Modeling and Simulation with ODE for MSE

The exercises can be handed in during the week 14.11. – 18-11.

Exercise 1 (Inhomogenous linear ODEs of first order)
Solve the initial value problems
(a) \( y'(x) + y(x) \cos(x) = \sin(x) \cos(x), \ y(0) = 1 \),
(b) \( y'(x) - \frac{2}{x} y(x) = x^2 \sin(3x), \ y(1) = 0 \).

Exercise 2 (Ricatti Differential Equation)
Let an equation of the form
\[
y'(x) = q_0(x) + q_1(x)y + q_2(x)y^2
\]
be given. Equations of this type are called Ricatti differential equation or of Ricatti-type. They can be solved by the following strategy.

First, assume a solution \( y_0 \) of the ODE is already known. Then the general solution can be found using the ansatz
\[
y(x) = u(x) + y_0(x).
\]

Inserting gives the equation of Bernoulli type
\[
u' = (q_1 + 2q_2y_1)u + q_2u^2.
\]
In turn, this equation reduces to a linear equation using the substitution
\[
z(x) = \frac{1}{u(x)}.
\]

Use this general strategy (i.e. find a particular solution \( y_0 \), make an ansatz for the general solution involving \( u(x) \), then substitution to \( z(x) \), solve the resulting linear ODE) to find all solutions of the differential equation
\[
y'(x) = y^2(x) + 1 - x^2.
\]

Hint: A particular solution is of the type \( y_1(x) = ax^s \) for some \( s \in \mathbb{N} \).

Exercise 3* (If time permits: Another type of nonlinear ODE)
Let an ODE of the form
\[
F(x, y) + G(x, y)y' = 0
\]
be given for some functions of two variables \( F \) and \( G \) (with appropriate domains of definition, which we will not further specify here). Such an equation is called exact, if the condition
\[
\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}
\]
is fulfilled. The solution of such an ODE can be derived by first determining a function $H(x, y)$ such that

$$F(x, y) = \frac{\partial H(x, y)}{\partial x} \quad \text{and} \quad G(x, y) = \frac{\partial H(x, y)}{\partial y}.$$ 

The solutions $y(x)$ of the ODE then are obtained from solving the equation

$$H(x, y(x)) = c$$

for $y(x)$ for some fixed constant $c \in \mathbb{R}$.

Now consider the equations

(a) \hspace{1cm} 2xy + (2y + x^2)y' = 0 \\
(b) \hspace{1cm} xy + x^2y' = 0 \\
(c) \hspace{1cm} \sin x - (\ln y)y' = 0

Which of these equations is exact? For the exact equations, determine their general solution as well as the solution for initial value $y(0) = 2$.

Information and material related to the lecture can be found at the lecture webpage

http://www-m15.ma.tum.de/Allgemeines/ModelingSimulation