IGDK 1754

Optimization and Numerical Analysis for Partial Differential Equations with Nonsmooth Structures

Sparse Control of Alignment Models in High Dimension

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Problem Setting

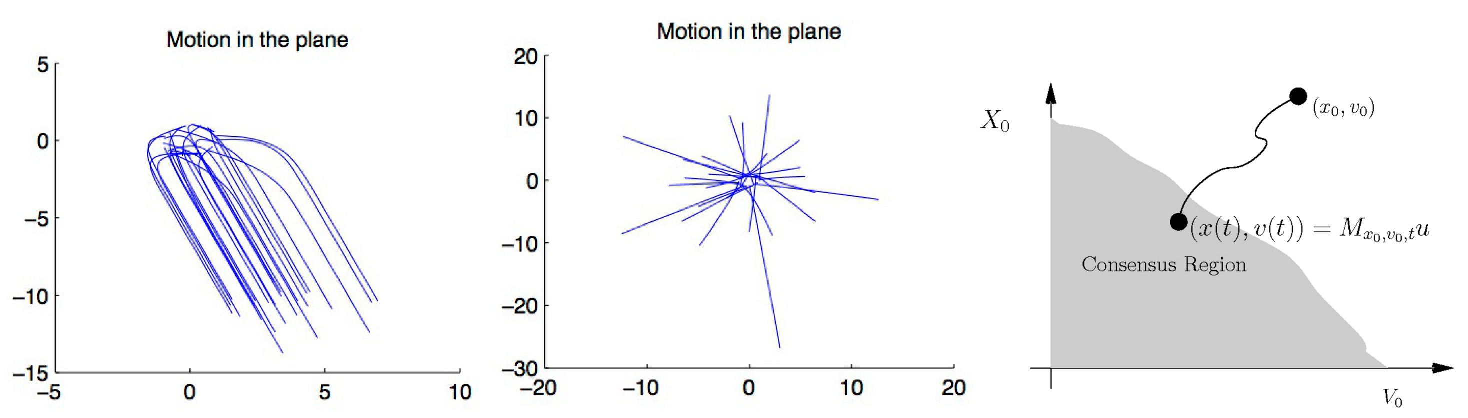
We study the control of Cucker-Smale-type systems (see [3])

$$\begin{cases} \dot{x}_i = v_i, & i = 1, \dots, N \\ \dot{v}_i = \frac{1}{N}\sum_{i=1}^N a(\|x_i - x_j\|)(v_j - v_i) + u_i, \end{cases} \quad (1)$$

where $a : [0, +\infty) \to [0, +\infty)$ is bounded and decreasing, $(x_i(t), v_i(t)) \in \mathbb{R}^{2d}$ for every $t \geq 0$, and the controls are supposed to be parsimonious, i.e., for given $\theta > 0$

$$\sum_{i=1}^N \|u_i(t)\| \leq \theta \quad \text{for every } t \geq 0.$$

We want to steer these systems to consensus configurations ($v_i(t) = \overline{v}(t) = \frac{1}{N}\sum_{j=1}^N v_j(t)$ for every i). It is possible as soon as the initial data can be mapped inside a consensus region (depending on a).



(a) Consensus (b) Non-consensus (c) Consensus region

When d is very large (e.g., when the agents are financial agents or indexed assets), the curse of dimensionality occurs: no efficient way of numerically tackling the above problem without a reformulation.

Dimension Reduction and Sparse Control

We first need a Johnson-Lindenstrauss' Lemma for continuous data.

Continuous JL Lemma (Generalizes [4, Theorem 3.3])

Let $\varphi : [0, 1] \to \mathbb{R}^d$ be a Lipschitz function (with bound L_φ), $0 < \varepsilon' < \frac{1}{2}$, $\delta > 0$ and $M \in \mathbb{R}^{k \times d}$ be a Johnson-Lindenstrauss matrix for $n \geq 4L_\varphi \frac{\sqrt{d}+2}{\delta\varepsilon'}$ points and $\varepsilon = \frac{\varepsilon'}{2}$ with high probability. Then for every $t \in [0, 1]$ one of the following holds (with the same high probability):

- $(1 - \varepsilon')|\varphi(t)| \leq |M\varphi(t)| \leq (1 - \varepsilon')|\varphi(t)|$ or
- $|\varphi(t)| \leq \delta$ and $|M\varphi(t)| \leq \delta$.

Theorem 1

Given the uncontrolled system

$$\begin{cases} \dot{x}_i = v_i, & i = 1, \dots, N \\ \dot{v}_i = \frac{1}{N}\sum_{i=1}^N a(\|x_i - x_j\|)(v_j - v_i), \end{cases}$$

let M be a continuous JL matrix for $\|x_i(t) - x_j(t)\|$, for given $\varepsilon, \delta > 0$, all $t \in [0, T]$ and $i \neq j$. Then the k-dimensional IVP

$$\begin{cases} \dot{y}_i = w_i, & i = 1, \dots, N \\ \dot{w}_i = \frac{1}{N}\sum_{i=1}^N a(\|y_i - y_j\|)(w_j - w_i), \\ y(0) = Mx_0, w(0) = Mv_0, \end{cases}$$

satisfies $\|y(t) - Mx(t)\| + \|w(t) - Mv(t)\| \lesssim (\varepsilon + \delta)e^{Ct}$.

Hence the trajectories of the original d-dimensional system and the ones of its k-dimensional projection stay close.

Theorem, [2]

For every θ there exists $\tau_0 > 0$ such that for all sampling times $\tau \in (0, \tau_0]$, the solution of (1) with the control

$$u_i(t) = \begin{cases} -\theta \frac{v_j^\perp(k\tau)}{\|v_j^\perp(k\tau)\|} & \text{if } j \text{ min index:} \|v_j^\perp(k\tau)\| \geq \|v_i^\perp(k\tau)\| \,\forall i, \\ 0 & \text{otherwise,} \end{cases} \quad (2)$$

where $k = \left\lfloor \frac{t}{\tau} \right\rfloor$ and $v_i^\perp(t) = v_i(t) - \overline{v}(t)$, reaches the consensus region in finite time.

Sparse Control in High Dimension

Motivated by the previous results, we design the control strategy for (1) to be sparse: the information about the agent to act on shall be given by a low-dimensional projection. Setting

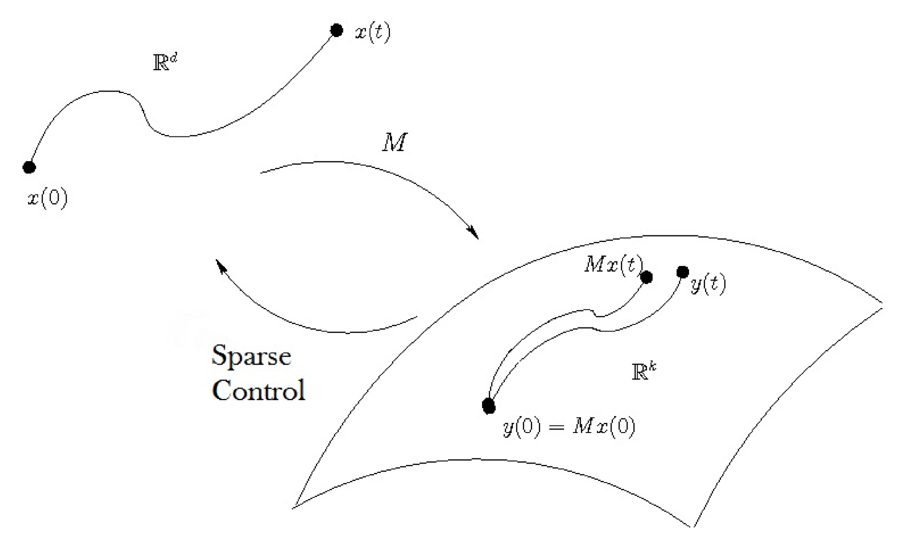
$$\begin{cases} \dot{x}_i = v_i, & i = 1, \dots, N \\ \dot{v}_i = \frac{1}{N}\sum_{i=1}^N a(\|x_i - x_j\|)(v_j - v_i) + u_i^h, \\ x(0) = x_0, v(0) = v_0, \end{cases} \quad (3) \qquad \begin{cases} \dot{y}_i = w_i, & i = 1, \dots, N \\ \dot{w}_i = \frac{1}{N}\sum_{i=1}^N a(\|y_i - y_j\|)(w_j - w_i) + u_i^\ell, \\ y(0) = Mx_0, w(0) = Mv_0, \end{cases} \quad (4)$$

we define

$$u_i^\ell(t) = \begin{cases} -\theta \frac{w_j^\perp(k\tau)}{\|w_j^\perp(k\tau)\|} & \text{if } \|w_j^\perp(k\tau)\| \geq \|w_i^\perp(k\tau)\| \,\forall i, \\ 0 & \text{otherwise,} \end{cases} \qquad u_i^h(t) = \begin{cases} -\theta \frac{v_j^\perp(k\tau)}{\|v_j^\perp(k\tau)\|} & \text{for the same } j \text{ of } u_i^\ell(t), \\ 0 & \text{otherwise.} \end{cases}$$

Main Theorem

Let $M \in \mathbb{R}^{k \times d}$, $\theta > 0$, (x, v) and (y, w) be solutions of (3) and (4), respectively, and assume that M is a continuous JL matrix for $\|x_i(t) - x_j(t)\|$ and $\|v_i(t) - v_j(t)\|$ for sufficiently long and for ε and δ sufficiently small. Then for τ sufficiently small, the trajectories of (3) and (4) stay close as in Theorem 1, and both systems reach the consensus region in finite time.



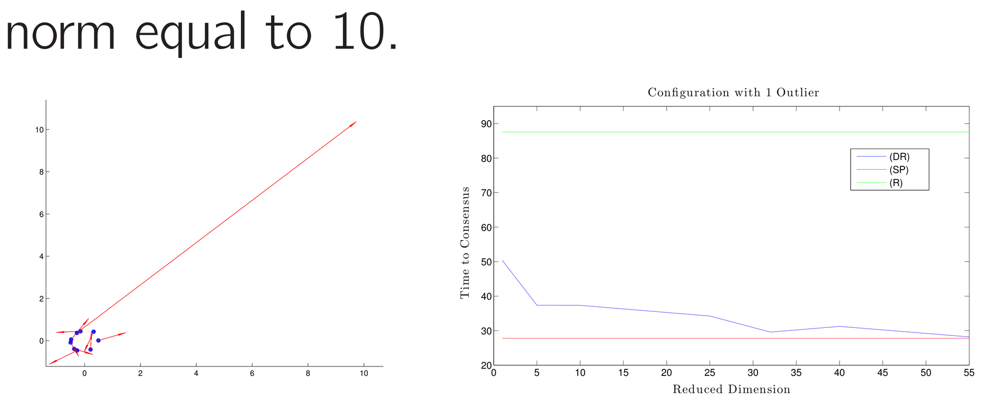
- The only high-dimensional information we need is the direction of the agent to control (the mean velocity can be easily computed from the past controls).
- The request that M is a JL matrix for the trajectories clashes with the fact that the initial values of the low-dimensional system (and hence the controls) depend on M. To overcome this issue, we take all possible trajectories into account (since the number of control switches is finite).

Numerical experiments

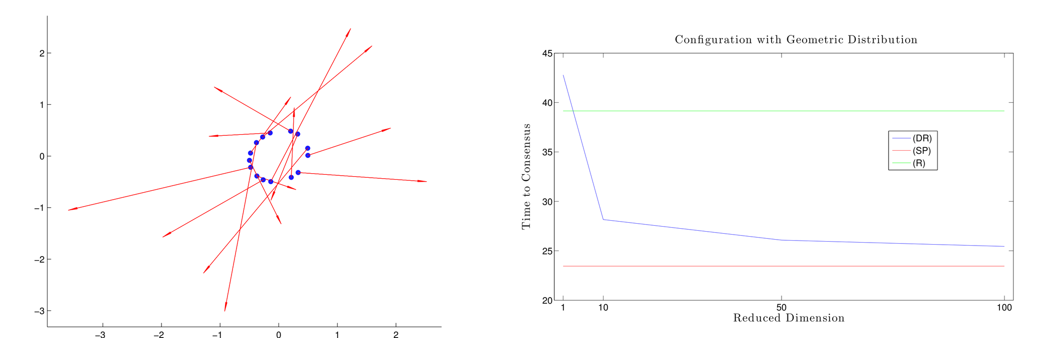
We compare the performance of the dimension-reduction control (blue line) with the sparse control (2) (red line) as benchmark, and the variation of it where the index j is chosen randomly following a uniform distribution (green line).

The experiments highlight that the more diverse are the norms of the initial velocities of the agents, the better the dimension-reduction strategy works. In this case, it shows to have comparable performances with the sparse strategy even if $k = d/10$.

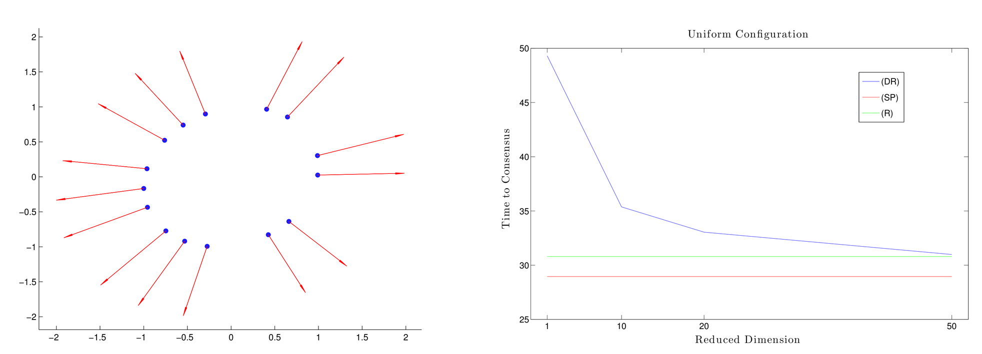
In this experiment we have $N = 9$ agents in dimension $d = 100$: all agents have velocities with norm equal to 1, and one with norm equal to 10.



We take $N = 15$ agents in dimension $d = 500$: the norm of the velocity of the i-th agent is proportional to $(1.2)^i$.

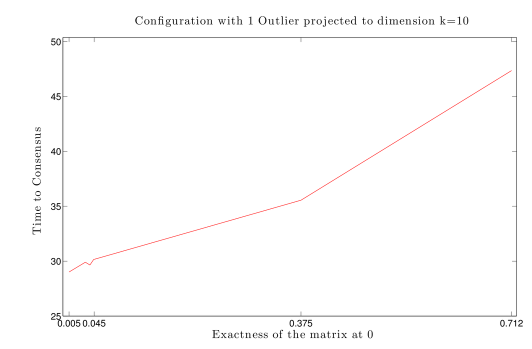


Here we have $N = 15$ agents in dimension $d = 200$: all agents have velocities with norm near 1. The strategy performs worse than in the other cases.



Since being a JL matrix is not verifiable at time 0, we propose the following quantity as a preliminary test on the matrix M. The graph shows a correlation between its closeness to 0 and the performance of the strategy.

$$\left| 1 - \frac{\sum_{i=1}^N v_i^\perp(0)}{\sum_{i=1}^N Mv_i^\perp(0)} \right|$$



Bibliography

- M. Bongini, M. Fornasier, O. Junge, and B. Scharf. Sparse Control of Alignment Models in High Dimension. *Netw. Heterog. Media,, to appear.*
- M. Caponigro, M. Fornasier, B. Piccoli, and E. Trelat. Sparse stabilization and control of alignment models. *Math. Models Methods Appl. Sci., to appear.*
- F. Cucker, S. Smale. Emergent behavior in flocks. *IEEE Trans. Automat. Control, 52(5):852–862, 2007.*
- M. Fornasier, J. Haskovec, and J. Vybiral. Particle systems and kinetic equations modeling interacting agents in high dimension. *Multiscale Model. Simul., 9(4):1727–1764, 2011.*