
\]

Of course, for an actual calculation of the order, one always uses a truncated version of the series expansion. Then a natural question is: Where should I truncate the series? If you have a conjecture for the order \( p \), then you need to take the terms up to \( \xi^{p+1} \). If you have no such conjecture, then (i) too many terms only make the calculation more lengthy but not false (more terms to deal with, more products to sort), and (ii) too few terms don’t directly hurt either, if you draw the right conclusion at the right point.

So let us demonstrate the argument when (wrongly) conjecturing the order of the 2-step method as 2. Truncating the Taylor expansion after \( \xi^3 \) we thus need to consider
\[
\rho(\xi + 1) - \ln(\xi + 1)\sigma(\xi + 1) = \xi^2 + 2\xi - \left( \xi - \frac{1}{2} \xi^2 + \frac{1}{3} \xi^3 + O(\xi^4) \right) \left( \frac{1}{3} \xi^2 + 2\xi + 2 \right).
\]

Therein the term \( O(\xi^4) \) may serve as a reminder that the next term in the Taylor expansion involved \( \xi^4 \). When multiplying the polynomial terms one shouldn’t just straightforward multiply everything term-by-term and then sort the products, but go about it more strategically. We already know that the end-result is again a polynomial, so we should make ordering the terms our priority: We start collecting every product which results in a term proportional to \( \xi \), then all products proportional to \( \xi^2 \), etc. Keep in mind the question: Which terms do I need to combine to get, e.g., \( \xi^3 \)? In our situation we find
\[
\rho(\xi + 1) - \ln(\xi + 1)\sigma(\xi + 1) = \xi^2 + 2\xi - \left( \xi - \frac{1}{2} \xi^2 + \frac{1}{3} \xi^3 + O(\xi^4) \right) \left( \frac{1}{3} \xi^2 + 2\xi + 2 \right)
\]
\[
= \xi \left( 2 - 1 \cdot 2 \right) + \xi^2 \left( 1 - \left( 1 \cdot 2 - \frac{1}{2} \cdot 2 \right) \right)
\]
\[
= \xi^3 \left( 0 - \left( 1 \cdot \frac{1}{3} - \frac{1}{2} \cdot 2 + \frac{1}{3} \cdot 2 \right) \right)
\]
\[
= \xi^4 \left( 0 - \left( -\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3} \cdot 2 \right) \right) + \xi^5 \left( 0 - \frac{1}{3} \cdot \frac{1}{3} \right) + O(\xi^4).
\]

Keep in mind that we can end our calculation as soon as we find one coefficient to be non-zero, hence after collecting all relevant products one should immediately calculate the resulting coefficient. Our calculation so far demonstrated that the coefficients for \( \xi \), \( \xi^2 \) and \( \xi^3 \) vanish, thus next we need to determine the coefficient for \( \xi^4 \); however, now the truncation of the Taylor expansion comes into play: We truncated after \( \xi^3 \), thus neglecting terms involving \( \xi^4 \) and higher powers. Since now products resulting in \( \xi^4 \) are of interest, we need to refine the Taylor expansion to now involve (at least) that term:
\[
\rho(\xi + 1) - \ln(\xi + 1)\sigma(\xi + 1)
\]
\[
= \xi^4 \left( 0 - \left( -\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3} \cdot 2 - \frac{1}{4} \cdot 2 \right) \right)
\]
\[
= \xi^5 \left( 0 - \left( \frac{1}{3} - \frac{1}{4} \cdot 2 \right) \right) + \xi^6 \left( 0 - \left( -\frac{1}{4} \cdot \frac{1}{3} \right) \right) + O(\xi^5).\]
We thus have found that also the $\xi^4$-term vanishes, so we once again have to extend the Taylor expansion to include the $\xi^5$-term:

\[
\rho(\xi + 1) - \ln(\xi + 1)\sigma(\xi + 1) \\
= \xi^2 + 2\xi - \left(\xi - \frac{1}{2}\xi^2 + \frac{1}{3}\xi^3 - \frac{1}{4}\xi^4 + \frac{1}{5}\xi^5 + \mathcal{O}(\xi^6)\right)\left(\frac{1}{3}\xi^2 + 2\xi + 2\right) \\
= \xi^5 \left(0 - \left(-\frac{1}{3} \cdot \frac{1}{3} - \frac{1}{4} \cdot 2 + \frac{1}{5} \cdot 2\right)\right) + \mathcal{O}(\xi^6). \\
= -\frac{1}{90} \xi^5
\]

We finally have found a non-vanishing coefficient, hence immediately all terms with higher powers of $\xi$ are no longer of interest. Since the non-vanishing coefficient belonged to the $\xi^5$-term, we ultimately conclude the consistency order to be 4, as claimed.

**Remark:** As this is a 2-step method with consistency order at least 3, it has to coincide with the general parametric version of Problem 1 for a specific choice of the parameter $a$. From the left-hand side $y_{n+2} - y_n$ we can immediately read off $a = -1$, and our solution for Problem 1 indeed tells us to use $a_1 = 0$, $a_2 = \frac{1}{3}$, $a_1 = \frac{4}{3}$ and $b_0 = \frac{1}{3}$, which as expected gives the stated method. Moreover, as we had seen that this method has consistency order 4, this is exactly the particular choice of the parameter $a$ mentioned at the end of the solution of Problem 1 (the one which guarantees an additional order of consistency).

**An excursus: Interpolation polynomials**

Let a continuous real- or complex-valued function $g$ on some interval $[a,b]$ be given. Further, let $a \leq x_0 < x_1 < \cdots < x_s \leq b$ be given (we will henceforth call these points the interpolation nodes). Together with the function values $g(x_j)$, $j = 0, \ldots, s$, we can construct a polynomial $p$ of degree at most $s$ with the property

\[
p(x_j) = g(x_j), \quad j = 0, \ldots, s,
\]

the so-called interpolation polynomial. It can be shown that the above interpolation property already uniquely determines the polynomial $p$ (intuitively, a polynomial of degree $s$ has $s + 1$ degrees of freedom – its coefficients – and the interpolation property consists of $s + 1$ conditions, leading to a system of linear equations which can be solved uniquely).

Due to Lagrange there is a specific and explicit representation of that interpolation polynomial (there is at least one more, different, representation which is commonly used, due to Newton). This representation is based on using the Lagrange interpolation polynomials $\lambda_j$, which are interpolation polynomials for the same nodes $x_k$, with the specific function values

\[
\lambda_j(x_k) = \delta_{j,k} = \begin{cases} 
1, & j = k, \\
0, & j \neq k, 
\end{cases}
\]

i.e. all function values in the nodes are 0, except in $x_j$, where it takes the value 1. These special polynomials can be written explicitly as

\[
\lambda_j(x) = \prod_{k \neq j} \frac{x - x_k}{x_j - x_k},
\]
the prescribed values in the nodes being checked easily.

With the help of the Lagrange polynomials, the interpolation polynomial for \( g \) now can be given as

\[
p(x) = \sum_{j=0}^{s} g(x_j) \lambda_j(x).
\]

Again, inserting \( x = x_k \) the interpolation property is verified immediately.

**Exercise 3** (Adams-Bashforth and Adams-Moulton methods)

Try to recall the constructions of the Adams-Bashforth and Adams-Moulton multistep methods from the lecture. In particular, show the iteration formula

\[
y_{n+1} = y_n + \frac{h}{12} \left( 5f(t_{n+1}, y_{n+1}) + 8f(t_n, y_n) - f(t_{n-1}, y_{n-1}) \right)
\]

for the 2-step Adams-Moulton method. Further determine its order of consistency.

**Solution:**

In the lecture it had been explained that the basic idea underlying the Adams-Bashforth methods consists in extrapolating the function \( g(t) = f(t, y(t)) \) with respect to the nodes \( t_j = t_0 + jh, \ j = n, \ldots, n + s - 1 \), and then integrating the extrapolated polynomial. More precisely, from the given nodes together with the respective values \( g(t_j) = f(t_j, y(t_j)) \) one constructs the interpolating polynomial (of degree \( s - 1 \)) on the interval \([t_n, t_{n+s-1}]\), and then extends that one to \([t_n+s-1, t_{n+s}]\). The integration of that polynomial can be done exactly, leading to the coefficients of the numerical scheme; note, however, that of course the exact values \( g(t_j) \) themselves are not known, so for the resulting numerical method they are replaced by \( g_j = f(t_j, y_j) \).

For the Adams-Moulton methods one proceeds quite similarly. The main difference consists in using the nodes \( t_j, \ j = n, \ldots, n + s \) (i.e. we added the node \( t_{n+s} \)) to directly construct the interpolation polynomial (now of degree \( s \)) on the whole interval \([t_n, t_{n+s}]\), which then is used in the same integration step as before. Due to the use of the node \( t_{n+s} \), the resulting numerical method then is an implicit one (in contrast to the explicit Adams-Bashforth methods).

So now let \( p \) be the interpolation polynomial for the nodes \( t_j, \ j = n, \ldots, n + s \), with function values \( p(t_j) = f(t_j, y(t_j)) \), as described above. The basic idea behind both the Adams-Bashforth and Adams-Moulton methods stems from the Fundamental Theorem of Calculus: Simply integrating the differential equation \( y'(t) = f(t, y(t)) \) over the interval \([t_{n+s-1}, t_{n+s}]\) leads to

\[
y(t_{n+s}) - y(t_{n+s-1}) = \int_{t_{n+s-1}}^{t_{n+s}} y'(t) \, dt = \int_{t_{n+s-1}}^{t_{n+s}} f(t, y(t)) \, dt.
\]

To approximate the right-hand side integral, we approximate its integrand by the interpolation polynomial \( p \) (in the hope that this approximation error becomes small for shrinking step size \( h \), i.e. denser interpolation nodes, which typically is indeed true), and instead calculate that one’s integral.

With the interpolation polynomial \( p \) represented as

\[
p(t) = \sum_{j=0}^{s} f(t_j, y(t_j)) \lambda_j(t),
\]
This leads to the polynomials

\[ \int_{t_{n+s-1}}^{t_{n+s}} p(t) \, dt = \sum_{j=0}^{s} f(t_j, y(t_j)) \int_{t_{n+s-1}}^{t_{n+s}} \lambda_j(t) \, dt \]

(note that \( f(t_j, y(t_j)) \) are just real numbers, i.e. constants w.r.t. integration). Thus to calculate the coefficients of the numerical method, we just need to calculate the integrals for the Lagrange interpolation polynomials.

**The concrete case for the 2-step Adams-Moulton method:**

In our case now we have \( s = 2 \), and we use the nodes \( t_n, t_{n+1} = t_n + h \) and \( t_{n+2} = t_n + 2h \). This leads to the polynomials

\[
\lambda_n(t) = \prod_{k \neq n} \frac{t - t_k}{t_n - t_k} = \frac{t - t_{n+1}}{t_n - t_{n+1}} \cdot \frac{t - t_{n+2}}{t_n - t_{n+2}} = \frac{(t - t_n) - h}{-h} \cdot \frac{(t - t_n) - 2h}{-2h} = \frac{(t - t_n)^2}{2h^2} - \frac{3(t - t_n)}{2h} + 1,
\]

\[
\lambda_{n+1}(t) = \prod_{k \neq n+1} \frac{t - t_k}{t_n - t_k} = \frac{t - t_n}{t_{n+1} - t_n} \cdot \frac{t - t_{n+2}}{t_{n+1} - t_{n+2}} = \frac{t - t_n}{h} \cdot \frac{(t - t_n) - 2h}{-h} = \frac{(t - t_n)^2}{h^2} + \frac{2(t - t_n)}{h},
\]

\[
\lambda_{n+2}(t) = \prod_{k \neq n+2} \frac{t - t_k}{t_n - t_k} = \frac{t - t_n}{t_{n+2} - t_n} \cdot \frac{t - t_{n+1}}{t_{n+2} - t_{n+1}} = \frac{t - t_n}{2h} \cdot \frac{(t - t_n) - h}{h} = \frac{(t - t_n)^2}{2h^2} - \frac{t - t_n}{2h}.
\]

In turn this results in

\[
\alpha_0 = \int_{t_{n+1}}^{t_{n+2}} \lambda_n(t) \, dt = \int_{t_{n+h}}^{t_{n+2h}} \left( \frac{(t - t_n)^2}{2h^2} - \frac{3(t - t_n)}{2h} + 1 \right) \, dt
\]

\[
= \int_{h}^{2h} \left( \frac{t^2}{2h^2} - \frac{3t}{2h} + 1 \right) \, dt = \left[ \frac{t^3}{6h^2} - \frac{3t^2}{4h} + t \right]_h^{2h}
\]

\[= h \left( \frac{8}{6} - \frac{12}{4} + 2 - \frac{1}{6} + \frac{3}{4} - 1 \right) = -\frac{h}{12},
\]

\[
\alpha_1 = \int_{t_{n+1}}^{t_{n+2}} \lambda_{n+1}(t) \, dt = \int_{t_{n+h}}^{t_{n+2h}} \left( -\frac{(t - t_n)^2}{h^2} + \frac{2(t - t_n)}{h} \right) \, dt
\]

\[
= \int_{h}^{2h} \left( -\frac{t^2}{h^2} + \frac{2t}{h} \right) \, dt = \left[ -\frac{t^3}{3h^2} + \frac{t^2}{h} \right]_h^{2h}
\]

\[= h \left( -\frac{7}{3} + 3 \right) = \frac{2h}{3},
\]

\[
\alpha_2 = \int_{t_{n+1}}^{t_{n+2}} \lambda_{n+2}(t) \, dt = \int_{t_{n+h}}^{t_{n+2h}} \left( \frac{(t - t_n)^2}{2h^2} + \frac{t - t_n}{2h} \right) \, dt
\]

\[
= \int_{h}^{2h} \left( \frac{t^2}{2h^2} - \frac{t}{2h} \right) \, dt = \left[ \frac{t^3}{6h^2} - \frac{t^2}{4h} \right]_h^{2h}
\]

\[= h \left( \frac{7}{6} - \frac{3}{4} \right) = \frac{5h}{12}.
\]
Everything put together, we then arrive at the numerical scheme
\[ y_{n+2} = y_{n+1} + \alpha_2 f(t_{n+2}, y_{n+2}) + \alpha_1 f(t_{n+1}, y_{n+1}) + \alpha_0 f(t_n, y_n), \]
which is exactly the formula for the 2-step Adams-Moulton method as claimed.

To determine its order, we check the order conditions for multistep methods. Recall:

For a method of the form
\[ \sum_{j=0}^{s} a_j y_j = h \sum_{k=0}^{s} b_k f(t_k, y_k) \]
these order conditions read as
\[ \sum_{j=0}^{s} a_j = 0 \quad \text{as well as} \quad q \sum_{k=0}^{s} k^{q-1} b_k = \sum_{k=0}^{s} k^q a_k. \]

For a method of order \( p \) the second condition has to hold for all \( q = 1, \ldots, p \).

In our situation we have \( s = 2 \) and \( a_2 = 1, a_1 = -1, a_0 = 0 \) as well as \( b_2 = \frac{5}{12}, b_1 = \frac{2}{3} \) and \( b_0 = -\frac{1}{12} \). Clearly, the first condition is fulfilled. The second one for \( q = 1 \) reads as
\[ b_0 + b_1 + b_2 \overset{!}{=} 2a_2 + a_1, \]
which is true as well (both sides add up to 1); for \( q = 2 \) we find
\[ 2(2b_2 + b_1) \overset{!}{=} 4a_2 + a_1, \]
again fulfilled (both sides are 3); for \( q = 3 \) we obtain
\[ 3(4b_2 + b_1) \overset{!}{=} 8a_2 + a_1, \]
also true (both sides here are 7); finally \( q = 4 \) gives
\[ 4(8b_2 + b_1) \overset{!}{=} 16a_2 + a_1, \]
which is NOT fulfilled \( (16 \neq 15) \). We conclude that the consistency order is 3 (as predicted by the general theory for the 2-step Adams-Moulton method).

Information and material related to the lecture can be found at the lecture webpage

http://www-m15.ma.tum.de/Allgemeines/ModelingSimulation