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SEQUENTS-OF-RELATIONS CALCULI: PROOF THEORY FOR SEMIPROJECTIVE LOGICS

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Abstract

Sequents-of-relations calculi provide a uniform and elegant method for introducing Co-NP, proof-search oriented analytic calculi based on hypersequents for a large class of logics, namely semiprojective logics. These logics are characterized by a special format of their connectives and it is seen that Gödel logic with an involutive negation, Nilpotent Minimum logic, Weak Nilpotent Minimum logic and n-contractive BL logics belong to this new class, together with all the logics already belonging to the broad class of Projective Logics.

In this work, we briefly recall the basic definitions and theorems in the first three chapters and, in the fourth one, we characterize the axioms of the sequents-of-relations calculi for Gödel logic with an involutive negation and Nilpotent Minimum logic, exhibiting a polynomial-time algorithm which recognizes whether a sequent of relations is an axiom of the calculus or not. In the fifth chapter, we study cut-elimination in the sequents-of-relations calculus for Gödel logic with an involutive negation, showing a procedure which transforms every proof of a sequent of relations into a cut-free proof of the same sequent. In the sixth chapter, we show that some of the most important constructions in Fuzzy Logic (namely, ordinal sum and rotation) preserve the semiprojectivity of the logics involved, and we show that other constructions do so at least in the format of their truth functions. Finally, in the last chapter, we explore a further generalization of semiprojectivity, which allows logics like Łukasiewicz, Product and BL logics to be proof-theoretically treated in the common framework of hypersequents-of-relations calculi, a generalization of the sequents-of-relations approach.

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Chapter 1

Introduction

The origins of Fuzzy Logic may be traced back to Aristotle, who was the first (at least, the first we know of) to question the *law of excluded middle*, pointing out that it does not hold for future events, but doubts about the expressive power of Classical Logic were already highlightened by the famous *Epimenide's paradox.* Nevertheless, it was only during the 1920's that the Polish mathematician Łukasiewicz sistematically developed those ideas in [28], using a third value, "possible", different from "true" and "false" to deal with Aristotle's *paradox of the sea battle*; a further generalization of this approach was pursued independently by Post [39], and by Łukasiewicz and Tarski [29], which formulated a family of logics with n truth values, where $n \geq 2$, called *many-valued logics.* Years later, Gödel proved in [21] that Intuitionistic Logic is not a many-valued logic, introducing a system of *intermediate logics* between Classical and Intuitionistic Logics, now called *Gödel-Dummett logics* since Dummett in [15] presented the first axiomatization matching Gödel's matrix characterization. A great boost in the studies inside this framework was given with the publication of Hájek's monograph [23], which gave birth to most of the nomenclature and opened problems that hold still today. It radically changed the approach to the subject too, shifting the attention to methods closer to that of Universal Algebra, instead of topological and geometric ones.

Besides these developments inside pure many-valued logic, Zadeh [43] started a (computer science oriented) approach toward the formalization of vague notions by generalized set theoretic means, which soon was related by Goguen [22] to philosophical applications, approach that culminated in the formalization of the notion of *fuzzy set*, see [44], which paved the way to massive applications in industry; Zadeh had observed that conventional computer logic couldn't manipulate data that represented subjective or vague ideas, so he created fuzzy logic to allow computers to determine the distinctions among data with shades of gray, similar to the process of human reasoning. From that moment, Fuzzy Logic has found its way toward most of nowadays' central research topic in computer science, providing a foundation for reasoning under vagueness.

Logic in application is often a synonym of *proof theory*: proof theory is a branch of mathematical logic that represents proofs as formal mathematical objects, facilitating their analysis by mathematical techniques. The story of modern proof theory is often seen as being established by David Hilbert, who initiated what is called *Hilbert's program* in the foundations of mathematics. In [24], he introduced the proof calculi called the *Hilbert systems*, which are still used nowadays. Gentzen, in [19], improved Hilbert systems allowing the drawing of conclusions from assumptions in the inference rules of the logic, introducing the so called calculi of *natural deduction*, and pointed out the centrality of symmetry between the grounds for asserting propositions, expressed in introduction rules, and the consequences of accepting propositions in the elimination rules, an idea that has proved to be crucial in proof theory. All these efforts were made in order to solve one of the most striking open problems at the time, namely the *consistency proof of Peano arithmetic*; Gentzen's idea was to extend natural deduction to a system of arithmetic by the addition of a rule that corresponds to the principle of complete induction. Consistency would then follow from the *normalization of derivations* and the subformula property. By early 1933, Gentzen realized that this proof strategy would not go through: the induction rule is schematic and has an infinity of instances, with no bound on the complexity of the induction formulas. It would be impossible to restrict these formulas in advance, thus no subformula property can hold. After this failure, Gentzen wrote that he was unable to prove a normalization theorem for a classical system of natural deduction. He therefore invented another logical calculus that he called *sequent calculus* and made it the central topic of his thesis. The name of the calculus comes from the representation of assumptions of a derivation as a list.

Sequent calculus can be seen as a formal representation of the derivability relation in natural deduction. A sequent consists of a pair of multisets of formulas, written $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$, and usually interpreted as "A_1 and

... and A_n implies B_1 or ... or B_m". This novelty led to the first satisfactory formulation of a proof system for classical logic. This calculus better expressed the duality of the logical connectives, went on to make fundamental advances in the formalisation of Intuitionistic Logic, and provided the first combinatorial proof of the consistency of Peano arithmetic.

A central task of logic in computer science is to supply analytic Gentzen-style calculi for a wide range of non-classical logic. *Analyticity* of the proof calculi is the key property: proofs of such systems have the feature that the formulas occurring are built from the same material (subformulas) as the formula proved. The existence of such proofs - established via the key proof-theoretic technique of *cut elimination* - has nice consequences, since we can deduce from it decidability and complexity results, obtain interpolation and conservative extension properties, or use the calculus as the basis for automated reasoning methods. Analytic proof calculi are important both as a theoretical tool, useful for understanding relationships between logics and proving properties like the so-called "standard completeness", which establishes that the semantic and syntactic approaches coincide, and admissibility of rules, and also as the key to potential applications. Proof search algorithms can be used as the basis for "inference engines" in Artificial Intelligence for (fuzzy) knowledge representation, and reasoning in contexts of uncertainty and vagueness, as query answering, consistency checking and revision. For these applications, analytic proof methods are not only good candidates for automated proof-search, but also, since they proceed by a stepwise decomposition of formulas, facilitate the understanding of proofs and allow the extraction of explanatory information.

In this context, one of the first succesful approach is due to Avron [2], which provided a calculus for Gödel logic **G** using *hypersequents*: a hypersequent is a *multiset of sequents* written as $\Gamma_1 \vdash \Delta_1 \mid \ldots \mid \Gamma_n \vdash \Delta_n$, and interpreted at the metalevel as "one of the $\Gamma_i \vdash \Delta_i$ holds". This calculus, however, has its drawbacks, since it is not suitable for proof search (the so-called *communication rule* of the calculus prevent the termination of the process), but it is particularly appropriate for logics with disjunctive axioms, such as prelinearity $(A \to B) \vee (B \to A)$. Recently, many hypersequent calculi for fuzzy logic were introduced, see [32] for Π, [33] for Ł, and [42] for BL, whose main fault is the lack of homogeneity and generality, many of them being ad hoc for the particular fuzzy logic under consideration. The calculus for Monoidal T-norm based logic MTL provided in [6], instead, does not allow proof search since termination is still an open problem, which does not help characterizing the computational complexity of the logic.

A step forward in the direction of the automated construnction of analytic and proof search oriented calculi for many-valued logics was done in [7], with the introduction of *sequents of relations* and of a methodology to construct such calculi for a large family of fuzzy logics, called *projective logics*. A sequent of relation is a disjunction of semantic predicate symbols over formulas, and generalize the concept of a sequent, considering the symbol $\vdash$ just as a particular predicate symbol of a certain *semantic theory* of the logic. These calculi are systematically derived from a specific presentation of the semantics of the connectives involved, which is a "piecewise" projection of the inputs of the connective, in the sense that, when making calculations, the connective projects one of its inputs (or return a constant) depending on which condition such inputs satisfies.; this format of presentation has inspired to call the corresponding class of logics "projective". The main example of a projective logic is *infinite-valued* Gödel logic $\mathbf{G}_\infty$, but all finite-valued logics are captured too. Sequents-of-relations calculi, as already said, can be introduced automatically and uniformly for projective logics and are characterized by sound and invertible rules, analiticity and efficiency, matching the time-complexity of most of the logics captured. This feature, however, is related to the availability of polynomial-time decision procedure for the axioms of the calculi, an aspect tackled in [4], in which a cut-elimination result for Gödel logic is also presented.

In order to widen the expressive power of projective logics, *semiprojective logics* were introduced in [37], aa notion which allowed semantic theories of the logics to include unary function symbols. This difference lead to a class of logics encompassing Nilpotent Minimum logic, Weak Nilpotent Minimum logic, Gödel logic with an involutive negation and n-contractive BL$^+$ logics for $n \geq 1$, and, of course, all the projective logics. With slight modifications, the same procedure which introduces sequents-of-relations calculi for projective logics is available for semiprojective logics too, which allows us to conclude that each formalized logic is Co-NP, provided that the validity of the axioms of the calculi can be checked in polynomial time.

Even if great generality is accomplished with the introduction of semiprojective logics, still two of the three fundamentals fuzzy logics, Łukasiewicz logic and Product logic, and Hájek's Basic Logic BL are not captured in this framework. In order to fill this gap, in the last section of this work we further generalize the concept of semiprojective logic into that of *hyperprojective logic*. This extension is motivated by [13], in which, using *relational hypersequents*, objects very similar to sequents of relations, analytic calculi for Gödel, Łukasiewicz and Product logics are introduced in a systematic way. The new objects of the calculi we shall introduce for hyperprojective logics will be *hypersequents of relations*, basically sequents of relations in which formulas are substituted with multisets of formulas. The procedure for the introduction of such calculi will be a generalization of that available for semiprojective logics, and we shall see that the most important features of the sequents-of-relations calculi, such as analyticity of the rules and Co-NP complexity for the formalized logics, will be carried over to the hypersequents-of-relations one.

Structure of the Thesis

The goal of this work is to report in full details the present knowledge on semiprojective logics and sequents-of-relations calculi, with in mind the aim to improve and extend this framework. A brief overview of the chapters is given below.

Chapter 2 treats the most basic knowledge the reader is required to deal with this work: the preliminary definitions and results of Universal Algebra, the notion of satisfiability and provability of first-order Classical Logic, the semantic and syntactic approaches to Fuzzy Logic.

Chapter 3 introduces the concepts of semiprojective logic and sequents-of-relations calculi, providing examples and case studies taken from the existing literature. A soundness and completeness result is also given, showing how it yields Co-NP complexity for the encompassed logics.

Chapter 4 explores the crucial problem of the availability of an efficient decision procedure for the axioms of the calculi, recalling already known results and developing a new approach, based on a polynomial-time algorithm, for the characterization of the axioms of the sequents-of-relations calculi for Gödel logic with an involutive negation and Nilpotent Minimum logic. A full list of the rules and axioms of known semiprojective logics is given at the end of the chapter.

Chapter 5 reports a detailed discussion about extended structural rules in sequents-of-relations calculi, and a cut-elimination result for Gödel logic with an involutive negation, obtained generalizing an existing cut-elimination procedure available for Gödel logic.

Chapter 6 deals with the theoretical task of finding logical constructions which preserve the semiprojectivity of the logics involved: we carry out detailed proofs for the *(disconnected) rotation* construction and the *ordinal sum*, giving a brief overlook to the *generalized ordinal sum construction*.

Chapter 7 tackles the challenge of extending the notion of semiprojective logic, analyzing ideas and results that brings to the brand-new concepts of *hyperprojective logic* and *hypersequents-of-relations calculus*. The similarity with already existing calculi is pointed out, together with examples of captured logics uncatchable with the semiprojective approach. In the end, a very general investigation on the complexity of the calculi introduced is reported.

Chapter 2

Basic Notions

In this chapter we introduce the essential notation and the basic concepts which are needed for the rest of the work. We will cover only the necessary topics, leaving to the reader the effort to fill the gaps we left for the sake of brevity. As references for the first section we suggest [30, 12], for the second one the classic [31] and [1], and for the third one [34], which is suggested to the interested reader for further reading. We will therefore assume some familiarity with naive set theory and propositional logic.

2.1 Universal Algebra

By an *algebra* we mean any model for a propositional language L whose connective symbols are operations f_i with arity $\rho(i) \in \mathbb{N}$, with $i \in I$. Thus an algebra is a pair $\mathbf{A} = \langle A, (f_i^{\mathbf{A}} : i \in I)\rangle$ where A is a nonempty set, called the *universe* of $\mathbf{A}$, and each $f_i^{\mathbf{A}}$ is a $\rho(i)$-ary operation on A. Whenever we deal with a class of algebras $\mathcal{K}$, it will be assumed that all the algebras are models of one language, i.e. that they are of the same *similarity type*. The set of all models of a propostional language L will be indicated as Mod(L), and if $\mathbf{A} \in$ Mod(L) as above, then $f_i^{\mathbf{A}}$ is called the *interpretation* of f_i in $\mathbf{A}$.

Given $\mathbf{A} \in$ Mod(L), if we consider a subset B of A which is closed with respect to the operations of $\mathbf{A}$ (i.e. if $b_1, \ldots, b_n \in B$ and $f_i^{\mathbf{A}}$ is such that $\rho(i) = n$, then $f^{\mathbf{A}}(b_1, \ldots, b_n) \in B$), then the structure $\mathbf{B} = \langle B, (f_i^{\mathbf{A}}|_B : i \in I)\rangle$ is a member of Mod(L) and it is called a *subalgebra of* $\mathbf{A}$. If $X \subseteq A$, with $\mathrm{Sg}_{\mathbf{A}}(X)$ we identify the smallest subalgebra of $\mathbf{A}$ whose universe contains the set X; of course $\mathrm{Sg}_{\mathbf{A}}(X) \in$ Mod(L). We say that an algebra $\mathbf{A}$ *is generated* *by a set* X if $X \subseteq A$ and $\mathrm{Sg}_{\mathbf{A}}(X) = \mathbf{A}$, and that it is *finitely generated* if it is generated by a finite set.

The set of *terms* (or *formulas*, depending on the context) in the variables $V = \{p_i\}_{i\in\omega}$ for a language L is the smallest set $Fm_{\mathrm{L}}(V)$ containing each p_i and such that if $t_1, \dots, t_n \in Fm_{\mathrm{L}}(V)$ and f_i is an operation symbol in L with $\rho(i) = n$, then $f_i(t_1, \dots, t_n) \in Fm_{\mathrm{L}}(V)$. Of course, if $\mathbf{A} \in \mathrm{Mod}(\mathrm{L})$ is an algebra, then to each term of L we can associate an operation on A in an obvious way: thus, if $t(p_0, \dots, p_{n-1})$ is a term in which only the (distinct) variables $p_0, \dots, p_{n-1}$ appear, then $t^{\mathbf{A}}$ denotes the corresponding n-ary operation on $\mathbf{A}$. Members of $Fm_{\mathrm{L}}(V)$ are also denoted with φ, ψ, ϕ, etc.

Suppose that $\mathbf{A} \in \mathrm{Mod}(\mathrm{L})$ and that $s = s(p_0, \dots, p_{n-1})$ and $t = t(p_0, \dots, p_{n-1})$ are terms of L. The string $s \approx t$ is called an *equation*; we denote with Eq(L) the set of all equations of L. We write $\mathbf{A} \models s \approx t$ to denote that $s^{\mathbf{A}} = t^{\mathbf{A}}$, i.e. $s^{\mathbf{A}}(a_0, \dots, a_{n-1}) = t^{\mathbf{A}}(a_0, \dots, a_{n-1})$ for all $a_0, \dots, a_{n-1} \in A$. When $\mathbf{A} \models s \approx t$ we say that $s \approx t$ is an *identity* of $\mathbf{A}$. $\mathcal{K} \models s \approx t$ means that $\mathbf{A} \models s \approx t$ for all $\mathbf{A} \in \mathcal{K}$, while if Γ is a set of equations, $\mathcal{K} \models \Gamma$ means that $\mathcal{K} \models \varepsilon$ for all $\varepsilon \in \Gamma$.

Let $\mathcal{K}$ be a class of algebras with similarity type L. If $\Gamma \cup \{\varepsilon\} \subseteq \mathrm{Eq}(\mathrm{L})$ we write $\Gamma \models_{\mathcal{K}} \varepsilon$ if $\mathcal{K} \models \Gamma$ implies that $\mathcal{K} \models \varepsilon$.

If $\Sigma \subseteq \mathrm{Eq}(\mathrm{L})$ then

$$\mathrm{Mod}(\Sigma) = \{\mathbf{A} \in \mathrm{Mod}(\mathrm{L}) \mid \mathbf{A} \models \varepsilon \text{ for all } \varepsilon \in \Sigma\}$$

Classes of algebras of the form $\mathrm{Mod}(\Sigma)$ for some set of equations Σ of some language, are called *varieties* (or *equational classes*). The celebrated HSP theorem of G. Birkhoff [11] states that a class of algebras $\mathcal{V}$ is a variety if and only if $\mathcal{V}$ is closed under the formation of homomorphic images, subalgebras and products. A variety $\mathcal{V}$ is said to be *locally finite* if every finitely generated algebra in $\mathcal{V}$ is finite.

Let $VAR(\mathrm{L})$ denote the set of propositional variables of the propositional language L. An assignment $\sigma : VAR(\mathrm{L}) \longrightarrow Fm_{\mathrm{L}}(VAR(\mathrm{L}))$ of formulas to variables extends naturally to a function from $Fm_{\mathrm{L}}(VAR(\mathrm{L}))$ (indicated simply by Fm_{L}) onto itself, also denoted by σ, by setting $\sigma(\varphi(p_0, \dots, p_n)) = \varphi(\sigma(p_0), \dots, \sigma(p_n))$. Such a σ is called a *substitution.* By a *finitary inference rule over* L we mean any pair $\langle \Gamma, \varphi \rangle$ where $\Gamma \cup \{\varphi\} \in Fm_{\mathrm{L}}$; a rule of the form $\langle \emptyset, \varphi \rangle$ is called an *axiom.* A formula ψ is *directly derivable* from a set of formulas Δ by the rule $\langle \Gamma, \varphi \rangle$ if there is a substitution σ such that $\sigma(\varphi) = \psi$

and $\sigma(\Gamma) \subseteq \Delta$. Given a set of rules and axioms S, the *deductive consequence* $\vdash_S$ is the smallest relation between a set of formulas Δ and a single formula ψ such that $\Delta \vdash_S \psi$ holds iff ψ is contained in the smallest set of formulas that includes Δ together with all the substitution instances of the axioms of S, and is closed under direct derivability by the inference rules of S. A *deductive system over* L is a couple $\langle \mathrm{L}, \vdash_S \rangle$, for some set of rules and formulas S, which sometimes we simply indicate with S.

Let $\langle \mathrm{L}, \vdash_S \rangle$ be a deductive system and $\mathcal{K}$ be a class of algebras. $\mathcal{K}$ is an *equivalent algebraic semantics for* $\langle \mathrm{L}, \vdash_S \rangle$ if there is a finite set $\delta_i(p) \approx \varepsilon_i(p)$, for $i < n$, of equations with a single variable p and a finite system $\Delta_j(p, q)$ of formulas with two variables, such that:

1. for all $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathrm{L}}$ and each $j < n$,

$$\Gamma \vdash_S \varphi \iff \{\delta_i(\psi) \approx \varepsilon_i(\psi) \mid i < n, \psi \in \Gamma\} \models_{\mathcal{K}} \delta_j(\varphi) \approx \varepsilon_j(\varphi)$$

2. for every $\varphi \approx \psi \in \mathrm{Eq(L)}$,

$$\varphi \approx \psi \dashv\models_{\mathcal{K}} \{\delta_i(\Delta(\varphi, \psi)) \approx \varepsilon_i(\Delta(\varphi, \psi))\}$$

The $\delta_i \approx \varepsilon_i$, $i < n$, are called *defining equations* for S and $\mathcal{K}$, while the system of formulas $\Delta_j(p, q)$ is called a system of *equivalence formulas* of S and $\mathcal{K}$. As an example, consider the deductive system generated by the Classical Propositional Calculus (see [31]) and the single Boolean algebra

$$\mathbf{2} = \langle \{\bot, \top\}, \vee, \wedge, \rightarrow, \neg, \bot, \top \rangle.$$

The class of algebras $\{\mathbf{2}\}$ is an equivalent algebraic semantics of Classical Propositional Calculus with defining equation $p \approx \top$ and equivalence formula $(p \rightarrow q) \wedge (q \rightarrow p)$, as it is easy to see.

2.2 Classical Logic

A *first order classical language* L consists of:

1. a denumerable set of symbols called *propositional variables*, indicated as $VAR(\mathrm{L})$, whose members are indicated by the metavariables $x, y, z, x_1, x_2, \ldots$;

2. a set of *constant symbols*, $C(\mathrm{L})$;
3. a set of *function symbols*, $F(\mathrm{L})$;
4. a nonempty set of *relation symbols*, $R(\mathrm{L})$;
5. a set of *connective symbols* $CON(\mathrm{L}) = \{\circ_1, \ldots, \circ_n\}$;
6. a set of *quantifier symbols*, $Q(\mathrm{L}) = \{\forall, \exists\}$.

The *arity* of each nonconstant symbol is given by the function

$$AR(\mathrm{L}) : F(\mathrm{L}) \cup R(\mathrm{L}) \cup CON(\mathrm{L}) \to \mathbb{N}.$$

The set of *terms* of a first order classical language L, Ter(L), is the smallest set such that

1. $VAR(\mathrm{L}) \cup C(\mathrm{L}) \subseteq \mathrm{Ter}(\mathrm{L})$;
2. if $t_1, \ldots, t_n \in \mathrm{Ter}(\mathrm{L})$ and $f \in F(\mathrm{L})$ is such that $AR(\mathrm{L})(f) = n$, then

$$f(t_1, \ldots, t_n) \in \mathrm{Ter}(\mathrm{L}).$$

The set of *well-formed formulas of* L (or, briefly, *formulas*), $FORM(\mathrm{L})$, is the smallest set such that

1. if $\circ \in CON(\mathrm{L})$ is such that $AR(\mathrm{L})(\circ) = 0$, then $\circ \in FORM(\mathrm{L})$;
2. if $t_1, \ldots, t_n \in \mathrm{Ter}(\mathrm{L})$ and $P \in R(\mathrm{L})$ is such that $AR(\mathrm{L})(P) = n$, then $P(t_1, \ldots, t_n) \in FORM(\mathrm{L})$;

these kind of formulas are called *atomic*, and with At(L) we indicate the set containing all the atomic formulas;

3. if $A_1, \ldots, A_n \in FORM(\mathrm{L})$ and $\circ \in CON(\mathrm{L})$ is such that $AR(\mathrm{L})(\circ) = n$ then $\circ(A_1, \ldots, A_n) \in FORM(\mathrm{L})$;
4. if $A \in FORM(\mathrm{L})$, $\Box \in Q(\mathrm{L})$ and $x \in VAR(\mathrm{L})$ then $(\Box x A) \in FORM(\mathrm{L})$; the variable x is called *eigenvariable* and the formula A the *scope of the quantifier*.

From now on we will focus on first order classical languages whose set of connectives is

$$CON(\mathrm{L}) = \{\wedge, \vee, \rightarrow, \neg, \bot, \top\}$$

the first four binary, the fifth unary and the last two 0-ary.

For the sake of simplicity, we will require that all the quantifiers use different eigenvariables and, if a variable x is an eigenvariable, then it can not appear outside the scope of the quantifier which involves x. So, for instance, we shall not allow formulas like

$$(\forall x P(x, y)) \rightarrow Q(x, z).$$

A variable is *bounded* in a formula if it is the eigenvariable of some quantifier occuring in the formula, otherwise it is *free*. If *some* of the variables $x_1, \ldots, x_n$ are free in the formula φ, we will write $\varphi(x_1, \ldots, x_n)$. If $t_1, \ldots, t_n$ are term we will write $\varphi[t_1/x_1, \ldots, t_n/x_n]$ to indicate the formula obtained substituting *simultaneously* each free x_i with t_i. For every formula $\varphi(x_1, \ldots, x_n)$, the *universal closure of* φ, indicated as $\forall\vec{x}\varphi$, is the formula $\forall x_1 \ldots \forall x_n \varphi(x_1, \ldots, x_n)$. Similarly we define $\exists\vec{x}\varphi$.

If φ is a formula, t a term and x a variable, we say that t is *free for x in φ* iff, for every variable y appearing in t, y is not in the scope of a quantifier with eigenvariable x. For example, the term x is free for z in the formula $\forall y P(z, y)$, but the terms $f(x, y)$ and y are not.

Clearly, terms and formulas alone have no meaning at all by itself: in fact, these definitions were introduced with the specific target to treat every kind of mathematical reasoning, to talk about and study *proofs* in general. In order to regain sense, we have to contextualize them, which is to introduce an *environment* to interpret the symbols of our language.

Let L be a first order language. A *first order structure over* L is a quadruple $\mathcal{A} = \langle A, C^{\mathcal{A}}, F^{\mathcal{A}}, R^{\mathcal{A}}\rangle$ such that A is a nonempty set and

1. $C^{\mathcal{A}} \subseteq A$ is such that for every $c \in C(\mathrm{L})$ there is a $c^{\mathcal{A}} \in C^{\mathcal{A}}$;
2. $C^{\mathcal{A}} \subseteq \bigcup_{k=1}^{\infty} A^{A^k}$ is such that, for every $f \in F(\mathrm{L})$ with $AR(\mathrm{L})(f) = n$, there is an $f^{\mathcal{A}} \in F^{\mathcal{A}} \cap A^{A^n}$;
3. $R^{\mathcal{A}} \subseteq \bigcup_{k=1}^{\infty} A^k$ is such that, for every $P \in R(\mathrm{L})$ with $AR(\mathrm{L})(P) = n$, there is a $P^{\mathcal{A}} \in R^{\mathcal{A}} \cap A^n$.

A *valuation of the structure* $\mathcal{A}$ over L is a function $v : VAR(\mathrm{L}) \longrightarrow A$. Every valuation can be extended uniquely to a function $v^\star : \mathrm{Ter}(\mathrm{L}) \longrightarrow A$ in the following way:

1. for every $x \in VAR(\mathrm{L})$, $v^\star(x) = v(x)$;
2. for every $c \in C(\mathrm{L})$, $v^\star(c) = c^{\mathcal{A}}$;
3. for every $f \in F(\mathrm{L})$ such that $AR(\mathrm{L})(f) = n$, and $t_1, \dots, t_n \in \mathrm{Ter}(\mathrm{L})$, then
$$v^\star(f(t_1, \dots, t_n)) = f^{\mathcal{A}}(v^\star(t_1), \dots, v^\star(t_n)).$$

Evaluations cast a meaning over terms, turning them into members of the structure. We can push the extension forward to formulas in order to say whether a formula is *true* or not in a given structure.

Definition 2.1. Let $\mathcal{A}$ be a first order structure over the language L, v be an evaluation of $\mathcal{A}$ and $\varphi \in FORM(\mathrm{L})\backslash\{\bot\}$. We write $\mathcal{A}, v \models \varphi$ in the following cases:

1. if $\varphi = \top$;
2. if $\varphi = P(t_1, \dots t_n) \in \mathrm{At}(\mathrm{L})$ and $(v^\star(t_1), \dots, v^\star(t_n)) \in P^{\mathcal{A}}$;
3. if $\varphi = \psi \wedge \chi$ and it is true that $\mathcal{A}, v \models \psi$ and $\mathcal{A}, v \models \chi$;
4. if $\varphi = \psi \vee \chi$ and it is true that $\mathcal{A}, v \models \psi$ or $\mathcal{A}, v \models \chi$;
5. if $\varphi = \psi \rightarrow \chi$ and it is true that $\mathcal{A}, v \models \psi$ implies $\mathcal{A}, v \models \chi$;
6. if $\varphi = \neg\psi$ and it is not true that $\mathcal{A}, v \models \psi$;
7. if $\varphi = \forall x \psi$ and for every valuation w of $\mathcal{A}$ such that $w(y) = v(y)$ for all $y \in VAR(\mathrm{L})\backslash\{x\}$ it is true that $\mathcal{A}, w \models \psi$;
8. if $\varphi = \forall x \psi$ and there exists a valuation w of $\mathcal{A}$ such that $w(y) = v(y)$ for all $y \in VAR(\mathrm{L})\backslash\{x\}$ for which it is true that $\mathcal{A}, w \models \psi$.

Furthermore, it is never true that $\mathcal{A}, v \models \bot$.

If $\Gamma \cup \{\varphi\} \subseteq FORM(\mathrm{L})$, we will write $A \models \varphi$ if $\mathcal{A}, v \models \varphi$ is true for every evaluation v of $\mathcal{A}$, and $\mathcal{A} \models \Gamma$ if $\mathcal{A} \models \varphi$ for every $\varphi \in \Gamma$. We say that a formula φ is *valid*, written $\models \varphi$, if, for every structure $\mathcal{A}$ over L, $\mathcal{A} \models \varphi$. Finally, we will write $\Gamma \models \varphi$ if, for every structure $\mathcal{A}$, whenever it is true that $\mathcal{A} \models \Gamma$ then it is true that $\mathcal{A} \models \varphi$.

This notion of truth, introduced in [41], unfortunately does not grant us any effective procedure in order to determine whether a formula is valid or not, because in general the truth of a formula has to be checked in *every* structure over L, even those with infinite domains. That is why we need to develop a *calculus*, i.e. a truth preserving procedure which produces valid formulas. Several calculi are available for first order logic, but the easiest to handle was introduced by Gentzen in [19]: *Natural Deduction* consists of several rules for introducing and eliminating connectives and quantifiers; these rules may be composed together to form a *tree* such that the formula lying on its root is the conclusion and the formulas on its"undischarged" leaves the premises. The rules are reported below.

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}\,(\wedge i) \qquad \frac{\varphi \wedge \psi}{\varphi}\,(\wedge e1) \qquad \frac{\varphi \wedge \psi}{\psi}\,(\wedge e2)$$

$$\frac{\varphi}{\varphi \vee \psi}\,(\vee i1) \qquad \frac{\psi}{\varphi \vee \psi}\,(\vee i2) \qquad \frac{\varphi \qquad \varphi \to \psi}{\psi}\,(\to e)$$

$$\frac{\varphi \qquad \neg\varphi}{\bot}\,(\neg e) \qquad \frac{\bot}{\varphi}\,(\bot e)$$

The following rules allow to *discharge* some of the assumptions, that is remove them from the premises of the tree. For instance, the rule for eliminating $\vee$

$$\frac{\varphi \vee \psi \qquad \begin{matrix}[\varphi]_n \\ \chi\end{matrix} \qquad \begin{matrix}[\psi]_n \\ \chi\end{matrix}}{\chi}\,(\vee e)[n]$$

should be read as: if I have $\varphi \vee \psi$ and from *both* φ and ψ I can get χ, then $\varphi \vee \psi$ yields χ *without assuming* φ and ψ; thus, I can discharge as many occurrences as I want from from leaves of the subderivations of χ. The brackets around the formulas allow us to identify the discharged leaves, and their label "n" serves to track which rule have discarded them.

$$\frac{\begin{matrix}[\varphi]_n \\ \psi\end{matrix}}{\varphi \to \psi}\,(\to i)[n] \qquad \frac{\begin{matrix}[\varphi]_n \\ \bot\end{matrix}}{\neg\varphi}\,(\neg i)[n] \qquad \frac{\begin{matrix}[\neg\varphi]_n \\ \bot\end{matrix}}{\varphi}\,(RAA)[n]$$

$$\frac{\varphi[y/x]}{\forall x\varphi}\ (\forall i) \qquad \frac{\forall x\varphi}{\varphi[t/x]}\ (\forall e)$$

In the rules above, x and y are variables and t is a generic term. In $(\forall i)$ we require that y is not free in any of the removed leaves of the subtree with root in $\varphi[y/x]$.

$$\frac{\varphi[t/x]}{\exists x\varphi}\ (\exists i) \qquad \frac{\exists x\varphi \qquad \begin{matrix}[\varphi[y/x]]_n \\ \psi\end{matrix}}{\psi}\ (\exists e)[n]$$

In $(\exists e)$, the variable y cannot be free neither in ψ nor in any of the removed leaves of the subtree with root in ψ (except, of course, $\varphi[y/x]$).

Definition 2.2. Let L be a first order classical language and $\Gamma \cup \{\varphi\} \subseteq FORM(\mathcal{L})$. A *proof tree with premises in* Γ *and conclusion* φ is defined inductively as follows:

1. for every $\varphi \in \Gamma$,

$$\frac{}{\varphi}$$

   is a proof tree with premises in Γ and conclusion φ;

2. if τ_1 and τ_2 are proof trees with premises Γ_1 and Γ_2, respectively, and conclusions φ and $\varphi \to \psi$, respectively, then

$$\frac{\begin{matrix}\tau_1 \\ \varphi\end{matrix} \qquad \begin{matrix}\tau_2 \\ \psi\end{matrix}}{\varphi \wedge \psi}$$

   is a proof tree with premises in $\Gamma_1 \cup \Gamma_2$ and conclusion $\varphi \wedge \psi$;

3. if τ is a proof tree with premises in Γ and conclusion $\varphi \wedge \psi$, then

$$\frac{\begin{matrix}\tau \\ \varphi \wedge \psi\end{matrix}}{\varphi}$$

   is a proof tree with premises in Γ and conclusion φ;

4. if τ is a proof tree with premises in Γ and conclusion $\varphi \wedge \psi$, then

$$\frac{\begin{array}{c}\tau \\ \varphi \wedge \psi\end{array}}{\psi}$$

is a proof tree with premises in Γ and conclusion ψ;

5. if τ is a proof tree with premises in Γ and conclusion φ, then

$$\frac{\begin{array}{c}\tau \\ \varphi\end{array}}{\varphi \vee \psi}$$

is a proof tree with premises in Γ and conclusion $\varphi \vee \psi$;

6. if τ is a proof tree with premises in Γ and conclusion ψ, then

$$\frac{\begin{array}{c}\tau \\ \psi\end{array}}{\varphi \vee \psi}$$

is a proof tree with premises in Γ and conclusion $\varphi \vee \psi$;

7. if τ_1 and τ_2 are proof trees with premises Γ_1 and Γ_2, respectively, and conclusions φ and $\varphi \to \psi$, respectively, then

$$\frac{\begin{array}{c}\tau_1 \\ \varphi\end{array} \qquad \begin{array}{c}\tau_2 \\ \varphi \to \psi\end{array}}{\psi}$$

is a proof tree with premises in $\Gamma_1 \cup \Gamma_2$ and conclusion ψ;

8. if τ_1 and τ_2 are proof trees with premises Γ_1 and Γ_2, respectively, and conclusions φ and $\neg\varphi$, respectively, then

$$\frac{\begin{array}{c}\tau_1 \\ \varphi\end{array} \qquad \begin{array}{c}\tau_2 \\ \neg\varphi\end{array}}{\bot}$$

is a proof tree with premises in $\Gamma_1 \cup \Gamma_2$ and conclusion $\bot$;

9. if τ is a proof tree with premises in Γ and conclusion $\bot$, then

$$\frac{\begin{array}{c}\tau \\ \bot\end{array}}{\varphi}$$

is a proof tree with premises in Γ and conclusion φ;

10. if τ_1, τ_2 and τ_3 are proof trees with premises Γ_1, Γ_2 and Γ_3, respectively, and conclusions $\varphi \vee \psi$, χ and χ, respectively, then

$$\frac{\begin{matrix}\tau_1 & \tau_2 & \tau_3 \\ \varphi \vee \psi & \chi & \chi\end{matrix}}{\chi}$$

is a proof tree with premises in $\Gamma_1 \cup \Gamma_2' \cup \Gamma_3'$ and conclusion χ, where Γ_2' and Γ_3' are obtained from Γ_2 and Γ_3 by removing φ and ψ respectively, though the removal is optional;

11. if τ is a proof tree with premises in Γ and conclusion ψ, then

$$\frac{\begin{matrix}\tau \\ \psi\end{matrix}}{\varphi \to \psi}$$

is a proof tree with premises in Γ' and conclusion $\varphi \to \psi$, where Γ' is obtained from Γ by removing φ, though the removal is optional;

12. if τ is a proof tree with premises in Γ and conclusion $\perp$, then

$$\frac{\begin{matrix}\tau \\ \perp\end{matrix}}{\neg\varphi}$$

is a proof tree with premises in Γ' and conclusion $\neg\varphi$, where Γ' is obtained from Γ by removing φ, though the removal is optional;

13. if τ is a proof tree with premises in Γ and conclusion $\perp$, then

$$\frac{\begin{matrix}\tau \\ \perp\end{matrix}}{\varphi}$$

is a proof tree with premises in Γ' and conclusion φ, where Γ' is obtained from Γ by removing $\neg\varphi$, though the removal is optional;

14. if τ is a proof tree with premises in Γ and conclusion $\varphi[y/x]$, and y is a variable not free in any of the formulas in Γ, then

$$\frac{\begin{matrix}\tau \\ \varphi[y/x]\end{matrix}}{\forall x\varphi}$$

is a proof tree with premises in Γ and conclusion $\forall x\varphi$;

15. if τ is a proof tree with premises in Γ and conclusion $\forall x\varphi$, then for any term t

$$\frac{\begin{matrix}\tau \\ \forall x\varphi\end{matrix}}{\varphi[t/x]}$$

is a proof tree with premises in Γ and conclusion $\varphi[t/x]$;

16. if τ is a proof tree with premises in Γ and conclusion $\varphi[t/x]$, then

$$\frac{\begin{matrix}\tau \\ \varphi[t/x]\end{matrix}}{\exists x\varphi}$$

is a proof tree with premises in Γ and conclusion $\exists x\varphi$;

17. if τ_1 and τ_2 are proof trees with premises Γ_1 and Γ_2, respectively, and conclusions $\exists x\varphi$ and ψ, respectively, then

$$\frac{\begin{matrix}\tau_1 \\ \exists x\varphi\end{matrix} \qquad \begin{matrix}\tau_2 \\ \psi\end{matrix}}{\psi}$$

is a proof tree with premises in $\Gamma_1 \cup \Gamma_2'$ and conclusion ψ, where Γ_2' is obtained from Γ_2 by removing a formula of the type $\varphi[y/x]$, where y is not free neither in any formula of Γ_2' nor in ψ, though the removal is optional.

If there exists a proof tree with premises in Γ and conclusion φ, we write $\Gamma \vdash \varphi$. In the case that $\Gamma = \emptyset$, instead of $\emptyset \vdash \varphi$, we usually write $\vdash \varphi$.

Given a first order language L and $\Gamma \subseteq FORM(\mathrm{L})$, the *first order theory of* Γ is the set $\mathbf{T}_\Gamma = \{\varphi \in FORM(\mathrm{L}) \mid \Gamma \vdash \varphi\}$. Usually, when we say that $\mathbf{T}$ is a first order theory we implicitly mean that $\mathbf{T} = \mathbf{T}_\Gamma$ for a certain set of formulas Γ, called the *axiom set of* $\mathbf{T}$ and which we shall indicate with $\mathrm{Ax}_\mathbf{T}$. In this spirit, $\mathbf{T} \vdash \varphi$ stands for $\mathrm{Ax}_\mathbf{T} \vdash \varphi$.

The celebrated *Gödel's Completeness Theorem* shows us that the two notions we have introduced, namely $\models$ and $\vdash$ are nothing but the two faces of the same medal.

Theorem 2.1 (Gödel's Completeness Theorem, [20]). *Given a first order language* L, *for every* $\Gamma \cup \{\varphi\} \subseteq FORM(\mathrm{L})$,

$$\Gamma \vdash \varphi \iff \Gamma \models \varphi.$$

So, for every first order theory **T** *and every formula* φ,

$$\mathbf{T} \models \varphi \iff \mathbf{T} \vdash \varphi.$$

A rule

$$\frac{\varphi_1 \quad \dots \quad \varphi_n}{\varphi}$$

is *derivable* in the first order theory **T**, if

$$\mathbf{T} \models \forall \vec{x}((\varphi_1 \wedge \ldots \wedge \varphi_n) \to \varphi).$$

Derivable rules can be used in proof trees as normal rules, and usually provide a remarkable speedup in writing them.

By virtue of Gödel's Completeness Theorem, in order to prove that a rule is derivable in a first order theory **T** it is sufficient to find a proof tree with premises in $\mathrm{Ax}_{\mathbf{T}} \cup \{\varphi_1, \ldots, \varphi_n\}$ and conclusion φ.

2.2.1 The theories $\mathbf{T}_{\leq,<}$ and $\mathbf{T}_{\leq,<,\sim}$

We now introduce two first order classical theories which will turn out to be of interest in later chapters.

$\mathbf{T}_{\leq,<}$ is the first order theory based on the language with two binary predicate symbols $\leq$ and $<$, and two constant symbols 0 and 1, whose axiom set contains the following formulas:

$$
\begin{array}{rl}
(\mathrm{T1}) & \forall x(\neg(x < x)) \\
(\mathrm{T2}) & \forall x \forall y \forall x((x < y \wedge y < z) \to x < z) \\
(\mathrm{T3}) & 0 < 1 \\
(\mathrm{T4}) & \forall x(x \leq x) \\
(\mathrm{T5}) & \forall x \forall y \forall x((x \leq y \wedge y \leq z) \to x \leq z) \\
(\mathrm{T6}) & \forall x \forall y(x \leq y \vee y \leq x) \\
(\mathrm{T7}) & \forall x \forall y(x < y \to \neg(y \leq x)) \\
(\mathrm{T8}) & \forall x(0 \leq x) \\
(\mathrm{T9}) & \forall x(x \leq 1) \\
(\mathrm{T10}) & \forall x \forall y(x < y \vee y \leq x)
\end{array}
$$

Basically $\mathbf{T}_{\leq,<}$ is the theory of *bounded total orders*, where 0 and 1 play the roles of the bottom and the top element, while $\leq$ and $<$ give the ordering between the elements. Notice that, by axiom (T5), the *transitive rule* is derivable in $\mathbf{T}_{\leq,<}$:

$$\dfrac{x \leq y \qquad y \leq z}{x \leq z}\ (\mathrm{tr})$$

Proposition 2.2. *The following hold:*

1. $\mathbf{T}_{\leq,<} \models \forall x \forall y \forall z((x \leq y \wedge y < z) \rightarrow x < z)$;
2. $\mathbf{T}_{\leq,<} \models \forall x \forall y(\neg(x \leq y) \rightarrow y < x)$;
3. $\mathbf{T}_{\leq,<} \models \forall x \forall y(\neg(x < y) \rightarrow y \leq x)$.

This means that the following rules are derivable in $\mathbf{T}_{\leq,<}$*:*

$$\dfrac{x \leq y \qquad y < z}{x < z}\ (\mathrm{tr} \leq <) \qquad \dfrac{\neg(x \leq y)}{y < x}\ (\neg \leq) \qquad \dfrac{\neg(x < y)}{y \leq x}\ (\neg <)$$

Proof. This is the proof tree for (1):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x < z \vee z \leq x}\ (\mathrm{T10}) \qquad \dfrac{\dfrac{[\neg(x < z)]_2 \qquad [x < z]_1}{\bot}\ (\neg e)}{z \leq x}\ (\bot e) \qquad [z \leq x]_1}{z \leq x}\ (\vee e)[1] \qquad x \leq y}{z \leq y}\ (\mathrm{tr}) \qquad \dfrac{y < z \qquad \dfrac{}{y < z \rightarrow \neg(z \leq y)}\ (\mathrm{T7})}{\neg(z \leq y)}\ (\mathrm{tr})}{\dfrac{\bot}{x < z}\ (RAA)[2]}\ (\neg e)$$

This, instead, is the proof tree for (2):

$$\dfrac{\dfrac{}{y < x \vee x \leq y}\ (\mathrm{T10}) \qquad \dfrac{\dfrac{\neg(x \leq y) \qquad [y < x]_1}{\bot}\ (\neg e)}{y < x}\ (\bot e) \qquad [y < x]_1}{y < x}\ (\vee e)[1]$$

The proof tree for (3) is similar. □

$\mathbf{T}_{\leq,<,\sim}$ is the first order theory based on the language with two binary relation symbols $\leq$ and $<$, one unary operation symbol $\sim$, and two constant symbols 0 and 1, whose axiom set contains the formulas (T1)-(T10) plus the following ones:

$$\begin{array}{ll} (\sim 1) & \forall x \forall y(x \leq y \rightarrow \sim y \leq \sim x) \\ (\sim 2) & \forall x \forall y(x < y \rightarrow \sim y < \sim x) \\ (\sim 3) & \forall x(\sim\sim x \leq x \wedge x \leq \sim\sim x) \end{array}$$

$\mathbf{T}_{\leq,<,\sim}$ is nothing but $\mathbf{T}_{\leq,<}$ enriched with a unary, involutive, order-reversing operation. Clearly, the following rules are derivable in $\mathbf{T}_{\leq,<,\sim}$:

$$\frac{x \leq y}{\sim y \leq \sim x}\,(\sim\leq) \qquad \frac{x < y}{\sim y < \sim x}\,(\sim<)$$

Proposition 2.3. *The following hold:*

1. $\mathbf{T}_{\leq,<,\sim} \models \forall x \forall y (x \lhd \sim y \rightarrow y \lhd \sim x)$*, where* $\lhd \in \{\leq, <\}$*;*

2. $\mathbf{T}_{\leq,<,\sim} \models \forall x \forall y (\sim x \lhd y \rightarrow \sim y \lhd x)$*, where* $\lhd \in \{\leq, <\}$*.*

Proof. We just show the proof tree of (1) in the case $\lhd = \leq$:

$$\cfrac{\cfrac{\overline{\sim\sim y \leq y \wedge y \leq \sim\sim y}}{y \leq \sim\sim y}\,(\wedge e1) \qquad \cfrac{x \leq \sim y}{\sim\sim y \leq \sim x}\,(\sim\leq)}{y \leq \sim x}\,(\mathrm{tr})$$

□

2.3 Fuzzy Logics

As pointed out in the introduction of [34], *Logics come in many guises*, that is any logic may arise as the same manifestation of very different backgrounds. Take for example *Classical Logic*: in the previous section we have seen how it can be presented *semantically* using the Tarski's concept of truth, or *syntactically* via Gentzen's Natural Deduction.

Fuzzy logics arise in the context of *reasoning under vagueness*, formalizing common natural language predicates such as "hot" and "tall": in contrast with Classical Logic, in which a formulas can only be true or false (that is, take as values 1 or 0 only), in Fuzzy Logic a formula may be assigned every real value between 0 and 1, interpreted as the *magnitude* of its truth; connectives are therefore interpreted as functions on $\mathbb{R}$. Typical design choices in this framework are made as to which real numbers to take as truth values and what kind of properties connectives should have. This aspect of Fuzzy Logic shall be exploited in the next section.

However, much like Classical Logic, together with this semantical approach there is also a syntactical one, based on Hilbert systems, a calculi popularized by [24], which provides a great kinship with Algebra. We shall deal with it after the next section.

From now on, our work shall focus on *propositional fuzzy logics*, that is, basically, fuzzy logics without quantifications.

2.3.1 Fuzzy logics as logics based on real numbers

Definition 2.3. A *propositional language* L consists of:

1. a denumerable set of symbols called *propositional variables*, indicated as $VAR(\mathrm{L})$, whose members are indicated by the metavariables $p, q, r, p_0, p_1, \ldots$;
2. a set of *connectives* $CON(\mathrm{L}) = \{\circ_1, \ldots, \circ_n\}$, with arities given by the function $AR(\mathrm{L}) : CON(\mathrm{L}) \longrightarrow \mathbb{N}$.

The *set of well-formed formulas of* L, $FORM(\mathrm{L})$, is the smallest set such that:

1. $VAR(\mathrm{L}) \subseteq FORM(\mathrm{L})$;
2. if $A_1, \ldots, A_m \in FORM(\mathrm{L})$ and $AR(\mathrm{L})(\circ_i) = m$ then $\circ_i(A_1, \ldots, A_m) \in FORM(\mathrm{L})$.

As pointed out earlier, in order to specify a fuzzy logic we have to make certain design choices. These are represented by *logical matrices*.

Definition 2.4. A *logical matrix* $\mathcal{M}$ for a propositional language L is a triple $\langle \mathcal{T}, \mathcal{D}, \mathcal{C} \rangle$, where:

1. $\mathcal{T}$ is a non-empty set of *truth values*;
2. $\mathcal{D}$ is a non-empty subset of $\mathcal{T}$ of *designated truth values*;
3. $\mathcal{C}$ is a set of *truth functions* for each connective of L:

$$\mathcal{C} = \{\widetilde{\circ} : \mathcal{T}^m \longrightarrow \mathcal{T} \mid \circ \in CON(\mathrm{L}) \text{ where } AR(\mathrm{L})(\circ) = m\}$$

A *many-valued logic* is a couple $\mathcal{L} = (\mathrm{L}, \mathcal{M})$ such that L is a propositional language and $\mathcal{M}$ a logical matrix for L. If the set of truth values $\mathcal{T}$ of $\mathcal{M}$ is finite, we say that $\mathcal{L}$ is a *finite-valued logic*; otherwise we say that $\mathcal{L}$ is an *infinite-valued logic*.

Definition 2.5. A *valuation* for a logic $\mathcal{L} = (\mathrm{L}, \mathcal{M})$ is a function $v : VAR(\mathrm{L}) \longrightarrow \mathcal{T}$ extended to $FORM(\mathrm{L})$ by:

$$v(\circ(\varphi_1, \ldots, \varphi_m)) = \widetilde{\circ}(v(\varphi_1), \ldots, v(\varphi_m)).$$

We say that a formula $\varphi \in FORM(\mathrm{L})$ is *valid* for $\mathcal{L}$, written $\models_{\mathcal{L}} \varphi$, if $v(\varphi) \in \mathcal{D}$ for all valuations v for $\mathcal{L}$.

Definition 2.6. We define the following properties for a generic binary function $* : I^2 \longrightarrow I$ with domain $I \subseteq \mathbb{R}$:

Associativity: $*$ is *associative* if $(x * y) * z = x * (y * z)$ for all $x, y, z \in I$;

Commutativity: $*$ is *commutative* if $x * y = y * x$ for all $x, y \in I$;

Monotonicity: $*$ is *increasing* if $x \leq y$ implies both $x*z \leq y*z$ and $z*x \leq z*y$ for all $x, y, z \in I$;

Identity: an element $e \in I$ is an *identity* for $*$ if $x * e = e * x = x$ for all $x \in I$;

Continuity: $*$ is *continuous* if for all $x, y \in I$, given a sequence $(x_i)_{i \leq 0}, x_i \in I$, such that $x = \lim_{i\to\infty} x_i$, then also $\lim_{i\to\infty}(x * x_i) = x * y$;

Left-Continuity: $*$ is *left-continuous* if for all $x, y \in I$, given a sequence $(x_i)_{i \leq 0}, x > x_i \in I$, such that $x = \lim_{i\to\infty} x_i$, then also $\lim_{i\to\infty}(x * x_i) = x * y$.

The very first requirement we impose on connectives falls upon the *product*: the fuzzy product shall be a *uninorm*.

Definition 2.7. A *uninorm* is a binary function $* : [0,1]^2 \longrightarrow [0,1]$ that is associative, commutative, increasing and has an identity $e \in [0,1]$. A uninorm $*$ is *conjunctive* if it satisfies $1 * 0 = 0 * 1 = 0$. A uninorm is a *t-norm* if $e = 1$.

It turns out, see [38], that every continuous t-norm is locally isomorphic to one of the following *fundamental t-norms*:

1. Łukasiewicz t-norm: $x *_{\text{Ł}} y = \max(0, x + y - 1)$;
2. Gödel t-norm: $x *_{\mathbf{G}} y = \min(x, y)$;
3. Product t-norm: $x *_{\Pi} y = x \cdot y$ (product of reals).

The standard example of a t-norm which is left-continuous but not continuous is that of the *Nilpotent Minimum* t-norm:

$$x *_{NM} y = \begin{cases} \min(x, y) & \text{if } x + y > 1 \\ 0 & \text{otherwise} \end{cases}$$

The fuzzy implication shall be a connective very close to the product: its *residuum*.

Definition 2.8. A binary function $* : [0,1]^2 \longrightarrow [0,1]$ is said to be *residuated* if there exists a binary function $\Rightarrow : [0,1]^2 \longrightarrow [0,1]$, called the *residuum of* $*$, such that $*$ and $\Rightarrow$ satisfy for all $x, y, z \in [0,1]$ the *residuation law*

$$x * y \leq z \text{ iff } x \leq y \Rightarrow z.$$

Proposition 2.4. *The following hold:*

1. *Łukasiewicz implication:* $x \Rightarrow_{L} y = \min(1, 1 - x + y)$
2. *Gödel implication:* $x \Rightarrow_{\mathbf{G}} y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } y < x \end{cases}$
3. *Product implication:* $x \Rightarrow_{\Pi} y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{if } y < x \end{cases}$

Proposition 2.5. *([23]) For every uninorm* $*$*:*

1. *if* $*$ *has a residuum* $\Rightarrow$ *and is conjunctive then for all* $x, y \in [0,1]$*:*

$$x \Rightarrow y = \max\{z \mid z * x \leq y\};$$

2. $*$ *is residuated if and only if it is left-continuous and conjunctive.*

Proof. Take any conjunctive uninorm $*$. Then:

1. by definition, $(x \Rightarrow y) * x \leq y$ so $x \Rightarrow y \leq \max\{z \mid z * x \leq y\}$; on the other hand, if $w * x \leq y$ then, by the residuation law, $w \leq x \Rightarrow y$ so we have
$$x \Rightarrow y = \max\{z \mid z * x \leq y\}$$
has required;
2. define, for a fixed z, $f(x) = x * z$. Since f is non-decreasing, f commutes with sups iff it is continuous iff $*$ is left-continuous. Thus, taking $x \Rightarrow y = \max\{z \mid z * x \leq y\}$, we have
$$x * (x \Rightarrow y) = x * \sup\{z \mid x * z \leq y\} = \sup\{x * z \mid x * z \leq y\} \leq y,$$
hence $x \Rightarrow y$ as above is the residuum of $*$ iff $*$ is left-continuous.

□

As stated in [34], for *continuous* t-norms we obtain the remarkable bonus of being able to define the functions min and max using just the t-norm and its residuum.

Proposition 2.6. *For all continuous t-norms* $*$ *with residuum* $\Rightarrow$*:*

1. $\min(x, y) = x * (x \Rightarrow y)$;
2. $\max(x, y) = \min((x \Rightarrow y) \Rightarrow y, (y \Rightarrow x) \Rightarrow x)$.

Putting things together, we get the definition of *propositional fuzzy logic.*

Definition 2.9. Take the propositional language $\text{L}_{\mathcal{F}}$ with connectives

$$CON(\text{L}_{\mathcal{F}}) = \{\&, \rightarrow, \wedge, \vee, t, f, \bot, \top\}.$$

For any conjunctive uninorm $* : [0,1]^2 \longrightarrow [0,1]$ with identity e and residuum $\Rightarrow$, we define the *fuzzy propositional logic*

$$PL(*) = (\text{L}_{\mathcal{F}}, \langle [0,1], [e,1], \{*, \Rightarrow, \min, \max, e, 0\}\rangle)$$

Notice that a *valuation* for $PL(*)$ is a function $v : VAR(\text{L}_{\mathcal{F}}) \longrightarrow [0,1]$ extended to $FORM(\text{L}_{\mathcal{F}})$ as follows:

$$\begin{array}{rl} v(\varphi\&\psi) = v(\varphi) * v(\psi) & v(\varphi \rightarrow \psi) = v(\varphi) \Rightarrow v(\psi) \\ v(\varphi \wedge \psi) = \min(v(\varphi), v(\psi)) & v(\varphi \vee \psi) = \max(v(\varphi), v(\psi)) \\ v(t) = e & v(f) \in [0,1] \\ v(\bot) = 0 & v(\top) = 1 \end{array}$$

$\varphi \in FORM(\text{L}_{\mathcal{F}})$ is valid in $PL(*)$ iff $v(\varphi) \geq e$ for all valuations v.

By definition, for every uninorm $*$ with residuum $\Rightarrow$, a formula φ is valid in $PL(*)$ iff it is valid in the algebra

$$\mathbf{A}(*, f) = \langle [0,1], \min, \max, *, \Rightarrow, e, f, 0, 1\rangle.$$

We note also that if $*$ is a continuous t-norm then, by Proposition (2.6), it is sufficient to base $PL(*)$ on a language with connectives $\&, \rightarrow$ and $\bot$, defining

$$\begin{array}{rcl} \varphi \wedge \psi & = & \varphi\&(\varphi \rightarrow \psi) \\ \varphi \vee \psi & = & ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi) \\ t & = & \bot \rightarrow \bot \end{array}$$

2.3.2 Fuzzy logics as axiomatic extensions of HUL

An *Hilbert system* for a logic $\mathcal{L}$ on a propositional language L, indicated as $\mathsf{H}\mathcal{L}$, is composed by a set of formulas of $FORM(\mathrm{L})$, called *axioms*, and a set of *rules*: a rule is a set of *inferences*, namely a tuple $(\varphi_1, \dots, \varphi_n, \varphi) \in FORM(\mathrm{L})^{n+1}$, where the formulas $\varphi_1, \dots, \varphi_n$ are called *premises* and φ is called *conclusion*. Rules are typically presented via *schemata*, i.e. formulas with variables replaced by formula meta-variables $\varphi, \psi, \chi, \dots$, while axioms are usually presented as schemata with no premises.

Hylbert systems for fuzzy logics where first introduced in [23], which was followed by other axiomatizations for several fuzzy logics, such as in [16]. The power of this approach consists in making fuzzy logics algebraizable in the sense of Blok and Pigozzi, with equivalent algebraic semantic the class of *bounded pointed commutative residuated lattices*, introduced later.

As an example, a famous Hilbert system for Classical Logic is based on the language $\mathrm{L} = \{\rightarrow, \neg\}$, have axiom schemata:

$$
\begin{array}{ll}
\text{(CL1)} & \varphi \rightarrow (\psi \rightarrow \varphi) \\
\text{(CL2)} & (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)) \\
\text{(CL3)} & (\neg\varphi \rightarrow \neg\psi) \rightarrow (\varphi \rightarrow \psi)
\end{array}
$$

and the “modus ponens” rule:

$$
\frac{\varphi \qquad \varphi \rightarrow \psi}{\psi}\ \text{(mp)}
$$

Given an Hilbert system $\mathsf{H}\mathcal{L}$ based on a language L and $\Gamma \cup \{\varphi\} \subseteq FORM(\mathrm{L})$, a formula φ is said to be *derivable in* $\mathsf{H}\mathcal{L}$ *from the set of formulas* Γ, written $\Gamma \vdash_{\mathsf{H}\mathcal{L}} \varphi$ if there is an upward tree rooted in φ such that every leaf is either an axiom of $\mathsf{H}\mathcal{L}$ or a formula in Γ, and every other formula is obtained from the ones standing immediately above it by application of one of the rules of $\mathsf{H}\mathcal{L}$.

Definition 2.10. The Hilbert system HMAILL (for *Multiplicative Additive Intuitionistic Linear Logic*) is an Hilbert system based on $\mathrm{L}_{\mathcal{F}}$ axiomatized by the following axiom schemata:

$$
\begin{array}{ll}
\text{(B)} & (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)) \\
\text{(C)} & (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi)) \\
\text{(I)} & \varphi \rightarrow \varphi
\end{array}
$$

(&1)	$\varphi \to (\psi \to (\varphi \& \psi))$
(&2)	$(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$
(e1)	e
(e2)	$\varphi \to (e \to \varphi)$
(∧1)	$(\varphi \wedge \psi) \to \varphi$
(∧2)	$(\varphi \wedge \psi) \to \psi$
(∧3)	$((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$
(∨1)	$\varphi \to (\varphi \vee \psi)$
(∨2)	$\psi \to (\varphi \vee \psi)$
(∨3)	$((\varphi \to \psi) \wedge (\varphi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$
(⊥)	$\bot \to \varphi$
(⊤)	$\varphi \to \top$

The *rules* of HMAILL are *modus ponens* and (adj):

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\ (mp) \qquad \frac{\varphi \qquad \psi}{\varphi \wedge \psi}\ (adj)$$

An $\mathsf{H}\mathcal{L}$-*extension* for a Hilbert system $\mathsf{H}\mathcal{L}$ based on $FORM(\mathrm{L}_{\mathcal{F}})$ consists of $\mathsf{H}\mathcal{L}$ extended with axiom schemata based on $FORM(\mathrm{L}_{\mathcal{F}})$.

The following axiom schemata give birth to the most interesting HMAILL-extensions, as reported in the table below (we write φ^n to mean $\varphi \& \ldots \& \varphi$ n times).

(W)	$(\varphi \to e) \wedge (f \to \varphi)$
(C_n)	$\varphi^{n-1} \to \varphi^n$
(INV)	$\neg\neg\varphi \to \varphi$
(PRL)	$(\varphi \to \psi) \vee (\psi \to \varphi)$
(DIS)	$(\varphi \wedge (\psi \vee \chi)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$
(RCAN)	$(\top \to \varphi) \vee (\varphi \to \bot) \vee ((\varphi \to (\varphi \& \psi)) \to \psi)$

Label	Name	Axiomatization
HIL	Intuitionistic Logic	HMAILL+(W)+(C_2)
HCL	Classical Logic	HIL+(INV)
HUL	Uninorm Logic	HMAILL+(PRL)+(DIS)
HMTL	Monoidal t-norm Logic	HUL+(W)
HBL	Basic Logic	HMTL+(DIV)
HcnBL	n-contractive Basic Logic	HBL+(C_{n+1})
HG	Gödel Logic	HMTL+(C_2)
HŁ	Łukasiewicz Logic	HBL+(INV)
HΠ	Product Logic	HBL+(RCAN)

As already announced, the equivalent algebraic semantics for HMAILL is the variety of *bounded pointed commutative residuated lattices*, indicated as $\mathbb{BPCRL}$.

Definition 2.11. A *bounded pointed commutative residuated lattice*, or *bpcrl*, is an algebra

$$\mathbf{L} = \langle L, \cap, \cup, *, \Rightarrow, e, f, \top, \bot \rangle$$

with four binary operations and four constants such that

1. $\langle L, \cap, \cup, \top, \bot \rangle$ is a bounded lattice;
2. $\langle L, *, e \rangle$ is a commutative semigroup with unit element e;
3. the *law of residuation* holds: for all $x, y, z \in [0, 1]$

$$x * y \leq z \text{ iff } y \leq x \Rightarrow z.$$

Though the law of residuation is far from being an identity, it could be seen that there exist identities equivalent to it, making the class of bpcrls a variety. A clue that we are going in the right direction is that, for every residuated uninorm $*$ with residuum $\Rightarrow$, and $f \in [0, 1]$, the algebra $\mathbf{A}(*, f)$ is a bpcrl.

We are now going to report a completeness result for every HMAILL-extension. First, if $\mathsf{H}\mathcal{L}$ is an HMAILL-extension, an $\mathcal{L}$-*algebra* is a bpcrl in which the axioms of $\mathsf{H}\mathcal{L}$ are valid. With $\mathsf{GEN}(\mathcal{L})$ we shall indicate the class of all $\mathcal{L}$-algebras.

Theorem 2.7. *([34]) For any* $\Gamma \cup \{\varphi\} \subseteq FORM(\mathrm{L}_{\mathcal{F}})$,

$$\Gamma \vdash_{\mathsf{H}\mathcal{L}} \varphi \textit{ iff } \Gamma \models_{\mathsf{GEN}(\mathcal{L})} \varphi.$$

A rather remarkable property for an HMAILL-extension $\mathsf{H}\mathcal{L}$ is that to be *standard complete*, which shortly means completeness with respect to the algebras of the equivalent algebraic semantics of the logic with universe $[0,1]$ and the usual order, called *the standard algebras* and indicated by $\mathsf{STAN}(\mathcal{L})$. Standard completeness for $\mathsf{H}\mathcal{L}$ is the following assertion:

$$\Gamma \vdash_{\mathsf{H}\mathcal{L}} \varphi \text{ iff } \Gamma \models_{\mathsf{STAN}(\mathcal{L})} \varphi.$$

This property is central, in order to recover the very essence of being a fuzzy logic, and it is also really helpful because, in order to prove something for the whole logic, it is sufficient to prove it for standard algebras, which is usually a rather simpler thing to do. Collecting several results from [34], [14], [23] and [27], we have the following theorem.

Theorem 2.8. *Every logic* $\mathcal{L} \in \{\mathrm{UL}, \mathrm{MTL}, \mathrm{BL}, \mathbf{G}, \text{Ł}, \Pi\}$ *is standard complete.*

The fact that UL (resp., MTL, BL) is standard complete implies also that UL is a complete axiomatization of the intersection of all logics given by residuated uninorms (resp. left-continuous t-norms, continuous t-norms). This is the meaning of following theorem, which states the equivalence of the two presentations of fuzzy logics (at least for the most prominent fuzzy logics).

Corollary 2.9. *For each* $\mathcal{L} \in \{\mathrm{UL}, \mathrm{MTL}, \mathrm{BL}, \mathbf{G}, \text{Ł}, \Pi\}$*,* φ *is provable in* $\mathsf{H}\mathcal{L}$ *iff* φ *is valid in* $PL(*_{\mathcal{L}})$*.*

Chapter 3

Semiprojective Logics and Sequents-of-Relations Calculi

In this chapter we introduce the main concepts and results, providing proofs and examples whenever they are needed for a full comprehension of the subject. The first section shall deal with the main class of logics we will focus on, namely *semiprojective logics*, and a proper subclass of it, that of *projective logics*. The second section introduces *sequents-of-relations calculi*, the uniform method we shall adopt and study for providing analytic calculi to all semiprojective logics. The final section will be focused on the classical results on soundness, completeness and computational complexity of the calculi provided.

We point out that the logics $\mathcal{L}$ we consider are many-valued, are algebraizle in the sense of Blok and Pigozzi and have a locally finite variety $\mathcal{V}_{\mathcal{L}}$ as their equivalent algebraic semantics. We further assume that $\mathcal{V}_{\mathcal{L}}$ is a suitable variety of residuated lattices possibly with additional operations.

3.1 Semiprojective Logics

First of all, having fixed a logic $\mathcal{L}$, we focus on first order classical theories $\mathbf{T}$, called *semantic theories for* $\mathcal{L}$, which satisfies the following conditions:

(F) $\mathbf{T}$ contains unary function symbols $f_1, \ldots, f_t$ corresponding to the homonym unary connectives of $\mathcal{L}$ and the constant symbols of $\mathbf{T}$ coincide with the constants of $\mathcal{L}$;

(U) for each model $\mathcal{M}$ of $\mathbf{T}$, each valuation μ on $\mathcal{M}$, and for each $p, q = 1, \ldots, t$,

$$f_p^{\mathcal{M}}(f_q(\mu(x))) = \mu(f_h(x)),$$

for some h in $\{1, \ldots, t\}$;

(D) the set

$$\Pi_1^{\mathbf{T}} = \{\varphi \mid \mathbf{T} \models \varphi \text{ and } \varphi \text{ is universal}\}$$

is decidable.

By virtue of conditions (F) and (U), every atomic formula of $\mathbf{T}$ is of the form $R(t_1, \ldots, t_k)$, where each t_i is either a variable x, a constant c or a term of the form $f_q(x)$ or $f_q(c)$, where $q = 1, \ldots, t$. This happens because every composition of two unary function symbols shrinks to another unary function symbol, thanks to condition (U).

This is not enough, although, because the logics under investigation are characterized by a special format of their connectives. We call an n-ary connective $\circ$ of $\mathcal{L}$ *semiprojective* (with respect to an interpretation based on a semantic theory $\mathbf{T}$) if the corresponding truth function $\widetilde{\circ}$ can be written in the following form:

$$\widetilde{\circ}(x_1, \ldots, x_n) = \begin{cases} t_1 & \text{if } A_1 \\ \vdots & \vdots \\ t_n & \text{if } A_n \end{cases}$$

where each t_i is either a truth constant, a variable among $x_1, \ldots, x_n$, or a term of the form $f_q(x)$ or $f_q(c)$, where $q = 1, \ldots, t$, $x \in \{x_1, \ldots, x_n\}$ and c is a truth constant. Moreover, the conditions A_i are simple formulas of the semantic theory $\mathbf{T}$ whose free variables are among $\{x_1, \ldots, x_n\}$. And, since $\widetilde{\circ}$ is a total function, these conditions have to satisfy the following properties:

Totality: $\mathbf{T} \models \forall x_1 \cdots \forall x_n \bigvee_{1 \leq i \leq m} A_i$

Functionality: for all $i \neq j$, $i, j \in \{1, \ldots, m\}$: $\mathbf{T} \models \forall x_1 \cdots \forall x_n \neg(A_i \wedge A_j)$

Finally, in order to specify a logic, we need a *notion of designated truth values*: for this purpose, any simple formula $\mathrm{Des}(x)$ of $\mathbf{T}$ with exactly one free variable may be chosen.

Informally, we shall say that a logic $\mathcal{L}$ is *semiprojective* if there exists a semantic theory $\mathbf{T}$ of $\mathcal{L}$ such that all the connectives of $\mathcal{L}$ (except the

f_p's, the unary connectives which have a unary function symbol counterpart in **T**) plus the connectives obtained by the compositions $f_p(\circ)$, where $\circ$ is an arbitrary connective of $\mathcal{L}$, are semiprojective with respect to **T**, and the predicate $\mathrm{Des}(x)$ "catches" exactly the designated truth values of $\mathcal{L}$

The following definition puts on a solid ground everything we have said above.

Definition 3.1 ([37])**.** A logic $\mathcal{L}$ is *semiprojective* if there is a first order classical theory **T** such that the following conditions hold:

1. **T** contains the unary function symbols $f_1, \ldots, f_t$ corresponding to the homonym unary connectives of $\mathcal{L}$, the constant symbols of **T** coincide with the constants of $\mathcal{L}$ and the set

$$\Pi_1^{\mathbf{T}} = \{\varphi \mid \mathbf{T} \models \varphi \text{ and } \varphi \text{ is universal}\}$$

is decidable;

2. for each n-ary connective $\circ$ of $\mathcal{L}$ there are terms s_i, t_j, s'_i, t'_j of **T** which are truth constants, variables in $\{x_1, \ldots, x_n\}$ or of the form $f_q(x_i)$, where $1 \leq q \leq t$, and simple formulas $P^1_\circ(s_1, \ldots, s_n), \ldots, P^m_\circ(s_1, \ldots, s_n)$, $P^1_{f_p(\circ)}(s'_1, \ldots, s'_n), \ldots, P^m_{f_p(\circ)}(s'_1, \ldots, s'_n)$, such that:

   (2.i) for each model $\mathcal{M}$ of **T** and each valuation μ on $\mathcal{M}$, exactly one of the $P^1_\circ(s_1, \ldots, s_n)$ (resp., $P^1_{f_p(\circ)}(s'_1, \ldots, s'_n)$) is satisfied in $(\mathcal{M}, \mu)$;

   (2.ii) let $\mathcal{M}^\star$ be the model obtained by extending $\mathcal{M}$ with the interpretation $\circ^{\mathcal{M}^\star}$ and $f_p^{\mathcal{M}^\star}$ of the connectives of $\mathcal{L}$ defined by

   (a) for each $p, q = 1, \ldots, t$,

$$f_p^{\mathcal{M}^\star}(\mu(x)) = \mu(f_p(x))$$

   and

$$f_p^{\mathcal{M}}(f_q(\mu(x))) = \mu(f_h(x)), \text{ for some } h \in \{1, \ldots, t\};$$

   (b) for each n-ary connective $\circ$ ($n \geq 1$, $\circ \neq f_p$),

$$\circ^{\mathcal{M}^\star}(\mu(x_1), \ldots, \mu(x_n)) = \mu(t_i) \text{ if } \mathcal{M}, \mu \models P^i_\circ(s_1, \ldots, s_n).$$

   Moreover,

$$f_p^{\mathcal{M}^\star, \mu}(x_1, \ldots, x_n) = \mu(t'_i) \text{ if } \mathcal{M}, \mu \models P^i_{f_p(\circ)}(s'_1, \ldots, s'_n);$$

$\mathcal{M}^\star$ is an algebraic model of $\mathcal{L}$;

3. let $\psi^{\mathcal{M}^\star,\mu}$ denote the truth value of ψ in the model $\mathcal{M}^\star$ under μ. There is a simple formula $\mathrm{Des}(x)$ of $\mathbf{T}$ such that for each $\mathcal{L}$-formula ψ, for each model $\mathcal{M}$ of $\mathbf{T}$ and for each valuation μ, one has: $\mathcal{M},\mu \models \mathrm{Des}(\psi^{\mathcal{M}^\star,\mu})$ iff $\psi^{\mathcal{M}^\star,\mu}$ is a designated value of $\mathcal{M}^\star$.

Given a semantic theory $\mathbf{T}$, the *semiprojective logic* $\mathcal{L}$ *associated with* $\mathbf{T}$ is

$$\mathcal{L}_{\mathbf{T}} = \{\varphi \mid \mathcal{M} \models \mathrm{Des}(\varphi^{\mathcal{M}^\star,\mu}) \text{ for all } \mathcal{M} \text{ and } \mu \text{ of } \mathbf{T}\}.$$

The requirements above implies that if a logic $\mathcal{L}$ is semiprojective then $\mathcal{L} = \mathcal{L}_{\mathbf{T}}$.

Easy, and yet very important, examples of semiprojective logics are *Gödel logics*. In fact, their connectives are clearly semiprojective with respect to the semantic theory $\mathbf{T}_{\leq,<}$, as the truth functions of their connectives (conjunction $\wedge$, disjunction $\vee$, and implication $\rightarrow$) show us:

$$\widetilde{\wedge}(x,y) = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } y < x \end{cases} \qquad \widetilde{\vee}(x,y) = \begin{cases} y & \text{if } x \leq y \\ x & \text{if } y < x \end{cases}$$

$$\widetilde{\rightarrow}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } y < x \end{cases}$$

(we do not consider negation, since it can be treated as a derived connective by defining $\neg A := A \rightarrow 0$). "1" is intended as the only designated truth value, so we take $\mathrm{Des}(x) = 1 \leq x$.

Since efficient decision procedure for $\mathbf{T}_{\leq,<}$ are given, for instance, in [8], then $\mathbf{T}_{\leq,<}$ fulfills condition (D) on semantic theories.

It can be seen that the infinite-valued Gödel logic $\mathbf{G}_\infty$ is exactly $\mathcal{L}_{\mathbf{T}_{\leq,<}}$. Instead, every finite-valued Gödel logic $\mathbf{G}_n$ is precisely $\mathcal{L}_{\mathbf{T}_n}$, where $\mathbf{T}_n$ is obtained from $\mathbf{T}_{\leq,<}$ by adding the extra axiom

$$(\mathrm{Fin}_n) \qquad \exists x_1 \cdots \exists x_n \forall y (y \equiv x_1 \vee \ldots \vee y \equiv x_n)$$

where $x \equiv y$ abbreviates $x \leq y \wedge y \leq x$.

However, these are rather peculiar examples of semiprojective logics, since no unary connectives are present in the signature of the logics; in fact, we shall see that Gödel logics belong to a proper subclass of semiprojective logics.

Before going further with more significative examples, we report an interesting fact about semiprojective logics, which shall be crucial later.

Proposition 3.1 ([37]). *Each semiprojective logic with finitely many connectives and constants is locally finite.*

3.1.1 Gödel logics with an involutive negation

A much more complete example is provided by *Gödel logics with an involutive negation*: $\mathbf{G}_{n\sim}$, where $1 \leq n \leq \infty$, is obtained from the Gödel logic $\mathbf{G}_n$ by adding an extra unary connective $\sim$ to the signature, and the following axioms to that of $\mathbf{G}_n$ (remember that $\neg$ is the negation associated to the implication $\neg x = x \to 0$):

$$\begin{array}{rl} (\neg\neg) & \sim\sim \varphi \leftrightarrow \varphi \\ (\text{Rev-S1}) & \neg \sim (\varphi \to \psi) \to \neg \sim (\sim \psi \to \sim \varphi) \\ (\text{Rev-S2}) & \neg\varphi \to \sim \psi \end{array}$$

There is an extra inference rule: $\neg\sim$-generalization

$$\frac{\varphi}{\neg \sim \varphi} \text{ (gen)}$$

In [9] it is shown that $\mathbf{G}_{n\sim}$ is standard complete, so we can take as its semantic theory $\mathbf{T}_{\leq,<,\sim}$ (extended with the proper axiom (Fin_n), if $n < \infty$). Below there is a list of the truth functions of all the connectives of the logic (from now on, we will confuse typographically a connective and its truth function, whenever it appears clear from the context).

$$x \wedge y = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } y < x \end{cases} \qquad \sim (x \wedge y) = \begin{cases} \sim x & \text{if } x \leq y \\ \sim y & \text{if } y < x \end{cases}$$

$$x \vee y = \begin{cases} y & \text{if } x \leq y \\ x & \text{if } y < x \end{cases} \qquad \sim (x \vee y) = \begin{cases} \sim y & \text{if } x \leq y \\ \sim x & \text{if } y < x \end{cases}$$

$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } y < x \end{cases} \qquad \sim (x \to y) = \begin{cases} 0 & \text{if } x \leq y \\ \sim y & \text{if } y < x \end{cases}$$

$$\sim (\sim x) = x$$

Notice that we had to check that the compositions $\sim \circ$, where $\circ \in \{\wedge, \vee, \to\}$, where semiprojective too, and that the composition of $\sim$ with itself was either the identity or $\sim$.

3.1.2 Nilpotent Minimum and Weak Nilpotent Minimum logics

Esteva and Godo, in [16], introduced Nilpotent Minimum logic NM as the logic of the nilpotent minimum t-norm. As Gödel logics with an involutive negation, a semantic theory for NM is given by $\mathbf{T}_{\leq,<,\sim}$; again, "1" shall be the only designated value, so we take $\mathrm{Des}(x) = 1 \leq x$. We state below the truth functions for the connectives of NM (this time & and $\wedge$ does not coincide):

$$x\&y = \begin{cases} 0 & \text{if } x \leq \sim y \\ x & \text{if } \sim y < x \wedge x \leq y \\ y & \text{if } \sim y < x \wedge y < x \end{cases}$$

$$\sim (x\&y) = \begin{cases} 1 & \text{if } x \leq \sim y \\ \sim x & \text{if } \sim y < x \wedge x \leq y \\ \sim y & \text{if } \sim y < x \wedge y < x \end{cases}$$

$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } y < x \wedge \sim x < y \\ \sim x & \text{if } y < x \wedge y \leq \sim x \end{cases}$$

$$\sim (x \to y) = \begin{cases} 0 & \text{if } x \leq y \\ \sim y & \text{if } y < x \wedge \sim x \leq y \\ x & \text{if } y < x \wedge y \leq \sim x \end{cases}$$

$$x \wedge y = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } y < x \end{cases} \qquad \sim (x \wedge y) = \begin{cases} \sim x & \text{if } x \leq y \\ \sim y & \text{if } y < x \end{cases}$$

$$x \vee y = \begin{cases} y & \text{if } x \leq y \\ x & \text{if } y < x \end{cases} \qquad \sim (x \vee y) = \begin{cases} \sim y & \text{if } x \leq y \\ \sim x & \text{if } y < x \end{cases}$$

$$\sim (\sim x) = x$$

Notice that we do not distinguish typographically between $\wedge$ of the logic and $\wedge$ of the semantic theory, since we believe it is clear from the context.

In [16], it is also introduced Weak Nilpotent Minimum logic WNM, which is basically NM without the involutivity of negation. WNM can be treated as a semiprojective logics extending its signature with a unary connective c, corresponding to the function $c(x) = \sim\sim x$ in its semantic theory. Informally, the semantic theory of WNM shall be that of NM without the involutivity of $\sim$ and with the additional function symbol c. The truth functions for $\rightarrow$, $\circ$ and $\sim\circ$, where $\circ \in \{\&, \wedge, \vee\}$, don't change from that of NM, so we will omit them. The remaining ones are:

$$\sim(x \rightarrow y) = \begin{cases} 0 & \text{if } x \leq y \\ \sim y & \text{if } y < x \wedge \sim x \leq y \\ c(x) & \text{if } y < x \wedge y \leq \sim x \end{cases}$$

$$c(x \& y) = \begin{cases} 0 & \text{if } x \leq \sim y \\ c(x) & \text{if } \sim y < x \wedge x \leq y \\ c(y) & \text{if } \sim y < x \wedge y < x \end{cases}$$

$$c(x \rightarrow y) = \begin{cases} 1 & \text{if } x \leq y \\ c(y) & \text{if } y < x \wedge \sim x \leq y \\ \sim x & \text{if } y < x \wedge y \leq \sim x \end{cases}$$

$$c(x \wedge y) = \begin{cases} c(x) & \text{if } x \leq y \\ c(y) & \text{if } y < x \end{cases} \qquad c(x \vee y) = \begin{cases} c(y) & \text{if } x \leq y \\ c(x) & \text{if } y < x \end{cases}$$

$$c(\sim x) = \sim c(x) = \sim x \quad c(c(x)) = c(x) \quad \sim(\sim x) = c(x)$$

3.1.3 n-contractive Basic Logics

cnBL logics where introduced in [10] as an "approximation" of Hajek's Basic Logic BL, in the sense that BL turns out to be the intersection of all cnBL. For every $n \geq 1$, as anticipated, they are obtained from BL by adding the *n-contraction axiom schema*

$$\varphi^n \rightarrow \varphi^{n+1},$$

and their equivalent algebraic semantics is the class of all BL-algebras that are subdirect products of BL-chains that are ordinal sums of MV-chains with at most $n+1$ elements: this means that the provability of a formula in cnBL is equivalent to its validity in all ordinal sums of MV-chains with at most $n+1$ elements.

In their original formulation, cnBL logics are not semiprojective but, as pointed out in [37], they have a conservative extension which is semiprojective, namely *cnBL*$^+$. This logic is obtained extending cnBL' signature with unary connectives $S^1, \ldots, S^n$, where $S^1(x)$ denotes 1 if $x = 1$ and the coatom of the component which x belongs to if $x < 1$, and $S^h(x)$ stands for $(S^1(x))^h$.

Definition 3.2. The language of cnBL$^+$ ($n \geq 1$) is that of cnBL extended with the unary connectives $S^1, \ldots, S^n$. An Hilbert-style axiomatization of cnBL$^+$ consists of (in the following $\varphi \leftrightarrow \psi$ stands for $\varphi \to \psi \wedge \psi \to \varphi$):

1. the axioms of BL plus $\varphi^n \to \varphi^{n+1}$;

2. $\varphi \to S^1(\varphi)$;

3. $(S^1(\varphi))^n \to \varphi^n$;

4. $(S^1(\varphi) \to \psi) \to (\psi \to S^1(\varphi)) \vee ((\psi \to \varphi) \to \varphi)$;

5. $S^h(\varphi) \leftrightarrow (S^1(\varphi))^h$, for $h = 2, \ldots, n$.

The rules are modus ponens and the congruence rule for S^1:

$$\frac{\varphi \leftrightarrow \psi}{S^1(\varphi) \leftrightarrow S^1(\psi)}$$

The logic cnBL$^+$ is algebraizable in the sense of Blok and Pigozzi, and its equivalent algebraic semantics consists of the variety of cnBL$^+$-algebras, that is the variety generated by the ordinal sums of MV-chains with cardinality not greater than $n+1$, with the connective S^1 interpreted as said before. As it is shown in [37], cnBL$^+$ is a conservative extension of cnBL.

The language of the semantic theory $\mathbf{T}^\star$ for cnBL$^+$ contains the binary predicate symbol $\preceq$, the unary predicate symbols $C_{h,k}$, $1 \leq h, k \leq n$, and the unary function symbols S^h, $1 \leq h \leq n$, together with the constants 0, 1. The negations of $\preceq$ and $C_{h,k}$, namely $\prec$ and $C^\star_{h,k}$, are in the language either. $\preceq$ shall be a total preorder which determines an equivalence relation $x \equiv y$ (standing for $x \preceq y \wedge y \preceq x$) on the components and a distribution of the equivalence classes with respect to $\equiv$ in the classes $C_{h,k}$ expressing the order of elements in each component. More precisely:

$x \preceq y$ means that either $y = 1$ or x and y are different from 1 and either they are in the same component or x is in a component below the component of y;

$x \prec y$ (the negation of $y \preceq x$) means that either $x < 1$ and $y = 1$ or x and y are not in the same component and x is in a component below the component of y;

$C_{h,k}(x)$ means that $x < 1$, x belong to a component with $k+1$ elements and $x = S^h(x)$;

$C^\star_{h,k}(x)$ (the negation of $C_{h,k}(x)$) means that either $x = 1$ or x belong to a component not having $k+1$ elements or x belongs to a component with $k+1$ elements and $x \neq S^h(x)$.

Finally we take $\mathrm{Des}(x) = 1 \preceq x$.

Notice that we can define the usual preorder $x \leq y$ as an abbreviation for

$$(x \preceq y) \vee \left((x \equiv y) \wedge \left(\bigvee_{1 \leq i \leq j \leq k \leq n} (C_{j,k}(x) \wedge C_{i,k}(y))\right)\right).$$

The typical model of $\mathbf{T}^\star$ is a totally ordered algebra with maximum 1 and minimum 0, partitioned into equivalence classes, each of which has cardinality at most n. The equivalence class of 1 is a singleton, and each class is convex (i.e., if x and y are members of the same class and $x \leq z \leq y$, then z also belongs to the class). If the equivalence class of x has k elements and $h \leq k$, then $S^h(x)$ is the h^{st} member of the class, counting from the top.

In [37] are reported the full motivations of why the connectives or cnBL^+ are semiprojective; here we shall just report them. Following the notation of that work, for the sake of simplicity (for $1 \leq h \leq m \leq n$) we use the abbreviations

$$\&^h_m(x,y) \text{ for } (x \equiv y) \wedge \left(\bigvee_{i+j=h; i,j \geq 1} (C_{i,m}(x) \wedge C_{j,m}(y))\right),$$

meaning intuitively that x and y are in the same component, the component has $m+1$ elements and $x \& y = (S^1)^h$, where S^1 is the coatom of the component, and

$$\rightarrow^h_m (x,y) \text{ for } (x \equiv y) \wedge (y < x) \wedge \left(\bigvee_{i-j=h; i \leq m; j \geq 1} (C_{i,m}(x) \wedge C_{j,m}(y))\right)$$

meaning that x and y are in the same component, the component has $m+1$ elements and $x \rightarrow y = (S^1)^h$, where S^1 is the coatom of the component.

The list of the truth function of the connectives of cnBL$^+$ follows:

$$x\&y = \begin{cases} x & \text{if } x \prec y \text{ or } 1 \preceq y \\ y & \text{if } y \prec x \text{ or } 1 \preceq x \\ S^h(x) & \text{if } \&^h_m(x,y) \end{cases} \qquad x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } y \prec x \\ S^h(x) & \text{if } \to^h_m (x,y) \end{cases}$$

$$x \wedge y = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } y < x \end{cases} \qquad x \vee y = \begin{cases} y & \text{if } x \leq y \\ x & \text{if } y < x \end{cases}$$

For all $h, k = 1, \ldots, n$:

$$S^h(x\&y) = \begin{cases} S^h(x) & \text{if } x \preceq y \\ S^h(y) & \text{if } y \prec x \end{cases} \qquad S^h(x \to y) = \begin{cases} 1 & \text{if } x \leq y \\ S^h(y) & \text{if } y < x \end{cases}$$

$$S^h(x \wedge y) = \begin{cases} S^h(x) & \text{if } x \leq y \\ S^h(y) & \text{if } y < x \end{cases} \qquad S^h(x \vee y) = \begin{cases} S^h(y) & \text{if } x \leq y \\ S^h(x) & \text{if } y < x \end{cases}$$

$$S^h(S^k(x)) = S^h(x)$$

3.1.4 Projective Logics

Hystorically, semiprojective logics arise after a first attempt to isolate a class of many-valued logics for which an analytic calculus could be automathically provided. In [7], the class of *projective logics* was introduced: basically, the only difference between projective logics and semiprojective logics is that semantic theories are based on function-free languages, so every connective $\circ$ of the logic must be "semiprojective", in the sense that its truth function must be of the following form:

$$\widetilde{\circ}(x_1, \ldots, x_n) = \begin{cases} t_1 & \text{if } A_1 \\ \vdots & \vdots \\ t_n & \text{if } A_n \end{cases}$$

where each t_i is either a truth constant or a variable in $\{x_1, \ldots, x_n\}$, and the conditions A_i are simple formulas of the underlying semantic theory whose free variables are among $\{x_1, \ldots, x_n\}$.

It is very important to note that each atomic formulas of a semantic theory of some projective logic is of the form $R(t_1, \ldots, t_k)$, where t_i are just variables or constants.

With this in mind, it is clear that Gödel logics are projective and, moreover, *every projective logic is also semiprojective.*

However, the class of projective logics is broad by itself, since every finite-valued propositional many-valued logic is projective. To see that, let $\mathcal{L}$ be a n-valued logic; as a semantic theory for $\mathcal{L}$ we take the first order classical theory $\mathbf{T}$ whose signature has a monadic predicate symbol C_i for each truth value c_i, and a constant symbol for every truth value. $\mathbf{T}$ is axiomatized by:

$$\forall x \bigvee_{1 \leq i \leq n} C_i(x)$$
$$\text{for all } i \neq j, 1 \leq i, j \leq n : \forall x \neg(C_i(x) \wedge C_j(x))$$
$$\text{for all } i, 1 \leq i \leq n : C_i(c_i)$$

Since $\mathcal{L}$ is finite, then, whatever connective $\circ$ we shall take, its truth function is clearly projective because every entry

$$\widetilde{\circ}(c_{i_1}, \ldots, c_{i_n}) = c_j$$

into the truth table for the n-ary connective $\circ$ translates into the part

$$\widetilde{\circ}(x_1, \ldots, x_n) = c_j \qquad \text{if } C_{i_1}(x_1) \wedge \ldots \wedge C_{i_n}(x_n)$$

of the definition of the truth function as above. Moreover, if $\{c_{d_1}, \ldots, c_{d_m}\}$ is the set of designated truth values, we shall take $\text{Des}(x) = C_{d_1}(x) \vee \ldots \vee C_{d_m}(x)$ as designating predicate. As a concluding remark, it is easy to see that $\mathcal{L} = \mathcal{L}_{\mathbf{T}}$.

3.2 Sequents-of-Relations Calculi

Semiprojective logics have, as a remarkable feature, the property that analytic calculus can be easily introduced for them thanks to the special format of the truth functions of their connectives. In this section we shall see how this can be done, presenting an homogeneous procedure for any semiprojective logic, first presented in [37] as an extension of the procedure given in [7] for projective logics.

Throughout the section, we fix a semiprojective logic $\mathcal{L}$ and its semantic theory $\mathbf{T}$. The main object of the calculus we will introduce for it, named $\mathbf{R}\mathcal{L}_{\mathbf{T}}$, shall be *sequents of relations*.

Definition 3.3. Let $\mathbf{T}$ be a semantic theory, with predicate symbols $R_1, \ldots, R_n$. A *sequent of relations* is a finite multiset of the form

$$R_{i_1}(\varphi_1^1, \ldots, \varphi_{r_1}^1) \mid \ldots \mid R_{i_k}(\varphi_1^k, \ldots, \varphi_{r_k}^k)$$

where, for $1 \leq j \leq k$, $i_j \in \{1, \ldots, n\}$ and r_j is the arity of R_{i_j}. Moreover, all φ_j^i are formulas of the logic $\mathcal{L}$.

Intuitively, a sequent of relations as above should be read as "*one among* $R_{i_1}(\varphi_1^1, \ldots, \varphi_{r_1}^1), \ldots, R_{i_k}(\varphi_1^k, \ldots, \varphi_{r_k}^k)$ *holds*": the atomic relations has to be seen as disjunctively connected at an external level. Since we are talking about multisets, the order of the atomic relations is not relevant and repetitions are allowed.

Definition 3.4 (Axiom sequents)**.** Let $\mathbf{T} \models \forall \vec{x} \bigvee_{1 \leq j \leq n} B_j$ where the B_j are atomic formulas and $\vec{x}$ are the free variables in $\bigvee_{1 \leq j \leq n} B_j$. Let σ be any substitution of the variables $\vec{x}$ with formulas of $\mathcal{L}$. Then

$$\sigma(B_1) \mid \ldots \mid \sigma(B_n)$$

is an axiom of $\mathbf{R}\mathcal{L}_{\mathbf{T}}$

In order to capture the intended interpretation of "|" as a disjunction, the following rules must be introduced in the calculi of every logic.

Definition 3.5 (Structural rules)**.** The following are the *structural rules* for $\mathbf{R}\mathcal{L}_{\mathbf{T}}$:

$$\frac{\mathcal{H}}{A \mid \mathcal{H}}\ (\text{wk}) \qquad \frac{A \mid A \mid \mathcal{H}}{A \mid \mathcal{H}}\ (\text{cn})$$

where A is an arbitrary atomic relation on formulas and $\mathcal{H}$ is an arbitrary (possibly empty) *side sequent*.

Definition 3.6 (Logical (projective) rules)**.** Let $\circ$ be any n-ary connective of $\mathcal{L}$ with truth function

$$\circ(x_1, \ldots, x_n) = \begin{cases} t_1 & \text{if } P_\circ^1(s_1, \ldots, s_n) \\ \vdots & \vdots \\ t_m & \text{if } P_\circ^m(s_1, \ldots, s_n) \end{cases}$$

For each predicate symbol with arity r and each p, $1 \leq p \leq r$, we have a rule $(\circ : R : p)$ for introducing $\circ$ at position p into an R-component of a sequent of relations. $(\circ : R : p)$ is obtained starting from the formula

$$\alpha_{(\circ:R:p)} = \bigvee_{1\leq \ell \leq m} P^{\ell}_{\circ}(s_1, \ldots, s_n) \wedge R(z_1, \ldots, z_r)\,[t_\ell/z_p]$$

Take any conjunction of disjunction of formulas

$$\bigwedge_{1\leq j\leq s} \bigvee_{1\leq k\leq u_s} B_{j,k}$$

which is equivalent in $\mathbf{T}$ to $\alpha_{(\circ:R:p)}$. Then, for every substitution of the variables in $\{x_1, \ldots, x_n\} \cup \{z_1, \ldots, z_r\} \backslash \{z_p\}$ with formulas of $\mathcal{L}$,

$$\frac{\sigma(B_{1,1}) \mid \ldots \mid \sigma(B_{1,u_1}) \mid \mathcal{H} \qquad \ldots \qquad \sigma(B_{s,1}) \mid \ldots \mid \sigma(B_{s,u_s}) \mid \mathcal{H}}{\sigma(R(z_1, \ldots, z_r)\,[\circ(x_1, \ldots, x_n)/z_p]) \mid \mathcal{H}}\ (\circ : R : p)$$

where $\mathcal{H}$ is an arbitrary side sequent, is an instance of the logical rule $(\circ : R : p)$.

Remark 1. We call the rules above *projective* because in the case of projective logics these are the only logical rules for the sequents-of-relations calculus of the logic.

Definition 3.7. For every unary connective f_p of $\mathcal{L}$ corresponding to a function of $\mathbf{T}$, for every connective $\circ$ of $\mathcal{L}$, for every predicate symbol R with arity r and each p', $1 \leq p' \leq r$, we have a rule $(f_p\circ : R : p')$ introducing $f_p(\circ(x_1, \ldots, x_n))$ at position p' into an R-component of a sequent of relations. We distinguish two cases:

1. if $\circ = f_q$ and $f_p(f_q(x)) = f_h(x)$, then we have the rule

$$\frac{\sigma(R(z_1, \ldots, z_r)[f_h(x)/z_{p'}]) \mid \mathcal{H}}{\sigma(R(z_1, \ldots, z_r)[f_p f_q(x)/z_{p'}]) \mid \mathcal{H}}\ (f_p f_q : R : p')$$

   where σ substitutes formulas of $\mathcal{L}$ for the variables $\{x, z_1, \ldots, z_r\} \backslash \{z_{p'}\}$, and $\mathcal{H}$ is a side sequent;

2. otherwise, if

$$f_p(\circ(x_1, \ldots, x_n)) = \begin{cases} t'_1 & \text{if } P^1_{f_p(\circ)}(s'_1, \ldots, s'_n) \\ \vdots & \vdots \\ t'_m & \text{if } P^m_{f_p(\circ)}(s'_1, \ldots, s'_n) \end{cases}$$

to define $(f_p\circ : R : p')$ we start from the $\mathbf{T}$-formula

$$\alpha_{(f_p\circ:R:p')} = \bigvee_{1\leq \ell\leq m} P^{\ell}_{f_p(\circ)}(s'_1,\ldots,s'_n) \wedge R(z_1,\ldots,z_r)\,[t'_\ell/z_{p'}]$$

and then we take any conjunction of disjunctions of atomic formulas of $\mathbf{T}$

$$\bigwedge_{1\leq j\leq s}\bigvee_{1\leq k\leq u_s} B_{j,k}$$

which is equivalent to it. Then we have the rule

$$\frac{\sigma(B_{1,1}) \mid \ldots \mid \sigma(B_{1,u_1}) \mid \mathcal{H} \qquad \ldots \qquad \sigma(B_{s,1}) \mid \ldots \mid \sigma(B_{s,u_s}) \mid \mathcal{H}}{\sigma(R(z_1,\ldots,z_r)\,[f_p\circ(x_1,\ldots,x_n)/z_{p'}]) \mid \mathcal{H}}\,(f_p\circ : R : p')$$

where σ and $\mathcal{H}$ are similar as above.

Remark 2. In order to pass from $\alpha_{(f_p\circ:R:p')}$ to a conjunction of disjunctions of atomic formulas, we heavily make use of the fact that the conditions $P^i_{f_p(\circ)}$ are simple. Moreover, there can be *many* conjunctive normal forms which are equivalent to $\alpha_{(f_p\circ:R:p')}$; whenever possible, is better to choose the most compact one.

Remark 3. It is worth noticing that the logical rules satisfy the subformula property "up to" the special unary connectives $f_1,\ldots,f_t$. Moreover, if $\mathcal{L}_\mathbf{T}$ is *projective*, the projective logical rules fully satisfy the subformula property.

The concept of *derivation with premises in* Σ *and conclusion* S, where Σ is a set of sequents of relations (closed with respect to substitution of variables with formulas of $\mathcal{L}$), is much the same as in Hilbert systems for fuzzy logics: briefly, it is an upward tree, rooted in S such that every leaf is an axiom or a member of Σ, and every other sequent is obtained from the ones standing immediately above it by application of one of the rules of $\mathbf{R}\mathcal{L}_\mathbf{T}$. In case there exists a proof tree with premises in Σ and conclusion S we write $\Sigma \vdash_{\mathbf{R}\mathcal{L}_\mathbf{T}}$ S. A sequent S is said to be *provable* if $\vdash_{\mathbf{R}\mathcal{L}_\mathbf{T}}$ S.

As usual, the *length of a derivation* is the number of inferences in a maximal branch of the derivation.

3.2.1 Gödel logics and $\mathbf{G}^\Delta_\infty$

The full sequents-of-relations calculi of some of the logics introduced in Section (3.1) will be given in the following chapter. Here we briefly report the logical rules of the projective logics $\mathbf{G}_n$, $1 \leq n \leq \infty$, and $\mathbf{G}_\infty^\Delta$ as in [5]. We notice that, since all $\mathbf{G}_n$ share the truth functions of their connectives, they share the logical rules too. For Gödel logics we have: $\circ \in \{\wedge, \vee, \to\}$, $R \in \{\leq, <\}$ and $p \in \{l, r\}$.

There are two ways to deal with the α's; the first one consists in disjunctively join the conjunctions $(z < t_i) \wedge P_\circ^i$ (as an example we take $\alpha_{(\to:<:r)}$), and then convert this formula in conjunctive normal form. Reduction allowed by the semantic theory are gladly welcome.

$$\begin{aligned} \alpha_{(\to:<:r)} &= (x \leq y \wedge z < 1) \vee (y < x \wedge z < y) \\ &\iff (x \leq y \vee y < x) \wedge (x \leq y \vee z < y) \wedge \\ &\quad \wedge (z < 1 \vee y < x) \wedge (z < 1 \vee z < y) \\ &\iff (x \leq y \vee z < y) \wedge (z < 1) \end{aligned}$$

The other way is based on the view of the entries of a truth function as implications like $P_\circ^i \to (t_i < z)$; so we can write every entry as $\neg P_\circ^i \vee (z < t_i)$, and then conjunctively join all the entries.

$$\begin{aligned} \alpha_{(\to:<:r)} &= (\neg(x \leq y) \vee z < 1) \wedge (\neg(y < x) \vee z < y) \\ &\iff (y < x \vee z < 1) \wedge (x \leq y \vee z < y) \\ &\iff (x \leq y \vee z < y) \wedge (z < 1) \end{aligned}$$

So, the rule for introducing implication at the right side of a $<$ is

$$\frac{A \leq B \mid C < B \mid \mathcal{H} \qquad C < 1 \mid \mathcal{H}}{C < A \to B \mid \mathcal{H}} \,(\to:<:r)$$

In the same fashion, we obtain the remaining rules for implication.

$$\frac{B < A \mid \mathcal{H} \qquad B < C \mid \mathcal{H}}{A \to B < C \mid \mathcal{H}} \,(\to:<:l)$$

$$\frac{A \leq B \mid C \leq B \mid \mathcal{H}}{C \leq A \to B \mid \mathcal{H}} \,(\to:\leq:r)$$

$$\frac{1 \leq C \mid B < A \mid \mathcal{H} \qquad B \leq C \mid \mathcal{H}}{A \to B \leq C \mid \mathcal{H}} \,(\to:\leq:l)$$

The following are the rules for conjunction and disjunction. Since they take the same form for both relations, we let $\lhd$ stands for $\leq$ and $<$.

$$\frac{C \lhd A \mid \mathcal{H} \qquad C \lhd B \mid \mathcal{H}}{C \lhd A \wedge B \mid \mathcal{H}} (\wedge : \lhd : r) \qquad \frac{A \lhd C \mid B \lhd C \mid \mathcal{H}}{A \wedge B \lhd C \mid \mathcal{H}} (\wedge : \lhd : l)$$

$$\frac{C \lhd A \mid C \lhd B \mid \mathcal{H}}{C \lhd A \vee B \mid \mathcal{H}} (\vee : \lhd : r) \qquad \frac{A \lhd C \mid \mathcal{H} \qquad B \lhd C \mid \mathcal{H}}{A \vee B \lhd C \mid \mathcal{H}} (\vee : \lhd : l)$$

In [23] the Δ connective is introduced as a unary operation on all BL-algebras with truth function

$$\widetilde{\Delta}(x) = \begin{cases} 1 & \text{if } 1 \leq x \\ 0 & \text{if } x < 1 \end{cases}$$

With $\mathbf{G}^{\Delta}_{\infty}$ we indicate the infinite-valued Gödel logic enriched with the Δ operator, studied intensively in [3]. Since Δ is clearly projective with respect to $\mathbf{T}_{\leq,<}$, $\mathbf{G}^{\Delta}_{\infty}$ is again projective, and the extra logical (projective) rules for Δ are:

$$\frac{A < 1 \mid 1 \leq B}{\Delta A \leq B} (\Delta :\leq: l) \qquad \frac{B \leq 0 \mid 1 \leq A}{B \leq \Delta A} (\Delta :\leq: r)$$

$$\frac{0 < B \mid A < 1}{\Delta A < B} (\Delta :<: l) \qquad \frac{B < 1 \mid 1 \leq A}{B < \Delta A} (\Delta :<: r)$$

Clearly, $\mathbf{G}^{\Delta}_{\infty}$ could be seen as properly semiprojective, taking into account the semantic theory $\mathbf{T}_{\leq,<,\Delta}$ obtained extending $\mathbf{T}_{\leq,<}$ with a unary function symbol Δ acting like the homonym connective. It is up to personal choice which vision is preferred, though a projective presentation is much more slender and neat.

3.3 Soundness, Completeness and Decidability

We now turn to the consistency of the calculi we have introduced. We will also give an efficient bound to the time complexity for the decision problem of a semiprojective logic, provided that the axioms of the calculus are recognizable in polynomial time. All the results present in this section can be found in [37].

Definition 3.8. Let S be a sequent of relations of the form

$$R_1(\varphi_{1,1}, \ldots, \varphi_{1,r_1}) \mid \ldots \mid R_n(\varphi_{n,1}, \ldots, \varphi_{n,r_n})$$

$\mathcal{M}$ a model of $\mathbf{T}$ and σ a valuation on $\mathcal{M}$. Then, *the formula associated to* S *under* σ is

$$\psi_{\mathrm{S}}^{\sigma} = \forall \vec{x} \bigvee_{1 \leq i \leq n} R_i(\varphi_{i,1}^{\mathcal{M}^\star,\sigma}, \ldots, \varphi_{i,r_i}^{\mathcal{M}^\star,\sigma}).$$

We write $\mathcal{M}, \sigma \models \mathrm{S}$ to mean $\mathcal{M}, \sigma \models \psi_{\mathrm{S}}^{\sigma}$.

Before we begin, let us state a useful property of the rules of the sequents-of-relations calculi of a semiprojective logic.

Definition 3.9. A rule

$$\frac{\mathrm{S}_1 \qquad \ldots \qquad \mathrm{S}_n}{\mathrm{S}}$$

is *sound* for $\mathcal{L}_{\mathbf{T}}$ if for each model $\mathcal{M}$ of $\mathbf{T}$ and each valuation σ on $\mathcal{M}$,

$$\mathcal{M}, \sigma \models \mathrm{S}_i \text{ for all } i = 1, \ldots, k \quad \text{then} \quad \mathcal{M}, \sigma \models \mathrm{S},$$

and *invertible* if for each model $\mathcal{M}$ of $\mathbf{T}$ and each valuation σ on $\mathcal{M}$,

$$\mathcal{M}, \sigma \models \mathrm{S} \quad \text{then} \quad \mathcal{M}, \sigma \models \mathrm{S}_i \text{ for all } i = 1, \ldots, k.$$

Lemma 3.2. *If the rule*

$$\frac{\mathrm{S}_1 \qquad \ldots \qquad \mathrm{S}_n}{\mathrm{S}}$$

is sound (invertible) for $\mathcal{L}_{\mathbf{T}}$*, then so is*

$$\frac{\mathrm{S}_1 \mid \mathcal{H} \qquad \ldots \qquad \mathrm{S}_n \mid \mathcal{H}}{\mathrm{S} \mid \mathcal{H}}$$

Proof. Obvious from the following facts:

1. it is true that

$$\mathcal{M}, \sigma \models \mathrm{S} \text{ iff } \mathcal{M}, \sigma \models \mathrm{S}_i \text{ for all } i = 1, \ldots, k \iff$$
$$\iff \mathcal{M}, \sigma \models \mathrm{S} \mid \mathcal{H} \text{ iff } \mathcal{M}, \sigma \models \mathrm{S}_i \mid \mathcal{H} \text{ for all } i = 1, \ldots, k,$$

2. for every sequent S and every side sequent $\mathcal{H}$,

$$\mathcal{M}, \sigma \models \psi_{\mathrm{S}}^{\sigma} \quad \text{implies} \quad \mathcal{M}, \sigma \models \psi_{\mathrm{S}|\mathcal{H}}^{\sigma}.$$

□

Proposition 3.3. *Every rule of* $\mathbf{R}\mathcal{L}_\mathbf{T}$ *is sound and invertible for* $\mathcal{L}_\mathbf{T}$*.*

Proof. We shall work out in details the case of a rule $(f_p\circ : R : p')$, using Lemma (3.2) to disregard side sequents, and claiming that the other cases are similar. Since $\mathcal{M}, \sigma \models \mathrm{S}_i$ stand for $\mathcal{M}, \sigma \models \psi^\sigma_{\mathrm{S}_i}$ then $\mathcal{M}, \sigma \models \mathrm{S}_i$ for all $i = 1, \ldots, k$ is equivalent to

$$\mathcal{M}, \sigma \models \bigwedge_{1\leq i\leq n} \psi^\sigma_{\mathrm{S}_i}.$$

This is in turn equivalent to $\mathcal{M}, \sigma \models \alpha_{(f_p\circ:R:p')}$ (see Definitions (3.6) and (3.7)), and so to $\mathcal{M}, \sigma' \models R(z_1, \ldots, z_n)$, where σ' differs from σ only in the assignment of $f_p(\circ(x_1, \ldots, x_n))$ to the variable $z_{p'}$. □

Turning back to soundness and completeness, let $\mathrm{Des}(x)$ be the designating predicate. Since Des is simple and $\mathbf{T}$ is classical, Des is equivalent to a formula in conjunctive normal form $\mathrm{Des}(x)'$ like

$$\bigwedge_{1\leq i\leq n} \bigvee_{1\leq j\leq m_i} A_{i,j}$$

where each $A_{i,j}$ has at most one free variable, namely x. If φ is a formula of $\mathcal{L}$, with

$$\mathcal{D}_1[\varphi/x], \ldots, \mathcal{D}_n[\varphi/x]$$

we denote the sequence of sequents that correspond to the conjuncts of $\mathrm{Des}(x)'$, with x replaced by φ.

Theorem 3.4 (Soundness)**.** *If* $\mathcal{D}_1[\varphi/x], \ldots, \mathcal{D}_n[\varphi/x]$ *are provable in* $\mathbf{R}\mathcal{L}_\mathbf{T}$ *then* φ *is valid in* $\mathcal{L}$*.*

Proof. We will prove that if a sequent S is provable then $\mathcal{M}, \sigma \models \mathrm{S}$ for all σ and $\mathcal{M}$ of $\mathbf{T}$. From this, the theorem would follow by the definition of $\mathcal{D}_1[\varphi/x], \ldots, \mathcal{D}_n[\varphi/x]$ and the fact that φ is valid if for all σ and $\mathcal{M}$

$$\mathcal{M}, \sigma \models \mathrm{Des}(\varphi^{\mathcal{M}^\star, \sigma}).$$

The proof is by induction on the length of the derivation which grants $\vdash_{\mathbf{R}\mathcal{L}_\mathbf{T}} \mathrm{S}$.

Base step: S is an axiom, so $\mathcal{M}, \sigma \models \mathrm{S}$ by definition;

Inductive step: immediately follows from the soundness of the rules of the calculi, stated in Proposition (3.3).

□

Theorem 3.5 (Completeness). *If φ is valid in $\mathcal{L}$ then $\mathcal{D}_1[\varphi/x], \dots, \mathcal{D}_n[\varphi/x]$ are provable in $\mathbf{R}\mathcal{L}_\mathbf{T}$.*

Proof. We shall use the technique of *Schütte's reduction tree* (see [40]), as shown in [5]: for every sequent S we will construct a reduction tree $\mathrm{RT_S}$ such that a proof of S can be extracted from $\mathrm{RT_S}$, if $\mathcal{M}, \mu \models \mathrm{S}$ for every $\mathcal{M}$ and μ of $\mathbf{T}$.

The upward tree of sequents $\mathrm{RT_S}$ is made up following the stages below:

Stage 0: Write S at the root of $\mathrm{RT_S}$.

Stage k: If the topmost sequent S' of a branch contains relations of the form $R_i(\phi_1, \dots, \phi_{r_i})$, where each ϕ_j is either an atomic formula of $\mathcal{L}$ or of the form $f_p(A)$, where A is atomic and f_p has an homonym unary function symbol in $\mathbf{T}$, then stop the reduction of this branch. Otherwise S' contains a relation $R_i(\phi_1, \dots, \phi_{r_i})$, where $\phi_{p'}$ is of the form

$$\circ(\psi_1, \dots, \psi_n) \text{ or } f_p(\circ(\psi_1, \dots, \psi_n)) \text{ or } f_p(f_q(\psi))$$

for some $1 \leq p' \leq r_i$. If the indicated occurrence of one the formulas above is not the result of a reduction at this stage and has not yet been reduced on this branch, then replace S' by

$$\frac{\sigma(B_{1,1}) \mid \dots \mid \sigma(B_{1,u_1}) \mid \mathrm{S}' \qquad \dots \qquad \sigma(B_{s,1}) \mid \dots \mid \sigma(B_{s,u_s}) \mid \mathrm{S}'}{\mathrm{S}'}$$

where the $B_{i,j}$ are as in the definition of the rule $(\Box : R_i : p')$, where $\Box \in \{\circ, f_p\circ, f_p f_q\}$, and σ is given by

$$\sigma(R(z_1, \dots, z_r)\,[\Box(x_1, \dots, x_n)/z_{p'}]) = R_i(\phi_1, \dots, \phi_{r_i}).$$

Notice that, since every occurrence of a formula is only reduced once in a branch, the construction of $\mathrm{RT_S}$ stops after finitely many steps.

We shall say that a sequent S' *contains an axiom* if there is an axiom S^A and a derivation $\{\mathrm{S}^A\} \vdash_{\mathbf{R}\mathcal{L}_\mathbf{T}} \mathrm{S}'$ involving structural rules only. Clearly, if each leaf of $\mathrm{RT_S}$ contains an axiom, then a derivation of S is easily obtained from $\mathrm{RT_S}$ by adding proper structural rules.

By definition of $\mathcal{L}$, $\mathcal{M} \models \mathrm{Des}(\varphi^{\mathcal{M}^\star,\mu})$ for all $\mathcal{M}$ and μ of $\mathbf{T}$, and therefore $\mathcal{M} \models \mathcal{D}_i(\varphi^{\mathcal{M}^\star,\mu})$, for each $i = 1, \dots, n$, because

$$\mathbf{T} \models \forall x \left(\mathrm{Des} \Leftrightarrow \bigwedge_{1 \leq i \leq n} \mathcal{D}_i \right).$$

So, if we take $\mathrm{RT}_{\mathcal{D}_i[\varphi/x]}$ then since the root is such that $\mathcal{M}, \mu \models \mathcal{D}_i[\varphi/x]$, by Proposition (3.3) (the *invertibility* part), we have that all the leafs of the reduction tree contain an axiom; hence a derivation of $\mathcal{D}_i[\varphi/x]$ is easily obtained from $\mathrm{RT}_{\mathcal{D}_i[\varphi/x]}$ for each $i = 1, \dots, n$. □

Since the construction of the reduction trees is effective and the axioms of the calculi are decidable by condition (1) in Definition (3.1), we obtain the following corollary.

Corollary 3.6. *All semiprojective logics are decidable.*

By *s-atomic sequents of relations* we mean sequents

$$R_1(\phi_{1,1}, \dots, \phi_{1,r_1}) \mid \dots \mid R_n(\phi_{n,1}, \dots, \phi_{n,r_n})$$

where each $\phi_{i,j}$ is either an atomic formula of $\mathcal{L}$ or of the form $f_p(A)$, where A is atomic and f_p has an homonym unary function symbol in $\mathbf{T}$. With this nomenclature, the corollary above can be reformulated as follows, highlightening one of the most important (if not the most important) features of the calculi we have introduced.

Corollary 3.7. *If the problem of determining the validity in* $\mathbf{T}$ *of s-atomic sequents of relations is polynomial then* $\mathcal{L}$ *is Co-NP.*

Proof. The following is a non-deterministic polynomial-time algorithm for the unprovable formulas of $\mathcal{L}$:

`INPUT:` `a formula` φ `of` $\mathcal{L}$;

`OUTPUT:` `NO if` φ `is not a valid formula of` $\mathcal{L}$;

`1. guess a` $\mathcal{D}_i[\varphi/x]$ `and a branch of its proof tree;`

`2. if there is a leaf of this branch which is not valid,`
`return NO.`

Since the leafs are s-atomic sequents (for which we have a polynomial-time checker for the validity problem) and the length of the derivation of $\mathcal{D}_i[\varphi/x]$ is linear in the number of connectives of φ, this verifier is polynomial. □

Remark 4. In the previous section, whenever a logic was shown to be semiprojective, condition (1) of Definition (3.1) was rarely checked. The reason lies in the corollary above, which grants us a "good" complexity bound of a logic as long as we have a *efficient* algorithm for the recognition (or, which is the same, an *efficient* characterization) of the valid s-atomic sequents. In the next section we will fill this gap by providing explicit descriptions of the s-atomic sequents for almost all the logics introduced in section (3.1), proving that they are indeed Co-NP.

Chapter 4

The Axioms of the Calculi

In the final section of the last chapter we saw how important is to have an explicit description of the valid s-atomic sequents of a semiprojective logic. In this chapter we shall introduce characterizations of such sequents for most of the semiprojective logics we have introduced so far, collecting several results from [5] and [37]. In the second section, though, we work out an alternative approach, providing a polynomial-time algorithm for the task and proving its efficiency and reliability. At the end of the chapter, we fully report the calculi for Gödel logic with an involutive negation and Nilpotent Minimum logic, together with the complete description of the axioms of n-contractive BL logics.

4.1 Gödel logics

In [4], the following explicit description of the axioms of $\mathbf{RG}_\infty$ was given. From now on, S′ *is a subsequent of* S shall mean that S corresponds to S′ up to external weakenings.

Definition 4.1. A sequent of relations S is said to be *sound* if it contains as subsequent one of the following:

1. $A_1 \lhd_n A_n \mid \ldots \mid A_3 \lhd_2 A_1 \mid A_2 \leq A_1$ if $n > 1$, where $\lhd_i \in \{\leq, <\}$, or $A_1 \leq A_1$ if $n = 1$;
2. $A_n \leq A_{n-1} \mid A_{n-1} < A_{n-2} \mid \ldots \mid A_1 < 1$ if $n > 1$, or $A_1 \leq 1$ if $n = 1$;
3. $0 < A_n \mid A_n < A_{n-1} \mid \ldots \mid A_2 \leq A_1$ if $n > 1$, or $0 \leq A_1$ if $n = 1$;

4. $0 < A_1 \mid A_1 < A_2 \mid \ \dots \ \mid A_n < 1$ if $n \geq 1$, or $0 < 1$ if $n = 0$.

Sequents of type (1) are called *cycles*, those of type (2) *1-chains*, those of type (3) *0-chains*, and finally those of type (4) *0-1-chains*.

Theorem 4.1. *([4]) A sequent is an axiom of* $\mathbf{RG}_\infty$ *if and only if it is sound.*

It is easy to check that the above axioms are valid in $\mathbf{G}_\infty$, so the "if" part is done. The "only if" part is, instead, trickier because we have to show that all the valid axioms are obtained from these axioms using external weakening only. Thus, some preliminary work is required.

Definition 4.2. A set of components id called *dual to axioms* if it does not contain any subset of one of the following forms:

(anti-cycle) $\{A_1 < A_2, A_2 \lhd_2 A_3, \dots, A_n \lhd_n A_1\}$ where $\lhd_i \in \{\leq, <\}$, and the case $n = 1$ is defined as $\{A_1 < A_1\}$;

(anti-1-chain) $\{1 \leq A_1, \dots, A_{n-2} \leq A_{n-1}, A_{n-1} < A_n\}$, where the case $n = 1$ is defined as $\{1 < A_1\}$;

(anti-0-chain) $\{A_1 < A_2, A_2 \leq A_3, \dots, A_n \leq 0\}$, and the case $n = 1$ is defined as $\{A_1 < 0\}$;

(anti-0-1-chain) $\{1 \leq A_1, A_2 \leq A_3, \dots, A_n \leq 0\}$, and the case $n = 0$ is defined as $\{1 \leq 0\}$.

The following properties of dual to axioms sets, whose proof we omit, are necessary.

Lemma 4.2. *Let* Γ *be a finite set of components* $A \lhd B$, $\lhd \in \{\leq, <\}$, *where* A *and* B *are either propositional variables or truth constants. The following hold:*

1. *if* Γ *is dual to axioms then* $\Gamma \cup \{A \leq B \mid A < B \in \Gamma\}$ *is dual to axioms too;*

2. *if* Γ *is dual to axioms then either* $\Gamma \cup \{A < B\}$ *or* $\Gamma \cup \{B \leq A\}$ *is dual to axioms too.*

Clearly, with the following lemma the theorem is proved.

Lemma 4.3. *Let* Γ *be a finite set of components* $A \lhd B$, $\lhd \in \{\leq, <\}$, *where* A *and* B *are either propositional variables or truth constants. If* Γ *is dual to axioms then* Γ *is satisfiable; i.e., there exists an interpretation that satisfies all components of* Γ.

Proof. We extend Γ to a "maximal" set $\Gamma^\star$ which is still dual to axioms. If we write

$$B \in [A] \iff \{A \leq B, B \leq A\} \subseteq \Gamma^\star$$

then Lemma (4.2) guarantees that this is an equivalence relation, and the set of equivalence classes

$$\overline{\Gamma^\star} = \{[A] \mid A \text{ occurs in } \Gamma^\star\}$$

exists and it is totally ordered with respect to the order relation

$$[A] < [B] \iff A < B.$$

The minimal element of the ordering is [0] and the maximal one is [1], if 0 and 1 occur in Γ; the ordering thus allows to match equivalence classes with truth values in a way that induces an interpretation satisfying $\Gamma^\star$, and therefore also Γ. □

4.1.1 An alternative approach

For proving completeness, we built a model of the theory $\mathbf{T}_{\leq,<}$ not satisfying the dual to axioms set of components in examination. We have outlined a procedure for doing so, but that was far from being computationally satisfactory: in other words, its time-complexity was not clear. What we shall do now is to develop a *formal algorithm* for the solution of the *countermodel problem*, i.e. the construction of the countermodel of any dual to axioms set, for which an efficient bound for the computational complexity is available.

Our strategy is based on the following observation: an s-atomic sequent S is a sequent of the form

$$\theta(A_1 \lhd_1 B_1) \mid \theta(A_2 \lhd_2 B_2) \mid \ \ldots \ \mid \theta(A_n \lhd_n B_n),$$

where each A_i and B_i is 0, 1 or a variable x, and θ substitutes atomic formulas for the variables appearing in S, and any countermodel of it must satisfy

$$\neg\mathcal{F}(\mathrm{S}) \equiv \exists\vec{x} \bigwedge_{i=1}^{n} (B_i \overline{\lhd_i} A_i).$$

So, if we find a model $\mathcal{C}$ of $\mathbf{T}_{\leq,<}$, and a valuation σ of the variables $\vec{x}$ appearing in S into the universe of $\mathcal{C}$ such that $\mathcal{C}$ satisfies $\sigma(B_i) \overline{\lhd_i} \sigma(A_i)$ for all $i = 1, \ldots, n$, then we are done. That is why, from now on, we focus on sets of inequalities.

Definition 4.3. A *set of inequalities with variables in the set* V is a set of the form

$$\Sigma = \{x_1 \lhd_1 y_1, x_2 \lhd_2 y_2, \ldots, x_k \lhd_k y_k\}$$

where, for all i, $x_i, y_i \in V$ and $\lhd_i \in \{\leq, <\}$.

Σ is said to be *transitive* if the following hold for every $x, y, z \in V$:

1. $x < y, y < z \in \Sigma \Rightarrow x < z \in \Sigma$;
2. $x < y, y \leq z \in \Sigma \Rightarrow x < z \in \Sigma$;
3. $x \leq y, y < z \in \Sigma \Rightarrow x < z \in \Sigma$;
4. $x \leq y, y \leq z \in \Sigma \Rightarrow x \leq z \in \Sigma$.

The *transitive closure of* Σ is the set $T(\Sigma)$ such that $x \leq y \in T(\Sigma)$ iff there exist a non negative integer n and variables $y_1, \ldots, y_n$ such that $x \leq y_1, y_1 \leq y_2, \ldots, y_n \leq x \in \Sigma$, and $x < y \in T(\Sigma)$ iff there exist a non negative integer n and variables $y_1, \ldots, y_n$ such that $x \lhd_1 y_1, y_1 \lhd_2 y_2, \ldots, y_n \lhd_{n+1} x \in \Sigma$ where $\lhd_i \in \{\leq, <\}$, and, for at least one i, $\lhd_i =<$.

Clearly, the transitive closure of a set of inequalities is transitive.

Definition 4.4. A set of inequalities with variables in the set V, Σ, is said to be *satisfiable* if, for every $x \in V$, $x < x \notin T(\Sigma)$.

Definition 4.5. Given a set of n variables V, consider a set of inequalities with variables in $V \cup \{0, 1\}$, Σ. Let $\mathcal{C} = (C, \leq_{\mathcal{C}}, <_{\mathcal{C}})$ be a model of $\mathbf{T}_{\leq,<}$ and $\sigma : V \cup \{0, 1\} \longrightarrow C$. We say that $(\mathcal{C}, \sigma)$ is *a model of* Σ, written $\mathcal{C}, \sigma \models \Sigma$, if the following hold:

1. for all $x \in V \cup \{0, 1\}$, $\sigma(0) \leq_{\mathcal{C}} \sigma(x)$;
2. for all $x \in V \cup \{0, 1\}$, $\sigma(x) \leq_{\mathcal{C}} \sigma(1)$;
3. $\sigma(0) <_{\mathcal{C}} \sigma(1)$;
4. $x < y \in T(\Sigma) \Rightarrow \sigma(x) <_{\mathcal{C}} \sigma(y)$;

5. $x \leq y \in T(\Sigma) \Rightarrow \sigma(x) \leq_{\mathcal{C}} \sigma(y)$.

We are going to introduce an algorithm which takes as input a set of inequalities Σ and produces (if Σ is satisfiable) a model $(\mathcal{C}, \sigma)$ of Σ. The model shall be a structure of the form $\mathcal{C} = (\{0, 1, \ldots, n\}, \leq_{\mathbb{N}}, <_{\mathbb{N}})$, where $\leq_{\mathbb{N}}$ and $<_{\mathbb{N}}$ are the usual orders on the positive integers $\mathbb{N}$.

The idea behind the algorithm is better grasped with an example: take the following set of inequalities

$$\{x \leq y, u < y, z \leq t, y \leq z, t \leq v, t < w, v \leq z\}.$$

First produce a matrix of the form

	x	y	z	u	v	w	t
x							
y							
z							
u							
v							
w							
t							

Next step: scan the inequalities. For every inequality of the form $x_i < x_j$ put a 0 in the (x_i, x_j)-cell; for every inequality of the form $x_i \leq x_j$ put a 1 in the (x_i, x_j)-cell.

	x	y	z	u	v	w	t
x		1					
y			1				
z							1
u		0					
v			1				
w							
t					1	0	

Now, the trickiest part: for every variable x_i scan the relative row; for every non-blank symbol a, let x_j be the variable such that the symbol is in the x_i row and x_j column; then scan the x_j row and, for every non-blank symbol b of the row (say it is in the x_k column) do the following: if there is a non-blank symbol c in the (x_i, x_k)-cell, put in that cell $a \cdot b \cdot c$, otherwise put $a \cdot b$ in it. Do it until the matrix does not change after these operations. This part corresponds to the transitive closure of Σ.

This is the result with our example:

	x	y	z	u	v	w	t
x		1	1		1	0	1
y			1		1	0	1
z			1		1	0	1
u		0	0		0	0	0
v			1		1	0	1
w							
t			1		1	0	1

At this point check if there are variables which have equal rows and columns: these are the variables that can be associated to the same integer, because they obey to the same inequalities. In our example we have to collapse the variables z, v and t together: so we remember it and remove their rows and columns, except z's.

	x	y	z	u	w
x		1	1		0
y			1		0
z			1		0
u		0	0		0
w					

Finally, we focus on columns: take the first variable whose column has at most one 1 in the diagonal cell (later we will show that it always exists), associate to it the integer 0 and remove its row and column; repeat these operations until there are no more variables left, always increasing the integer by 1. Of course, if there are variables which were collapsed together, associate to them the same integer associated to the "survived" variable. Our example gives us the interpretation $x = 0$, $u = 1$, $y = 2$, $z = v = t = 3$ and $w = 4$, so the model is the five element chain $\{0, 1, 2, 3, 4\}$.

Notice that if an inequality of the type $x < x$ is derivable from our set of inequalities, then a 0 appears in the diagonal of the matrix, making easy to check if a set of inequalities is satisfiable or not.

Here follows the algorithm:

INPUT: A set of inequalities Σ with variables $x_1, \ldots, x_n$;

OUTPUT: An interpretation $(\mathcal{C}, \sigma)$ of Σ if Σ is satisfiable, an error message if it is not;

1. $\Sigma := \Sigma \cup \{0 \leq x_i, x_i \leq 1 \mid i = 1, \ldots, n\} \cup \{0 < 1\}$;
2. $x_{n+1} := 0$;
3. $x_{n+2} := 1$;
4. define an $n \times n$ matrix M with blank entries;
5. for every inequality $x_i \lhd x_j$
    6. if $\lhd = \leq$ define $M(i,j) := 1$;
    7. if $\lhd = <$ define $M(i,j) := 0$;
8. define $next := 0$;
9. for every $i = 1, \ldots, n+2$
    10. $next := 1$;
    11. do until $next \neq 1$
        12. for every $j = 1, \ldots, n+2$
            13. for every $k = 1, \ldots, n+2$
                14. if $M(i,j)$ and $M(j,k)$ are not blank
                    15. if $M(i,k)$ is blank, then define $M(i,k) := M(i,j) \cdot M(j,k)$ and $next := 0$;
                    16. if $M(i,k)$ is not blank and

$$M(i,k) \neq M(i,j) \cdot M(j,k),$$

then define $M(i,k) := M(i,j) \cdot M(j,k) \cdot M(i,k)$ and $next := 0$;

17. check the diagonal of the matrix if there is a 0: if so, end the procedure with an error message; else, go on;
18. do until all the rows are different

19. `take the first variable` x_j `such that there exists another variable which have row and column equal to that of` x`, find all the variables with this property, and delete them all except` x_j`;`

20. `define` $N :=$ `number of variables left;`

21. `for every` $i = 0, \ldots N$

22. `take the first variable` x_j `such that its column contains all blank symbols, except, at most, one` 1 `in the diagonal cell, and define` $\sigma(x_j) = i$`;`

23. `if the columns and rows of the variables` $x_{j_1}, \ldots, x_{j_k}$ `where removed at step (19) together with` x_j`, define`

$$\sigma(x_{j_1}) := i, \ldots, \sigma(x_{j_k}) := i;$$

24. `delete the` x_j `row and column;`

25. `define` $C = \{0, 1, \ldots, N\}$ `and return` $(\mathcal{C}, \sigma)$`.`

Step (1) adds to Σ the necessary inequalities involving 0 and 1 in order to force them to be assigned the minimum integer possible and the maximum one, respectively. Steps (9)-(16) correspond to the transitive closure of the set of inequalities, as stated by Lemma (4.4). Step (17) ensures that the set of inequalities is satisfiable. Doing steps (18) and (19) is equivalent to quotient the set of variables V with the equivalence relation described in Proposition (4.5). Finally, steps (21)-(24) produce the valuation σ following the criterion: assign the first variable you find which is not greater than any other variable not yet assigned to the first non-assigned positive integer.

Remark 5. The time-complexity of the algorithm is clearly polynomial, the transitive closure of the set of inequalities being the slowest task the algorithm has to face.

Lemma 4.4. *The following hold:*

1. $x \leq y \in T(\Sigma)$ *iff the* (x, y)*-entry of the matrix after the cycle at step* (9) *contains a* 1*;*

2. $x < y \in T(\Sigma)$ *iff the* (x, y)*-entry of the matrix after the cycle at step* (9) *contains a* 0.

Proof. The left-to-right implications are done by induction on the length of the derivation of $x \leq y \in T(\Sigma)$ and $x < y \in T(\Sigma)$, respectively. The right-to-left implications are done by induction on the number of cycles at step (9). □

So, as anticipated, the cycle at step (9) has the function to close transitively Σ. The following proposition, instead, states that the valuation σ does not distinguish between variables satisfying the same inequalities.

Proposition 4.5. *For every $x, y \in V \cup \{0,1\}$, $\sigma(x) = \sigma(y)$ iff, for every $z \in V \cup \{0,1\}$ and $\lhd \in \{\leq, <\}$, the following hold:*

1. $x \lhd z \in T(\Sigma) \iff y \lhd z \in T(\Sigma)$;

2. $z \lhd x \in T(\Sigma) \iff z \lhd y \in T(\Sigma)$.

So, if $x \leq y, y \leq x \in T(\Sigma)$, then $\sigma(x) = \sigma(y)$.

Proof. Two variables are assigned to the same integer iff they have equal rows and columns, which is equivalent to asking (1) and (2), by Lemma (4.4). □

Proposition 4.6. *If the input of the algorithm Σ is satisfiable, then step (22) can always be done.*

Proof. If Σ is satisfiable then Lemma (4.4) assures that the algorithm does not reject at step (17). So suppose that step (22) can not be done: it means that for every variable $x \in V \cup \{0,1\}$ there is a variable y such that $x \lhd y \in T(\Sigma)$, where $\lhd \in \{\leq, <\}$, by Lemma (4.4). Since there are only a finite number of variables, then there exist x, y such that $x \lhd_1 y, y \lhd_2 x \in T(\Sigma)$; but $\lhd_1 = \lhd_2 = \leq$ (otherwise we would have $x < x \in T(\Sigma)$, a contradiction). So, by Proposition (4.5), x and y have equal rows and columns, which is absurd, since the cycle at step (18) removes one of them. □

Theorem 4.7. *Let Σ be a satisfiable set of inequalities, and $(\mathcal{C}, \sigma)$ be the output that the algorithm returns taking Σ as input. Then, $(\mathcal{C}, \sigma)$ is a model of Σ.*

Proof. Lemma (4.4) assures that if the algorithm takes Σ as input, then it produces as output $(\mathcal{C}, \sigma)$. Now, suppose $x \leq y \in T(\Sigma)$: if $y \leq x \in T(\Sigma)$ then Proposition (4.5) guarantees that x and y are assigned to the same integer, so $\sigma(x) = \sigma(y)$; if $y \leq x \notin T(\Sigma)$ then Lemma (4.4) tells us that the (y, x)-entry of the matrix is blank (if there were a 0 then $y < x \in T(\Sigma)$, which implies $x < x \in T(\Sigma)$, a contradiction), so y can be assigned only after x (since, until the x row is present, the y column has a 1 not in the diagonal cell), which means $\sigma(x) < \sigma(y)$. The case $x < y \in T(\Sigma)$ is similar.

For what concerns 0 and 1, the inequalities added at step (1) and the arguments above make sure that for all $x \in V \cup \{0, 1\}$,

$$\sigma(0) \leq \sigma(x) \leq \sigma(1) \text{ and } \sigma(0) < \sigma(1).$$

□

As a corollary of the above theorem, we have that the construction of a countermodel for a non-sound sequent is polynomial.

Corollary 4.8. *The countermodel problem for infinite-valued Gödel logic is polynomial.*

4.1.2 Finite-valued Gödel logics

We conclude the section with a last remark on finite-valued Gödel logics $\mathbf{G}_n$. We have already seen that $\mathbf{RG}_\infty$ and $\mathbf{RG}_n$ shares all the logical rules; but since the semantic theory of $\mathbf{RG}_n$, $\mathbf{T}_n$, is a proper extension of $\mathbf{T}_{\leq,<}$, all the axioms for $\mathbf{RG}_\infty$ are also axioms for the calculus $\mathbf{RG}_n$. The question is: what kind of axioms complete the explicit description of the axioms of $\mathbf{RG}_n$? As pointed out in [5], all we have to do is to add to the axioms of $\mathbf{RG}_\infty$ all axioms of the form

$$A_1 \lhd_1 A_2 \mid A_2 \lhd_2 A_3 \mid \ldots \mid A_\ell \lhd_\ell A_{\ell+1}$$

where $\lhd_i = \leq$ for at least n different $i \in \{1, \ldots, \ell\}$, where $\ell > n$. The meaning of these axioms is clear: no more than n different elements are allowed for models of $\mathbf{T}_n$, because any countermodel of such sequents shall have at least $n + 1$ elements. Since this request is the only difference between $\mathbf{T}_n$ and $\mathbf{T}_{\leq,<}$, the characterization of the axioms is complete.

4.2 Gödel logics with an involutive negation

In this section we will give a characterization of the s-atomic (or, briefly, *atomic*) sequents of the sequents-of-relations calculi for infinite-valued Gödel logics with an involutive negation, indicated in the following by $\mathbf{G}_\sim$.

We begin with interlocutory definitions providing the main tools we shall work with all along the section.

Definition 4.6. A *literal* of $\mathbf{G}_\sim$ is a term of the form 0, 1, A or $\sim A$, where A is a variable. A sequent of relations of $\mathbf{RG}_\sim$, S, is said to be *atomic* if it is of the form

$$\theta(A_1 \lhd_1 B_1) \mid \theta(A_2 \lhd_2 B_2) \mid \ldots \mid \theta(A_n \lhd_n B_n)$$

where, for all $i \in \{1, \ldots, n\}$, $\lhd_i \in \{\leq, <\}$, A_i and B_i are literals of $\mathbf{G}_\sim$, and θ substitutes atomic formulas for the variables appearing in S.

Definition 4.7. A sequent of relations S is said to be *sound* if it contains as subsequent one of the following:

1. $A_1 \lhd_n A_n \mid \ldots \mid A_3 \lhd_2 A_1 \mid A_2 \leq A_1$ if $n > 1$, where $\lhd_i \in \{\leq, <\}$, or $A_1 \leq A_1$ if $n = 1$;
2. $A_n \leq A_{n-1} \mid A_{n-1} < A_{n-2} \mid \ldots \mid A_1 < 1$ if $n > 1$, or $A_1 \leq 1$ if $n = 1$;
3. $0 < A_n \mid A_n < A_{n-1} \mid \ldots \mid A_2 \leq A_1$ if $n > 1$, or $0 \leq A_1$ if $n = 1$;
4. $0 < A_1 \mid A_1 < A_2 \mid \ldots \mid A_n < 1$ if $n \geq 1$, or $0 < 1$ if $n = 0$.

where each A_i is a literal of $\mathbf{G}_\sim$.

Following the nomenclature developed in Section (4.1), sequents of type (1) are called *cycles*, those of type (2) *1-chains*, those of type (3) *0-chains*, and finally those of type (4) *0-1-chains*.

Definition 4.8. Let S be an atomic sequent of relations of the form

$$\theta(A_1 \lhd_1 B_1) \mid \theta(A_2 \lhd_2 B_2) \mid \ldots \mid \theta(A_n \lhd_n B_n).$$

We define *the formula associated to* S as

$$\mathcal{F}(\mathrm{S}) = \forall \vec{x} \bigvee_{i=1}^{n} (A_i \lhd_i B_i).$$

Definition 4.9. Let S be an atomic sequent of relations of the form

$$\theta(A_1 \lhd_1 B_1) \mid \ldots \mid \theta(A_n \lhd_n B_n)$$

The *pumped* sequent of S, indicated as $\overline{\mathrm{S}}$, is the sequent

$$\theta(A_1 \lhd_1 B_1) \mid \ldots \mid \theta(A_n \lhd_n B_n) \mid \theta(\overline{B}_1 \lhd_1 \overline{A}_1) \mid \ldots \mid \theta(\overline{B}_n \lhd_n \overline{A}_n),$$

where

$$\overline{A}_i = \begin{cases} \sim x_j & \text{if } A_i = x_j \\ x_j & \text{if } A_i = \sim x_j \end{cases}$$

S is said to be *pumped* if, for every literals A and B of S and for every $\lhd \in \{\leq, <\}$, $\theta(A \lhd B)$ is a relation appearing in S, then the relation $\theta(\overline{B} \lhd \overline{A})$ appears in S too.

In the next sections we will be concerned in proving the following theorem, which provides the explicit description of the atomic axioms of the calculi we are looking for.

Theorem 4.9 ([37])**.** *Let* S *be an atomic sequent of relations. Then*

$$\overline{\mathrm{S}} \textit{ is sound} \iff \mathbf{T}_{\leq,<,\sim} \models \mathcal{F}(\mathrm{S}).$$

We will call the "only if" part *soundness*, and the "if" part *completeness*: we deal with each implication in the homonym section.

4.2.1 Soundness

The following lemma will grant us soundness easily.

Lemma 4.10. *If* S *is a sound sequent then* $\mathbf{T}_{\leq,<,\sim} \models \mathcal{F}(\mathrm{S})$.

Proof. Let S be a sound sequent and $\mathcal{C}$ a model of $\mathbf{T}_{\leq,<,\sim}$ such that $\mathcal{C} \not\models \mathcal{F}(\mathrm{S})$; so $\mathcal{C} \models \neg\mathcal{F}(\mathrm{S})$. Let's suppose that S contains a *cycle* (i.e. a subsequent like the one in clause (1) of Definition (4.7)): so, there exist a valuation σ, a positive integer n and literals $A_1, \ldots, A_n$ such that

$$\mathcal{C}, \sigma \models \neg(A_1 \lhd_n A_n) \wedge \ldots \wedge \neg(A_3 \lhd_2 A_2) \wedge \neg(A_2 \leq A_1).$$

If we take

$$\overline{\lhd} = \begin{cases} < & \text{if } \lhd = \leq \\ \leq & \text{if } \lhd = < \end{cases}$$

then we get, using clauses (2) and (3) of Proposition (2.2),

$$\mathcal{C}, \sigma \models (A_n \overline{\lhd_n} A_1) \wedge (A_{n-1} \overline{\lhd_2} A_{n-2}) \wedge \ldots \wedge (A_2 \overline{\lhd_2} A_3) \wedge (A_1 < A_2).$$

Using repeatedly clause (1) of Proposition (2.2) we obtain

$$\mathcal{C}, \sigma \models A_1 < A_1,$$

which is an absurd, since $\mathcal{C}$ is a model of $\mathbf{T}_{\leq,<,\sim}$. Now suppose S contains a *1-chain*: it means that there exist a valuation σ, a positive integer n and literals $A_1, \ldots, A_n$ such that

$$\mathcal{C}, \sigma \models \neg(A_n \leq A_{n-1}) \wedge \neg(A_{n-1} < A_{n-2}) \wedge \ldots \wedge \neg(A_1 \leq 1).$$

So, using clauses (2) and (3) of Proposition (2.2), we get

$$\mathcal{C}, \sigma \models (A_n < A_{n-1}) \wedge (A_{n-1} \leq A_{n-2}) \wedge \ldots \wedge \neg(A_1 < 1),$$

which, thanks to clause (1) of Proposition (2.2), leads us to

$$\mathcal{C}, \sigma \models 1 < A_n,$$

a contradiction. The cases of *0-chains* and *0-1-chains* are handled as above. So, if S is a sound sequent and $\mathcal{C}$ is a model of $\mathbf{T}_{\leq,<,\sim}$ then $\mathcal{C} \models \mathcal{F}(\mathrm{S})$. □

Theorem 4.11 (Soundness)**.** *Let* S *be an atomic sequent of relations. Then*

$$\overline{\mathrm{S}} \textit{ is sound } \Longrightarrow \mathbf{T}_{\leq,<,\sim} \models \mathcal{F}(\mathrm{S}).$$

Proof. Suppose that $\overline{\mathrm{S}}$ is a sound sequent; Lemma (4.10) tells us that

$$\mathbf{T}_{\leq,<,\sim} \models \mathcal{F}(\overline{\mathrm{S}}),$$

which means that for every model $\mathcal{C}$ of $\mathbf{T}_{\leq,<,\sim}$ and for every interpretation σ there exists a relation $\theta(A_i \lhd_i B_i)$ of $\overline{\mathrm{S}}$ such that $\mathcal{C}, \sigma \models A_i \lhd_i B_i$. Now, if the relation belonged already to S then it is clear that $\mathcal{C}, \sigma \models \mathcal{F}(\mathrm{S})$; but if it is not the case, then it was introduced in $\overline{\mathrm{S}}$ by the operation of pumping, which implies that the relation $\overline{B}_i \lhd_i \overline{A}_i$ belongs to S. Using clauses (1) and (2) of Proposition (2.3) we then get $\mathcal{C}, \sigma \models \overline{B}_i \lhd_i \overline{A}_i$. So, in every case we get

$$\mathcal{C}, \sigma \models \mathcal{F}(\mathrm{S}),$$

as we wanted. □

4.2.2 Completeness

The strategy for proving the "only if" implication of Theorem (4.9) is to provide a model $\mathcal{C}$ of $\mathbf{T}_{\leq,<,\sim}$ which satisfies $\neg\mathcal{F}(\overline{\mathrm{S}})$ (and so, also satisfies $\neg\mathcal{F}(\mathrm{S})$), under the assumption that $\overline{\mathrm{S}}$ is not sound. As already pointed out in Section (4.1), if the sequent $\overline{\mathrm{S}}$ is of the form

$$\theta(A_1 \lhd_1 B_1) \mid \theta(A_2 \lhd_2 B_2) \mid \ \dots \ \mid \theta(A_n \lhd_n B_n),$$

then

$$\neg\mathcal{F}(\overline{\mathrm{S}}) \equiv \exists\vec{x} \bigwedge_{i=1}^{n} (B_i \overline{\lhd_i} A_i),$$

so, if we find a model of $\mathbf{T}_{\leq,<,\sim}$, $\mathcal{C}$, and an assignment σ of the variables $\vec{x}$ into the universe of $\mathcal{C}$ such that $\mathcal{C}$ satisfies $\sigma(B_i) \overline{\lhd_i} \sigma(A_i)$ for all $i = 1, \dots, n$, then we are done. This is why, in this section, we will slightly modify the algorithm introduced in Section (4.1) for our purposes.

Definition 4.10. Let S be an atomic sequent of relations of the form

$$\theta(A_1 \lhd_1 B_1) \mid \theta(A_2 \lhd_2 B_2) \mid \ \dots \ \mid \theta(A_n \lhd_n B_n).$$

The *set of inequalities associated to* S is the set

$$\Sigma(\mathrm{S}) = \{B_i \overline{\lhd_i} A_i\}_{i=1}^{n}.$$

Proposition 4.12. *If an atomic sequent of relations* S *is not sound, then no inequalities of the form* $1 < 0$*,* $A < A$*,* $1 < A$*,* $A < 0$ *for any literal* A *of* $\mathbf{G}_\sim$ *belong to* $T(\Sigma(\mathrm{S}))$*.*

Proof. Suppose $A < A \in T(\Sigma(\mathrm{S}))$ for some literal A. By definition, there exist a non negative integer n and literals $A_1, \dots, A_n$ such that $A \lhd_1 A_1, A_1 \lhd_2 A_2, \dots, A_n \lhd_{n+1} A \in \Sigma(\mathrm{S})$ where $\lhd_i \in \{\leq, <\}$, and, for at least one i, $\lhd_i =<$. But this means that S contains the subsequent

$$A \overline{\lhd_{n+1}} A_n \mid \cdots \mid A_2 \overline{\lhd_2} A_1 \mid A_1 \overline{\lhd_1} A,$$

where, for at least one i, $\overline{\lhd_i} =\leq$; but this just means that S contains a cycle, and so it is sound, a contradiction. The other cases are handled similarly. □

The idea is to run the algorithm on the set of inequalities $\Sigma(\overline{\mathrm{S}})$ treating every literal as a variable; but, in order to be able to define an involutive, order reversing operation $\sim$ in the model, we have to modify the algorithm. This is how we modify the last cycle (note that N is even, since if two literals A and B are collapsed together, then $\overline{A}$ and $\overline{B}$ are collapsed together too):

21. `for every` $i = 0, \ldots, N/2$

 22. `take the first literal` A `such that its column contains all blank symbols, except, at most, one` 1 `in the diagonal cell, and that the literal` $\overline{A}$ `is not assigned, and define`

$$\sigma(A) := i \text{ and } \sigma(\overline{A}) := n - i;$$

 23. `if the columns and rows of the literals` $B_1, \ldots, B_k$ `where removed at step (17) together with` A, `define`

$$\sigma(B_1) := i, \ldots, \sigma(B_k) := i, \sigma(\overline{B_1}) := n - i, \ldots, \sigma(\overline{B_k}) := n - i;$$

 24. `delete rows and columns of` A `and` $\overline{A}$`;`

25. `define` $C = \{0, 1, \ldots, N\}$ `and return` $(\mathcal{C}, \sigma)$.

This is to prevent situations like the following one: take the atomic sequent S

$$\sim A \leq A \mid \sim B \leq B,$$

(where A and B are atomic formulas) which has as associated set of inequalities

$$\{x <\sim x, y <\sim y\}.$$

If we apply the algorithm to this set of inequalities, after the cycle at step (9), we get the matrix

	x	$\sim x$	y	$\sim y$
x		0		
$\sim x$				
y				0
$\sim y$				

If we proceed with the assignment as before, we obtain $x = 0$, $\sim x = 1$, $y = 2$ and $\sim y = 3$, so $\sim$ is not order reversing. With the new cycle, the algorithm produces the assignment $x = 0$, $y = 1$, $\sim y = 2$ and $\sim x = 3$, which is suitable for our purposes.

Lemma 4.13. *If* S *is a pumped, not sound sequent, then, for every literal* A, B*, we have that:*

1. *Proposition (4.5) holds, so* $\sigma(A) = \sigma(B) \iff \sigma(\overline{A}) = \sigma(\overline{B})$*;*

2. $A \leq B \in T(\Sigma(\mathrm{S})) \Longrightarrow \sigma(A) \leq \sigma(B)$*;*

3. $A < B \in T(\Sigma(\mathrm{S})) \Longrightarrow \sigma(A) < \sigma(B)$*;*

4. $\sigma(A) \leq \sigma(B) \Longrightarrow \sigma(\overline{B}) \leq \sigma(\overline{A})$*.*

Proof. 1. Proposition (4.5) holds because the cycle at step (18) of the algorithm is not touched. So, suppose that A and B are literals such that $\sigma(A) = \sigma(B)$; then Proposition (4.5) tells us that, for every literal C of S and for every $\triangleleft \in \{\leq, <\}$, we have

$$\begin{array}{lcl} A \triangleleft C \in T(\Sigma(\mathrm{S})) & \iff & B \triangleleft C \in T(\Sigma(\mathrm{S})) \text{ and} \\ C \triangleleft A \in T(\Sigma(\mathrm{S})) & \iff & C \triangleleft A \in T(\Sigma(\mathrm{S})). \end{array}$$

But, since S is a pumped sequent then we have

$$X \triangleleft Y \in T(\Sigma(\mathrm{S})) \iff \overline{Y} \triangleleft \overline{X} \in T(\Sigma(\mathrm{S})),$$

for every couple of literals X, Y of S. This implies that, for every literal C,

$$\begin{array}{lcl} C \triangleleft \overline{A} \in T(\Sigma(\mathrm{S})) & \iff & C \triangleleft \overline{A} \in T(\Sigma(\mathrm{S})) \text{ and} \\ \overline{A} \triangleleft C \in T(\Sigma(\mathrm{S})) & \iff & \overline{B} \triangleleft C \in T(\Sigma(\mathrm{S})), \end{array}$$

which is equivalent to $\sigma(\overline{A}) = \sigma(\overline{B})$, by Proposition (4.5). The reverse implication is straightforward.

2. Suppose $A \leq B \in T(\Sigma(\mathrm{S}))$; if $B \leq A \in T(\Sigma(\mathrm{S}))$ too, then (1) guarantees that $\sigma(A) = \sigma(B)$. Otherwise we have two cases: if A is assigned before than B, the thesis holds; but B can be assigned before than A, because of the inequality $\overline{B} \leq \overline{A} \in T(\Sigma(\mathrm{S}))$. In this case, the algorithm defines

$$\sigma(\overline{B}) := i \text{ and } \sigma(B) := N - i,$$

for a certain i, and where N is the number of variables we assign different integers to. If, at this point, the algorithm chooses to assign A, then

$$\sigma(A) := j \text{ and } \sigma(\overline{A}) := N - j,$$

where $i < j \leq N/2 < N - i$, so $\sigma(A) < \sigma(B)$; if, instead, the algorithm decides to assign $\overline{A}$, then

$$\sigma(\overline{A}) := j \text{ and } \sigma(A) := N - j,$$

where $i < j$, so $N - j < N - i$, which means $\sigma(A) < \sigma(B)$.

3. Same as point (2).

4. Suppose $\sigma(A) < \sigma(B)$. Then, two cases are possible: A is assigned before than B, so $\sigma(\overline{B}) < \sigma(\overline{A})$, by construction; or $\overline{B}$ is assigned before than $\overline{A}$, so, again, $\sigma(\overline{B}) < \sigma(\overline{A})$, by construction. □

Theorem 4.14 (Completeness)**.** *Let* S *be an atomic sequent of relations. Then*

$$\mathbf{T}_{\leq,<,\sim} \models \mathcal{F}(\mathrm{S}) \Longrightarrow \overline{\mathrm{S}} \textit{ is sound.}$$

Proof. Suppose $\overline{\mathrm{S}}$ is not sound and run the modified algorithm on $\Sigma(\overline{\mathrm{S}})$, treating any literal (included 0 and 1) as a variable. Note that $\Sigma(\overline{\mathrm{S}})$ is satisfiable by Proposition (4.12), so the algorithm produces a model of $\Sigma(\overline{\mathrm{S}})$, $(\mathcal{C}, \sigma)$. Clearly, 0 is assigned the minimum of the chain, and 1 the maximum (by Lemma (4.13)). It remains to see if we can define a unary, involutive and order reversing operation on $\mathcal{C}$ which is compatible with the inequalities of $\Sigma(\overline{\mathrm{S}})$.

For every integer n in the universe of $\mathcal{C}$, let A be a literal such that $\sigma(A) = n$, and define

$$\ominus n = \sigma(\overline{A}).$$

We have to check the following properties:

1. $\ominus$ is well defined, because of point (1) of Lemma (4.13);

2. the idempotency of $\ominus$ follows from

$$\ominus(\ominus\sigma(A)) = \ominus(\sigma(\overline{A})) = \sigma(A);$$

3. $\ominus$ is order reversing because of point (4) of Lemma (4.13).

So $(\mathcal{C}, \leq, <, \ominus) \models \mathbf{T}_{\leq,<,\sim}$, but Lemma (4.13) assures also that $\mathcal{C}, \sigma \models \Sigma(\overline{\mathrm{S}})$, yielding

$$\mathbf{T}_{\leq,<,\sim} \not\models \mathcal{F}(\mathrm{S}),$$

as we wanted. □

Since the changing made to the algorithm does not alter its time complexity, the above theorem implies the following as a corollary.

Corollary 4.15. *The countermodel problem for infinite-valued Gödel logic with an involutive negation is polynomial.*

4.2.3 A complete presentation for $\mathbf{RG}_{\sim}$

We now present in detail the full sequents-of-relations calculus for $\mathbf{RG}_{\sim}$, listing all the logical rules and reviewing the form of the axioms of the calculus.

These are the rules for $\wedge$:

$$\frac{A \leq C \mid B < A \mid \mathcal{H} \qquad B \leq C \mid A \leq B \mid \mathcal{H}}{A \wedge B \leq C \mid \mathcal{H}}\ (\wedge, \leq, l)$$

$$\frac{C \leq A \mid B < A \mid \mathcal{H} \qquad C \leq B \mid A \leq B \mid \mathcal{H}}{C \leq A \wedge B \mid \mathcal{H}}\ (\wedge, \leq, r)$$

$$\frac{A < C \mid B < A \mid \mathcal{H} \qquad B < C \mid A \leq B \mid \mathcal{H}}{A \wedge B < C \mid \mathcal{H}}\ (\wedge, <, l)$$

$$\frac{C < A \mid B < A \mid \mathcal{H} \qquad C < B \mid A \leq B \mid \mathcal{H}}{C < A \wedge B \mid \mathcal{H}}\ (\wedge, <, r)$$

The rules for $\vee$ are:

$$\frac{B \leq C \mid B < A \mid \mathcal{H} \qquad A \leq C \mid A \leq B \mid \mathcal{H}}{A \vee B \leq C \mid \mathcal{H}}\ (\vee, \leq, l)$$

$$\frac{C \leq B \mid B < A \mid \mathcal{H} \qquad C \leq A \mid A \leq B \mid \mathcal{H}}{C \leq A \vee B \mid \mathcal{H}}\ (\vee, \leq, r)$$

$$\frac{B < C \mid B < A \mid \mathcal{H} \qquad A < C \mid A \leq B \mid \mathcal{H}}{A \wedge B < C \mid \mathcal{H}}\ (\vee, <, l)$$

$$\frac{C < B \mid B < A \mid \mathcal{H} \qquad C < A \mid A \leq B \mid \mathcal{H}}{C < A \vee B \mid \mathcal{H}}\ (\vee, <, r)$$

The rules for $(\sim \wedge, \triangleleft, l)$ can be obtained from those above substituting $C \triangleleft A$, $C \triangleleft B$, and $C \triangleleft A \wedge B$ with $C \triangleleft \sim A$, $C \triangleleft \sim B$, $C \triangleleft \sim (A \wedge B)$ respectively, for any $\triangleleft \in \{\leq, <\}$. A similar argument is valid for $(\sim \vee, \triangleleft, r)$, where $\triangleleft \in \{\leq, <\}$.

The following are the rules for $\rightarrow$ and $\sim\rightarrow$:

$$\frac{1 \leq C \mid B < A \mid \mathcal{H} \qquad B \leq C \mid A \leq B \mid \mathcal{H}}{A \rightarrow B \leq C \mid \mathcal{H}}\ (\rightarrow, \leq, l)$$

$$\frac{C \leq B \mid A \leq B \mid \mathcal{H}}{C \leq A \rightarrow B \mid \mathcal{H}}\ (\rightarrow, \leq, r) \qquad \frac{\sim B \leq C \mid A \leq B \mid \mathcal{H}}{\sim (A \rightarrow B) \leq C \mid \mathcal{H}}\ (\sim\rightarrow, \leq, l)$$

$$\frac{C \leq 0 \mid B < A \mid \mathcal{H} \qquad C \leq \sim B \mid A \leq B \mid \mathcal{H}}{C \leq \sim (A \to B) \mid \mathcal{H}} \ (\sim\to, \leq, r)$$

Finally, the rules for $\sim$:

$$\frac{A \leq B \mid \mathcal{H}}{\sim\sim A \leq B \mid \mathcal{H}} \ (\sim\sim, \leq, l) \qquad \frac{B \leq A \mid \mathcal{H}}{B \leq \sim\sim A \mid \mathcal{H}} \ (\sim\sim, \leq, r)$$

$$\frac{A < B \mid \mathcal{H}}{\sim\sim A < B \mid \mathcal{H}} \ (\sim\sim, <, l) \qquad \frac{B < A \mid \mathcal{H}}{B < \sim\sim A \mid \mathcal{H}} \ (\sim\sim, <, r)$$

Theorem (4.9) states that an atomic sequent of relations is an axiom of the calculi if and only if it is equivalent (up to external weakening) to one of the following sequents:

1. $A_1 \lhd_n A_n \mid \ldots \mid A_3 \lhd_2 A_1 \mid A_2 \leq A_1$ if $n > 1$, where $\lhd_i \in \{\leq, <\}$, or $A_1 \leq A_1$ if $n = 1$;

2. $A_n \leq A_{n-1} \mid A_{n-1} < A_{n-2} \mid \ldots \mid A_1 < 1$ if $n > 1$, or $A_1 \leq 1$ if $n = 1$;

3. $0 < A_n \mid A_n < A_{n-1} \mid \ldots \mid A_2 \leq A_1$ if $n > 1$, or $0 \leq A_1$ if $n = 1$;

4. $0 < A_1 \mid A_1 < A_2 \mid \ldots \mid A_n < 1$ if $n \geq 1$, or $0 < 1$ if $n = 0$.

4.3 Nilpotent Minimum

As already pointed out in Section (4.2), since Nilpotent Minimum logics and Gödel logics with an involutive negation share the semantic theory $\mathbf{T}_{\leq,<,\sim}$, a characterization of the axiom of the sequent of relation calculus for NM is given by Theorem (4.9).

The following are the logical rules of the calculus (we omit the rules for $\wedge$, $\vee$, $\sim\wedge$, $\sim\vee$ and $\sim\sim$ because they coincide with those of $\mathbf{RG}_{\sim}$).

$$\frac{C \leq 0 \mid \sim B < A \mid \mathcal{H} \qquad C \leq A \mid A \leq \sim B \mid B < A \mid \mathcal{H} \qquad C \leq B \mid A \leq \sim B \mid A \leq B \mid \mathcal{H}}{C \leq A \& B \mid \mathcal{H}} \ (\&, \leq, r)$$

$$\frac{A \leq C \mid A \leq \sim B \mid B < A \mid \mathcal{H} \qquad B \leq C \mid A \leq \sim B \mid A \leq B \mid \mathcal{H}}{A \& B \leq C \mid \mathcal{H}} \ (\&, \leq, l)$$

$$\frac{C < 0 \mid \sim B < A \mid \mathcal{H} \qquad C < A \mid A \leq \sim B \mid B < A \mid \mathcal{H} \qquad C < B \mid A \leq \sim B \mid A \leq B \mid \mathcal{H}}{C < A \& B \mid \mathcal{H}} \ (\&, <, r)$$

$$\frac{0 < C \mid \sim B < A \mid \mathcal{H} \qquad A < C \mid A \leq \sim B \mid B < A \mid \mathcal{H} \qquad B < C \mid A \leq \sim B \mid A \leq B \mid \mathcal{H}}{A \& B < C \mid \mathcal{H}}\ (\&, <, l)$$

$$\frac{C \leq \sim A \mid A \leq \sim B \mid B < A \mid \mathcal{H} \qquad C \leq \sim B \mid A \leq \sim B \mid A \leq B \mid \mathcal{H}}{C \leq \sim (A \& B) \mid \mathcal{H}}\ (\sim \&, \leq, r)$$

$$\frac{1 \leq C \mid \sim B < A \mid \mathcal{H} \qquad \sim A \leq C \mid A \leq \sim B \mid B < A \mid \mathcal{H} \qquad \sim B \leq C \mid A \leq \sim B \mid A \leq B \mid \mathcal{H}}{\sim (A \& B) \leq C \mid \mathcal{H}}\ (\sim \&, \leq, l)$$

$$\frac{C < 1 \mid \sim B < A \mid \mathcal{H} \qquad C < \sim A \mid A \leq \sim B \mid B < A \mid \mathcal{H} \qquad C < \sim B \mid A \leq \sim B \mid A \leq B \mid \mathcal{H}}{C < \sim (A \& B) \mid \mathcal{H}}\ (\&, <, r)$$

$$\frac{1 < C \mid \sim B < A \mid \mathcal{H} \qquad \sim A < C \mid A \leq \sim B \mid B < A \mid \mathcal{H} \qquad \sim B < C \mid A \leq \sim B \mid A \leq B \mid \mathcal{H}}{\sim (A \& B) < C \mid \mathcal{H}}\ (\sim \&, <, l)$$

$$\frac{C \leq B \mid A \leq B \mid B \leq \sim A \mid \mathcal{H} \qquad C \leq \sim A \mid A \leq B \mid \sim A < B \mid \mathcal{H}}{C \leq A \to B \mid \mathcal{H}}\ (\to, \leq, r)$$

$$\frac{1 \leq C \mid B < A \mid \mathcal{H} \qquad B \leq C \mid A \leq B \mid B \leq \sim A \mid \mathcal{H} \qquad \sim A \leq C \mid A \leq B \mid \sim A < B \mid \mathcal{H}}{A \to B \leq C \mid \mathcal{H}}\ (\to, \leq, l)$$

$$\frac{C < 1 \mid B < A \mid \mathcal{H} \qquad C < B \mid A \leq B \mid B \leq \sim A \mid \mathcal{H} \qquad C < \sim A \mid A \leq B \mid \sim A < B \mid \mathcal{H}}{C < A \to B \mid \mathcal{H}}\ (\to, <, r)$$

$$\frac{1 < C \mid B < A \mid \mathcal{H} \qquad B < C \mid A \leq B \mid B \leq \sim A \mid \mathcal{H} \qquad \sim A < C \mid A \leq B \mid \sim A < B \mid \mathcal{H}}{A \to B < C \mid \mathcal{H}}\ (\to, <, l)$$

$$\frac{C \leq 0 \mid B < A \mid \mathcal{H} \qquad C \leq A \mid A \leq B \mid \sim A < B \mid \mathcal{H} \qquad C \leq \sim B \mid A \leq B \mid B \leq \sim A \mid \mathcal{H}}{C \leq \sim (A \to B) \mid \mathcal{H}}\ (\sim\to, \leq, r)$$

$$\frac{A \leq C \mid A \leq B \mid \sim A < B \mid \mathcal{H} \qquad \sim B \leq C \mid A \leq B \mid B \leq \sim A \mid \mathcal{H}}{\sim (A \to B) \leq C \mid \mathcal{H}}\ (\sim\to, \leq, l)$$

$$\frac{C < 0 \mid B < A \mid \mathcal{H} \qquad C < A \mid A \leq B \mid \sim A < B \mid \mathcal{H} \qquad C < \sim B \mid A \leq B \mid B \leq \sim A \mid \mathcal{H}}{C < \sim (A \to B) \mid \mathcal{H}}\ (\sim\to, <, r)$$

$$\frac{0 < C \mid B < A \mid \mathcal{H} \qquad A < C \mid A \leq B \mid \sim A < B \mid \mathcal{H} \qquad \sim B < C \mid A \leq B \mid B \leq \sim A \mid \mathcal{H}}{\sim (A \to B) < C \mid \mathcal{H}}\ (\sim\to, <, l)$$

4.4 n-contractive BL$^+$-logics

We now present the full description of the axioms for n-contractive BL$^+$-logics, as reported in [37]. We will adopt the following abbreviations:

1. if $\varphi \lhd_1 \psi_1 \mid \psi_2 \lhd_2 \psi_3 \mid \ldots \mid \psi_n \lhd_n \psi$, where $\lhd_i \in \{\preceq, \prec\}$, we write $\varphi \prec^* \psi$ if, for some i, $\lhd_i = \prec$, while we write $\varphi \preceq^* \psi$ otherwise;
2. $||_{i=1}^{n} \mathrm{S}_i$ stands for $\mathrm{S}_1 \mid \ldots \mid \mathrm{S}_n$.

Axioms are all s-atomic sequents of relations containing

(Ax0) $0 \preceq^* \varphi$, $\varphi \preceq^* 1$, $0 \prec^* 1$, $0 \preceq^* 1$,

(Ax1) a cycle $\varphi_1 \lhd_1 \varphi_2 \mid \varphi_2 \lhd_2 \varphi_3 \mid \ldots \mid \varphi_n \lhd_n \varphi_1$ where for $i = 1, \ldots, n$, $\lhd_i$ is either $\preceq$ or $\prec$ and for at least one i, $\lhd_i$ is $\preceq$,

(Ax2) $C_{h,k}(\varphi) \mid C^{\star}_{h,k}(\varphi)$ for some $1 \leq h \leq k \leq n$, and any component $C^{\star}_{h,k}(S^p(\varphi))$ with $h \neq k$ and $h \neq p$,

(Ax3) $\varphi \prec^* 1 \mid C^{\star}_{h,k}(\varphi)$, and $C^{\star}_{h,k}(1)$, for $1 \leq h \leq k \leq n$,

(Ax4) $C^{\star}_{h,k}(\varphi) \mid C^{\star}_{h',k'}(\varphi)$ when either $h \neq h'$ or $k \neq k'$,

(Ax5) $\varphi \preceq^* \psi \mid \psi \preceq^* \varphi \mid C^{\star}_{h,k}(\varphi) \mid C^{\star}_{h,k'}(\psi)$, with $h \leq k \leq n$, $h \leq k' \leq n$ and $k \neq k'$,

(Ax6) $\varphi \preceq^* \psi \mid \psi \preceq^* \varphi \mid C^{\star}_{h,k}(\varphi) ||_{i=1}^{k} C^{\star}_{i,k}(\psi)$,

(Ax7) $C^{\star}_{h,k}(0)$ for some $h < k$, or $||_{i=1}^{n} C_{i,i}(0)$,

(Ax8) $\varphi \preceq^* \psi \mid \psi \preceq^* \varphi \mid C^{\star}_{h,k}(\varphi) \mid C_{i,k}(S^i(\psi))$,

(Ax9) $1 \prec^* \varphi ||_{i=1}^{h} C^{\star}_{i,i}(S^h(\varphi)) ||_{i=h+1}^{n} C_{h,i}(S^h(\varphi))$.

Let $k \leq n$, $P = \{U_1, \ldots, U_n\}$ be a partition of $\{1, \ldots, n\}$ into k nonempty pairwise disjoint sets, and let $\Sigma(U_i) = ||_{j' \in U_i}^{j \leq j'} C_{j,j'}(\varphi_i)$. For every P

(Ax10) $||_{i,j=1}^{k,(i \neq j)} \varphi_i \prec^* \varphi_j ||_{i=1}^{k} 1 \prec^* \varphi_i \mid \Sigma(U_1) \mid \ldots \mid \Sigma(U_k)$ is an axiom (in the particular case $k = 1$ this is $1 \preceq^* \varphi \mid C_{1,1}(\varphi) \mid C_{1,2}(\varphi) \mid \ldots \mid C_{n,n}(\varphi)$),

(Ax11) all s-atomic sequents obtained from any of the previous axioms by replacing φ in any $\prec$ or $\preceq$ component by $S^h(\varphi)$, or $S^h(\varphi)$ by either φ or $S^k(\varphi)$, and by replacing in any component 1 by $S^h(1)$, or $S^h(1)$ by either 1 or $S^k(1)$. Moreover (Ax3), (Ax4) (in the case $k \neq k'$), (Ax5), (Ax6), (Ax8) and (Ax9) in which (some) φ are replaced by $S^h(\varphi)$.

Chapter 5

Cut-Elimination

The procedure we have presented so far introduces calculi whose rules are only logical or structural. We now study the possibility for introducing "conservative" cut rules, i.e. cut rules which do not alter the provable sequents of the calculi, as done in [4]. These rules are not needed for proof search, by the Completeness Theorem (3.5), but are really helpful because of the speedup they provide when building a proof tree.

5.1 Extended Structural Rules

This section is inspired by the homonym section in [7]: we introduce the background, the main definitions and examples of the arguments at the core of the next section.

Definition 5.1. Let $\mathcal{L}$ be a semiprojective logic with semantic theory $\mathbf{T}$. A rule is called *extended structural rule in* $\mathbf{R}\mathcal{L}_{\mathbf{T}}$ if it is of the form

$$\frac{\theta[\Gamma_1] \mid \mathcal{H} \qquad \dots \qquad \theta[\Gamma_n] \mid \mathcal{H}}{\theta[\Gamma] \mid \mathcal{H}}$$

where $\Gamma_1, \dots, \Gamma_n, \Gamma$ are sequences of atomic formulas of $\mathbf{T}$, θ is a substitution of variables by formulas and $\mathcal{H}$ is a side sequent.

An extended structural rule in $\mathbf{R}\mathcal{L}_{\mathbf{T}}$ is *analytic* if, denoting with $var(\Delta)$ the set of variables occurring in the sequent Δ, we have

$$\bigcup_{1 \leq i \leq n} var(\Gamma_i) \subseteq var(\Gamma).$$

An extended structural rule in $\mathbf{R}\mathcal{L}_{\mathbf{T}}$ is *admissibile* in the calculus if

$$\mathbf{T} \models \forall \vec{x}(\widehat{\Gamma}_1 \wedge \ldots \wedge \widehat{\Gamma}_n) \rightarrow \widehat{\Gamma}$$

where $\widehat{\Delta}$ is the disjunction of the atomic formulas of Δ (if Δ is empty then $\widehat{\Delta} \equiv True$).

It is worth noticing that it is possible to decide whether an extended rule is admissible or not because of the decidability of the Π_1-formulas of $\mathbf{T}$. Furthermore, all provable sequents in $\mathbf{R}\mathcal{L}_{\mathbf{T}}$ extended by admissible structural rules are already derivable in $\mathbf{R}\mathcal{L}_{\mathbf{T}}$, as follows from the definition.

Examples of admissible analytic extended structural rules can be given for semiprojective logics whose semantic theory is an extension of $\mathbf{T}_{\leq,<}$

$$\frac{1 \leq B \mid \mathcal{H}}{A \leq B \mid \mathcal{H}} \qquad \frac{A \leq 0 \mid \mathcal{H}}{A \leq B \mid \mathcal{H}} \qquad \frac{1 \leq 0 \mid \mathcal{H}}{\mathcal{H}}$$

Notice that the first two correspond to *internal* weakening in standard sequent calculi. In [5], it is shown that the following analytic structural rule, admissible for Gödel logics,

$$\frac{A \leq B \mid \mathcal{H} \qquad C \leq D \mid \mathcal{H}}{A \leq D \mid C \leq B \mid \mathcal{H}}$$

is an instance of Avron's communication rule. In that work, a much deeper connection between hypersequent calculi for Gödel logic (see [2]) and sequents-of-relations calculi was established.

Definition 5.2. An extended structural rule in $\mathbf{R}\mathcal{L}_{\mathbf{T}}$ is a *cut rule* if

$$var(\Gamma) \subsetneq \bigcup_{1 \leq i \leq n} var(\Gamma_i).$$

Concrete examples of admissible cut rules for semiprojective logics which have for semantic theory an extension of $\mathbf{T}_{\leq,<}$ can be obtained by Axioms (T2) and (T5):

$$\frac{A \leq B \mid \mathcal{H} \qquad B \leq C \mid \mathcal{H}}{A \leq C \mid \mathcal{H}} \qquad \frac{A < B \mid \mathcal{H} \qquad B < C \mid \mathcal{H}}{A < C \mid \mathcal{H}}$$

In general, if the semantic theory of a semiprojective logic contains a transitive relation $\prec$, then the following cut is an admissible cut rule, called "transitivity cut":

$$\frac{A \prec B \mid \mathcal{H} \qquad B \prec C \mid \mathcal{H}}{A \prec C \mid \mathcal{H}}$$

Proposition (2.2) enables us to list even the following "mixed" transitivity cut:

$$\frac{A \leq B \mid \mathcal{H} \qquad B < C \mid \mathcal{H}}{A < C \mid \mathcal{H}}$$

Other examples of cut rules are the following:

$$\frac{A \leq 0 \mid \mathcal{H} \qquad 1 \leq A \mid \mathcal{H}}{\mathcal{H}}\,(1<0) \qquad \frac{0 < A \mid \mathcal{H}}{\mathcal{H}} \qquad \frac{A < 1 \mid \mathcal{H}}{\mathcal{H}}$$

which follow from the fact that $\leq$ and $<$ have distinct maximum and minimum element, and that $<$ is irreflexive. Clearly, they can be generalized to semantic theories having a partial order $\prec$ satisfying these properties.

For the rest of the chapter we will be interested in a particular admissible cut rule for semantic theories extending $\mathbf{T}_{\leq,<}$:

$$\frac{A \leq B \mid \mathcal{H} \qquad B < A \mid \mathcal{H}}{\mathcal{H}}\,(\text{cut}_{\leq\backslash>})$$

A and B are called *cut-formulas* and the indicated components are referred to as *cut-components*.

Our interest is motivated by the fact that this type of cut is able to simulate other cuts straightforwardly, such as the $\leq$-transitivity cut, which can be obtained as follows:

$$\frac{\dfrac{B < A \mid A \leq C \mid C < B \mid \mathcal{H} \qquad A \leq B \mid \mathcal{H}}{C < B \mid A \leq C \mid \mathcal{H}}\,(\text{cut}_{\leq\backslash>}) \qquad B \leq C \mid \mathcal{H}}{A \leq C \mid \mathcal{H}}\,(\text{cut}_{\leq\backslash>})$$

The $(1 < 0)$-cut is also derivable from $(\text{cut}_{\leq\backslash>})$:

$$\frac{\dfrac{0 < A \mid A < 1 \mid \mathcal{H} \qquad A \leq 0 \mid \mathcal{H}}{A < 1 \mid \mathcal{H}}\,(\text{cut}_{\leq\backslash>}) \qquad 1 \leq A \mid \mathcal{H}}{\mathcal{H}}\,(\text{cut}_{\leq\backslash>})$$

5.2 Cut-elimination for Gödel logics with an involutive negation

Since $\mathbf{G}_\sim$ has as semantic theory an extension of $\mathbf{T}_{\leq,<}$, $\mathbf{RG}_\sim$ admit $(\text{cut}_{\leq\backslash>})$ as an extended structural rule. In this section we will prove a cut-elimination result in this framework, allowing us to use $(\text{cut}_{\leq\backslash>})$ at our pleasure, aware of the fact that a procedure for obtaining a cut-free derivation (at the end of the section) is available. To prove this fact, we will follow the strategy outlined in [4] (basically extending it, since it dealt with Gödel logics), which consists in four steps:

1. Lemma (5.1): replacement of compound axioms by atomic ones.
2. Lemma (5.3): replacement of cuts involving compound formulas with atomic cuts.
3. Lemma (5.4): moving atomic cuts up to atomic sequents.
4. Lemma (5.5): elimination of cuts involving only axioms.

Definition 5.3. Let d be a derivation. The *length of* d, $|d|$, is the maximal number of sequents occurring in every branch of d.

Definition 5.4. The *complexity* of a cut is the number of $\wedge, \vee, \rightarrow$ and $\sim\sim$ occurring in a cut-component of it plus 1. A cut of complexity 1 is called *atomic*. With $\rho(d)$ we denote the maximal complexity of cuts in d.

Lemma 5.1. *Non-atomic axioms are derivable from atomic ones.*

Proof. By induction on the number of connectives of formulas. Let us just consider the case of a non-atomic cycle of the form

$$A_1 \lhd_n A_n \mid \cdots \mid A_{i+1} < P \mid P \leq A_{i-1} \mid \cdots \mid A_2 \leq A_1$$

where $P = \sim (B \rightarrow C)$. This axiom can be derived from the axioms

$$A_1 \lhd_n A_n \mid \cdots \mid A_{i+1} < 0 \mid 0 \leq A_{i-1} \mid \cdots \mid A_2 \leq A_1,$$

$$A_1 \lhd_n A_n \mid \cdots \mid A_{i+1} < \sim B \mid \sim B \leq A_{i-1} \mid \cdots \mid A_2 \leq A_1,$$

and $B \leq C \mid C < B$ as follows: calling with $\mathcal{H}$ the sequent

$$A_1 \lhd_n A_n \mid \cdots \mid A_{i+2} \lhd_{i+1} A_{i+1} \mid A_{i-1} \lhd_{i-2} A_{i-2} \mid \cdots \mid A_2 \leq A_1,$$

we first obtain trees

$$\frac{A_{i+1} < 0 \mid 0 \leq A_{i-1} \mid C < B \mid \mathcal{H} \qquad A_{i+1} < \sim B \mid 0 \leq A_{i-1} \mid B \leq C \mid C < B \mid \mathcal{H}}{A_{i+1} <\sim (B \rightarrow C) \mid 0 \leq A_{i-1} \mid C < B \mid \mathcal{H}}$$

$$\frac{A_{i+1} < 0 \mid \sim B \leq A_{i-1} \mid C < B \mid B \leq C \mid \mathcal{H} \qquad A_{i+1} < \sim B \mid \sim B \leq A_{i-1} \mid B \leq C \mid \mathcal{H}}{A_{i+1} <\sim (B \rightarrow C) \mid \sim B \leq A_{i-1} \mid B \leq C \mid \mathcal{H}}$$

and then we apply again the rule $(\sim\rightarrow, \leq, l)$, obtaining the initial sequent

$$A_{i+1} <\sim (B \rightarrow C) \mid \sim (B \rightarrow C) \leq A_{i-1} \mid \mathcal{H}.$$

The other cases follow in a similar fashion. □

Lemma 5.2 (Inversion Lemma). *If* $d \vdash A \circ B \lhd C \mid \mathcal{H}$*, or* $d \vdash \sim(A \circ B) \lhd C \mid \mathcal{H}$*, or* $d \vdash C \lhd A \circ B \mid \mathcal{H}$*, or* $d \vdash C \lhd \sim(A \circ B) \mid \mathcal{H}$ *(where* $\circ \in \{\wedge, \vee, \rightarrow\}$ *and* $\lhd \in \{\leq, <\}$*), then it is possible to find a derivation* d_1 *of a sequent that is the instance of the premise of the rule for introducing* $A \circ B$ *or* $\sim(A \circ B)$*, if the rule is unary, or derivations* d_1 *and* d_2 *of sequents that are instances of the premises of the rule for introducing* $A \circ B$ *or* $\sim(A \circ B)$*, if the rule is binary, such that* $\rho(d_i) \leq \rho(d)$*, for* $i = 1, 2$*.*

Proof. By induction on the length of $|d|$. By Lemma (5.1), we can assume that all the axioms in d are atomic, so the initial step is done. For the inductive step, suppose the thesis holds for derivations d' such that $|d'| < |d|$; we have to prove the thesis for all $\circ \in \{\wedge, \vee, \rightarrow\}$ and all $\lhd \in \{\leq, <\}$, but we do it in full details only for the case $d \vdash C \leq \sim(A \rightarrow B) \mid \mathcal{H}$. So, we have to find derivations d_1 and d_2 of the sequents $C \leq 0 \mid B < A \mid \mathcal{H}$ and $C \leq \sim B \mid A \leq B \mid \mathcal{H}$, respectively, such that $\rho(d_i) \leq \rho(d)$, for $i = 1, 2$. Let R be the last inference rule of d:

1. R is a logical rule.
   (a) R introduces the indicated occurrence of $\sim(A \rightarrow B)$: then d_1 and d_2 are the subderivations of the premises of R.
   (b) R introduces C: we have to explore all the possible forms of C. If $C = C_1 \wedge C_2$ then by inductive hypothesis we have derivations

$$d_1' \vdash C_1 \leq 0 \mid C_2 < C_1 \mid B < A \mid \mathcal{H},$$

$$d_2' \vdash C_1 \leq \sim B \mid C_2 < C_1 \mid A \leq B \mid \mathcal{H},$$

$$d_1'' \vdash C_2 \leq 0 \mid C_1 \leq C_2 \mid B < A \mid \mathcal{H},$$

$$d_2'' \vdash C_2 \leq \sim B \mid C_1 \leq C_2 \mid A \leq B \mid \mathcal{H},$$

so we define d_1 and d_2 to be

$$\frac{C_1 \leq 0 \mid C_2 < C_1 \mid B < A \mid \mathcal{H} \qquad C_2 \leq 0 \mid C_1 \leq C_2 \mid B < A \mid \mathcal{H}}{C_1 \wedge C_2 \leq 0 \mid B < A \mid \mathcal{H}} (\wedge, \leq, l)$$

$$\frac{C_1 \leq \sim B \mid C_2 < C_1 \mid A \leq B \mid \mathcal{H} \qquad C_2 \leq \sim B \mid C_1 \leq C_2 \mid A \leq B \mid \mathcal{H}}{C_1 \wedge C_2 \leq \sim B \mid A \leq B \mid \mathcal{H}} (\wedge, \leq, l)$$

If $C = C_1 \vee C_2$, again by inductive hypothesis we have derivations

$$d_1' \vdash C_2 \leq 0 \mid C_2 < C_1 \mid B < A \mid \mathcal{H},$$

$$d_2' \vdash C_2 \leq \sim B \mid C_2 < C_1 \mid A \leq B \mid \mathcal{H},$$
$$d_1'' \vdash C_1 \leq 0 \mid C_1 \leq C_2 \mid B < A \mid \mathcal{H},$$
$$d_2'' \vdash C_1 \leq \sim B \mid C_1 \leq C_2 \mid A \leq B \mid \mathcal{H},$$

so we define d_1 and d_2 to be

$$\frac{C_2 \leq 0 \mid C_2 < C_1 \mid B < A \mid \mathcal{H} \qquad C_1 \leq 0 \mid C_1 \leq C_2 \mid B < A \mid \mathcal{H}}{C_1 \vee C_2 \leq 0 \mid B < A \mid \mathcal{H}}\ (\vee, \leq, l)$$

$$\frac{C_2 \leq \sim B \mid C_2 < C_1 \mid A \leq B \mid \mathcal{H} \qquad C_1 \leq \sim B \mid C_1 \leq C_2 \mid A \leq B \mid \mathcal{H}}{C_1 \vee C_2 \leq \sim B \mid A \leq B \mid \mathcal{H}}\ (\vee, \leq, l)$$

If $C = C_1 \to C_2$, by inductive hypothesis we have derivations

$$d_1' \vdash 1 \leq 0 \mid C_2 < C_1 \mid B < A \mid \mathcal{H},$$
$$d_2' \vdash 1 \leq \sim B \mid C_2 < C_1 \mid A \leq B \mid \mathcal{H},$$
$$d_1'' \vdash C_2 \leq 0 \mid C_1 \leq C_2 \mid B < A \mid \mathcal{H},$$
$$d_2'' \vdash C_2 \leq \sim B \mid C_1 \leq C_2 \mid A \leq B \mid \mathcal{H},$$

and we define d_1 and d_2 as

$$\frac{1 \leq 0 \mid C_2 < C_1 \mid B < A \mid \mathcal{H} \qquad C_2 \leq 0 \mid C_1 \leq C_2 \mid B < A \mid \mathcal{H}}{C_1 \to C_2 \leq 0 \mid B < A \mid \mathcal{H}}\ (\to, \leq, l)$$

$$\frac{1 \leq \sim B \mid C_2 < C_1 \mid A \leq B \mid \mathcal{H} \qquad C_2 \leq \sim B \mid C_1 \leq C_2 \mid A \leq B \mid \mathcal{H}}{C_1 \to C_2 \leq \sim B \mid A \leq B \mid \mathcal{H}}\ (\to, \leq, l)$$

If $C = \sim (C_1 \wedge C_2)$ then by inductive hypothesis we have derivations

$$d_1' \vdash \sim C_1 \leq 0 \mid C_2 < C_1 \mid B < A \mid \mathcal{H},$$
$$d_2' \vdash \sim C_1 \leq \sim B \mid C_2 < C_1 \mid A \leq B \mid \mathcal{H},$$
$$d_1'' \vdash \sim C_2 \leq 0 \mid C_1 \leq C_2 \mid B < A \mid \mathcal{H},$$
$$d_2'' \vdash \sim C_2 \leq \sim B \mid C_1 \leq C_2 \mid A \leq B \mid \mathcal{H},$$

and we can define d_1 and d_2 to be

$$\frac{\sim C_1 \leq 0 \mid C_2 < C_1 \mid B < A \mid \mathcal{H} \qquad \sim C_2 \leq 0 \mid C_1 \leq C_2 \mid B < A \mid \mathcal{H}}{\sim (C_1 \wedge C_2) \leq 0 \mid B < A \mid \mathcal{H}}\ (\sim \wedge, \leq, l)$$

$$\frac{\sim C_1 \leq \sim B \mid C_2 < C_1 \mid A \leq B \mid \mathcal{H} \qquad \sim C_2 \leq \sim B \mid C_1 \leq C_2 \mid A \leq B \mid \mathcal{H}}{\sim (C_1 \wedge C_2) \leq \sim B \mid A \leq B \mid \mathcal{H}}\ (\sim \wedge, \leq, l)$$

If $C = \sim (C_1 \vee C_2)$, by inductive hypothesis we have derivations

$$d_1' \vdash \sim C_2 \leq 0 \mid C_2 < C_1 \mid B < A \mid \mathcal{H},$$

$$d_2' \vdash \sim C_2 \leq \sim B \mid C_2 < C_1 \mid A \leq B \mid \mathcal{H},$$

$$d_1'' \vdash \sim C_1 \leq 0 \mid C_1 \leq C_2 \mid B < A \mid \mathcal{H},$$

$$d_2'' \vdash \sim C_1 \leq \sim B \mid C_1 \leq C_2 \mid A \leq B \mid \mathcal{H},$$

so as before we define d_1 and d_2 as

$$\frac{\sim C_2 \leq 0 \mid C_2 < C_1 \mid B < A \mid \mathcal{H} \qquad \sim C_1 \leq 0 \mid C_1 \leq C_2 \mid B < A \mid \mathcal{H}}{\sim (C_1 \vee C_2) \leq 0 \mid B < A \mid \mathcal{H}} \; (\sim \vee, \leq, l)$$

$$\frac{\sim C_2 \leq \sim B \mid C_2 < C_1 \mid A \leq B \mid \mathcal{H} \qquad \sim C_1 \leq \sim B \mid C_1 \leq C_2 \mid A \leq B \mid \mathcal{H}}{\sim (C_1 \vee C_2) \leq \sim B \mid A \leq B \mid \mathcal{H}} \; (\sim \vee, \leq, l)$$

If $C = \sim (C_1 \to C_2)$ then by inductive hypothesis we have derivations

$$d_1' \vdash \sim C_2 \leq 0 \mid C_1 \leq C_2 \mid B < A \mid \mathcal{H},$$

$$d_2' \vdash \sim C_2 \leq \sim B \mid C_1 \leq C_2 \mid A \leq B \mid \mathcal{H},$$

so we define d_1 and d_2 to be

$$\frac{\sim C_2 \leq 0 \mid C_1 \leq C_2 \mid B < A \mid \mathcal{H}}{\sim (C_1 \to C_2) \leq 0 \mid B < A \mid \mathcal{H}} \; (\sim\to, \leq, l)$$

$$\frac{\sim C_2 \leq \sim B \mid C_1 \leq C_2 \mid A \leq B \mid \mathcal{H}}{\sim (C_1 \to C_2) \leq \sim B \mid A \leq B \mid \mathcal{H}} \; (\sim\to, \leq, l)$$

Finally, if $C = \sim\sim C_1$ by inductive hypothesis we have derivations

$$d_1' \vdash C_1 \leq 0 \mid B < A \mid \mathcal{H},$$

$$d_2' \vdash C_1 \leq \sim B \mid A \leq B \mid \mathcal{H},$$

so we define d_1 and d_2 as

$$\frac{C_1 \leq 0 \mid B < A \mid \mathcal{H}}{\sim\sim C_1 \leq 0 \mid B < A \mid \mathcal{H}} \; (\sim\sim, \leq, l)$$

$$\frac{C_1 \leq \sim B \mid A \leq B \mid \mathcal{H}}{\sim\sim C_1 \leq \sim B \mid A \leq B \mid \mathcal{H}} \; (\sim\sim, \leq, l)$$

(c) the formula which R introduces is in $\mathcal{H}$: the proof is analogous to the one of the point above, so we just show the case in which R is $(\sim\wedge, <, r)$. This means that we have

$$\frac{F < \sim D \mid E < D \mid C \leq \sim(A \to B) \mid \mathcal{H}_1 \qquad F < \sim E \mid E \leq D \mid C \leq \sim(A \to B) \mid \mathcal{H}_1}{F < \sim(D \wedge E) \mid C \leq \sim(A \to B) \mid \mathcal{H}_1} R$$

so, by inductive hypothesis, we have derivations

$$d_1' \vdash F < \sim D \mid E < D \mid C \leq 0 \mid B < A \mid \mathcal{H}_1$$

$$d_2' \vdash F < \sim D \mid E < D \mid C \leq \sim B \mid A \leq B \mid \mathcal{H}_1$$

$$d_1'' \vdash F < \sim E \mid D \leq E \mid C \leq 0 \mid B < A \mid \mathcal{H}_1$$

$$d_2'' \vdash F < \sim E \mid D \leq E \mid C \leq \sim B \mid A \leq B \mid \mathcal{H}_1$$

and we can define derivations d_1 and d_2 as

$$\cfrac{\cfrac{F < \sim D \mid E < D \mid C \leq 0 \mid B < A \mid \mathcal{H}_1 \qquad F < \sim E \mid D \leq E \mid C \leq 0 \mid B < A \mid \mathcal{H}_1}{F < \sim(D \wedge E) \mid C \leq 0 \mid B < A \mid \mathcal{H}_1} R}{C \leq 0 \mid B < A \mid \mathcal{H}}$$

$$\cfrac{\cfrac{F < \sim D \mid E < D \mid C \leq \sim B \mid A \leq B \mid \mathcal{H}_1 \qquad F < \sim E \mid D \leq E \mid C \leq \sim B \mid A \leq B \mid \mathcal{H}_1}{F < \sim(D \wedge E) \mid C \leq \sim B \mid A \leq B \mid \mathcal{H}_1} R}{C \leq \sim B \mid A \leq B \mid \mathcal{H}}$$

2. R is (EW).

(a) R introduces the indicated occurrence of $C \leq \sim(A \to B)$. So we are in the case

$$\frac{\mathcal{H}}{C \leq \sim(A \to B) \mid \mathcal{H}} (EW)$$

and from the derivation of $\mathcal{H}$ we can define d_1 and d_2 as

$$\cfrac{\cfrac{\mathcal{H}}{B < A \mid \mathcal{H}} (EW)}{C \leq 0 \mid B < A \mid \mathcal{H}} (EW)$$

$$\cfrac{\cfrac{\mathcal{H}}{A \leq B \mid \mathcal{H}} (EW)}{C \leq \sim B \mid A \leq B \mid \mathcal{H}} (EW)$$

(b) R introduces the relation $D \lhd E$ of $\mathcal{H}$, where $\lhd \in \{\leq, <\}$. This means that

$$\frac{C \leq \sim(A \to B) \mid \mathcal{H}_1}{D \lhd E \mid C \leq \sim(A \to B) \mid \mathcal{H}_1} (EW)$$

By inductive hypothesis, we have derivations

$$d_1' \vdash C \leq 0 \mid B < A \mid \mathcal{H}_1$$

$$d_2' \vdash C \leq\sim B \mid A \leq B \mid \mathcal{H}_1$$

so we define d_1 and d_2 as

$$\dfrac{\dfrac{C \leq 0 \mid B < A \mid \mathcal{H}_1}{D \lhd E \mid C \leq 0 \mid B < A \mid \mathcal{H}_1}\,(EW)}{C \leq 0 \mid B < A \mid \mathcal{H}}$$

$$\dfrac{\dfrac{C \leq\sim B \mid B < A \mid \mathcal{H}_1}{D \lhd E \mid C \leq\sim B \mid B < A \mid \mathcal{H}_1}\,(EW)}{C \leq\sim B \mid B < A \mid \mathcal{H}}$$

3. R is $(\text{cut}_{\leq\backslash>})$. Suppose that we have

$$\dfrac{D \leq E \mid C \leq\sim(A \to B) \mid \mathcal{H} \qquad E < D \mid C \leq\sim(A \to B) \mid \mathcal{H}}{C \leq\sim(A \to B) \mid \mathcal{H}}\,(\text{cut}_{\leq\backslash>})$$

then, by inductive hypothesis, we have derivations

$$d_1' \vdash D \leq E \mid C \leq 0 \mid B < A \mid \mathcal{H}$$

$$d_2' \vdash D \leq E \mid C \leq\sim B \mid A \leq B \mid \mathcal{H}$$

$$d_1'' \vdash E < D \mid C \leq 0 \mid B < A \mid \mathcal{H}$$

$$d_2'' \vdash E < D \mid C \leq\sim B \mid A \leq B \mid \mathcal{H}$$

so we can define d_1 and d_2 as

$$\dfrac{D \leq E \mid C \leq 0 \mid B < A \mid \mathcal{H} \qquad E < D \mid C \leq 0 \mid B < A \mid \mathcal{H}}{C \leq 0 \mid B < A \mid \mathcal{H}}\,(\text{cut}_{\leq\backslash>})$$

$$\dfrac{D \leq E \mid C \leq\sim B \mid A \leq B \mid \mathcal{H} \qquad E < D \mid C \leq\sim B \mid A \leq B \mid \mathcal{H}}{C \leq\sim B \mid A \leq B \mid \mathcal{H}}\,(\text{cut}_{\leq\backslash>})$$

The other cases are similar. □

Lemma 5.3 (Reduction Lemma)**.** *Let $d \vdash \mathcal{H}$ be a derivation ending in a cut of maximal complexity with a cut formula $A \circ B$ or $\sim (A \circ B)$, where $\circ \in \{\wedge, \vee, \rightarrow\}$. Then one can find a derivation $d' \vdash \mathcal{H}$ where this cut is replaced by cuts involving as cut formulas only $A, B, \sim A, \sim B, 0$ or 1, and such that $\rho(d') \leq \rho(d)$. In particular, if the subderivations d_1 and d_2 of the premises of the cut satisfies $\rho(d_i) < \rho(d)$ for $i = 1, 2$, then $\rho(d') < \rho(d)$.*

Proof. We just do it for four cases involving negative formulas. Suppose we have

$$\frac{\overset{d_1}{C \leq \sim (A \rightarrow B) \mid \mathcal{H}} \qquad \overset{d_2}{\sim (A \rightarrow B) < C \mid \mathcal{H}}}{\mathcal{H}} \, (\text{cut}_{\leq \backslash >})$$

Applying the Inversion Lemma, we obtain derivations

$$d_1' \vdash C \leq 0 \mid B < A \mid \mathcal{H}$$
$$d_1'' \vdash C \leq \sim B \mid A \leq B \mid \mathcal{H}$$
$$d_2' \vdash 0 < C \mid B < A \mid \mathcal{H}$$
$$d_2'' \vdash \sim B < C \mid A \leq B \mid \mathcal{H}$$

which can be rearranged to form the following tree

$$\frac{\dfrac{C \leq \sim B \mid A \leq B \mid \mathcal{H} \quad \sim B < C \mid A \leq B \mid \mathcal{H}}{A \leq B \mid \mathcal{H}} \, (\text{cut}_{\leq \backslash >}) \quad \dfrac{C \leq 0 \mid B < A \mid \mathcal{H} \quad 0 < C \mid B < A \mid \mathcal{H}}{B < A \mid \mathcal{H}} \, (\text{cut}_{\leq \backslash >})}{\mathcal{H}} \, (\text{cut}_{\leq \backslash >})$$

Note that, if $\rho(d_i) < \rho(d)$ then, by the Inversion Lemma we have

$$\rho(d_i'), \rho(d_i'') \leq \rho(d_i) \text{ for } i = 1, 2,$$

so, since the cuts of d' highlightened in the tree have complexity strictly smaller than the complexity of the removed cut, $\rho(d') < \rho(d)$.

Now, suppose we have

$$\frac{\sim (A \wedge B) \leq C \mid \mathcal{H} \qquad C < \sim (A \rightarrow B) \mid \mathcal{H}}{\mathcal{H}} \, (\text{cut}_{\leq \backslash >})$$

Using again the Inversion Lemma, we get derivations

$$d_1' \vdash \sim A \leq C \mid B < A \mid \mathcal{H}$$
$$d_1'' \vdash \sim B \leq C \mid A \leq B \mid \mathcal{H}$$
$$d_2' \vdash C < \sim A \mid B < A \mid \mathcal{H}$$
$$d_2'' \vdash C < \sim B \mid A \leq B \mid \mathcal{H}$$

which give us the tree

$$\dfrac{\dfrac{\sim B \leq C \mid A \leq B \mid \mathcal{H} \quad C < \sim B \mid A \leq B \mid \mathcal{H}}{A \leq B \mid \mathcal{H}}(\text{cut}_{\leq\backslash>}) \quad \dfrac{\sim A \leq C \mid B < A \mid \mathcal{H} \quad C < \sim A \mid B < A \mid \mathcal{H}}{B < A \mid \mathcal{H}}(\text{cut}_{\leq\backslash>})}{\mathcal{H}}(\text{cut}_{\leq\backslash>})$$

Then, suppose we have

$$\dfrac{C \leq \sim(A \vee B) \mid \mathcal{H} \qquad \sim(A \to B) < C \mid \mathcal{H}}{\mathcal{H}}(\text{cut}_{\leq\backslash>})$$

Applying the Inversion Lemma, we obtain derivations

$$d_1' \vdash C \leq \sim B \mid B < A \mid \mathcal{H}$$

$$d_1'' \vdash C \leq \sim A \mid A \leq B \mid \mathcal{H}$$

$$d_2' \vdash \sim B < C \mid B < A \mid \mathcal{H}$$

$$d_2'' \vdash \sim A < C \mid A \leq B \mid \mathcal{H}$$

which yields the following tree

$$\dfrac{\dfrac{C \leq \sim A \mid A \leq B \mid \mathcal{H} \quad \sim A < C \mid A \leq B \mid \mathcal{H}}{A \leq B \mid \mathcal{H}}(\text{cut}_{\leq\backslash>}) \quad \dfrac{C \leq \sim B \mid B < A \mid \mathcal{H} \quad \sim B < C \mid B < A \mid \mathcal{H}}{B < A \mid \mathcal{H}}(\text{cut}_{\leq\backslash>})}{\mathcal{H}}(\text{cut}_{\leq\backslash>})$$

Finally, suppose we have

$$\dfrac{\sim\sim A \leq C \mid \mathcal{H} \qquad C < \sim\sim A \mid \mathcal{H}}{\mathcal{H}}(\text{cut}_{\leq\backslash>})$$

The Inversion Lemma give us derivations

$$d_1 \vdash A \leq C \mid \mathcal{H}$$

$$d_2 \vdash C < A \mid \mathcal{H}$$

which can be rearranged to form the following tree

$$\dfrac{A \leq C \mid \mathcal{H} \qquad C < A \mid \mathcal{H}}{\mathcal{H}}(\text{cut}_{\leq\backslash>})$$

The other cases are similar. □

Lemma 5.4. *Let d be a derivation of $\mathcal{H}$ from atomic axioms and whose last inference rule is an atomic cut. Then one can find a derivation d' of $\mathcal{H}$ with $\rho(d') \leq \rho(d)$, where this cut is replaced by cuts applied to atomic sequents.*

Proof. This is the situation we are considering:

$$\dfrac{\begin{matrix} d_1 \\ A \leq B \mid \mathcal{H} \end{matrix} \qquad \begin{matrix} d_2 \\ B < A \mid \mathcal{H} \end{matrix}}{\mathcal{H}} \ (\text{cut}_{\leq\backslash>})$$

where the cut is atomic. We will prove the lemma by induction on the number of connectives in $\mathcal{H}$. Pick a non atomic relation in $\mathcal{H}$ (e.g. suppose it is $\sim (C \to D) < E$, and call with $\mathcal{H}_1$ the sequent obtained from $\mathcal{H}$ removing this relation) and apply the Inversion Lemma to derivations d_1 and d_2: then we have derivations

$$d_1' \vdash 0 < E \mid D < C \mid A \leq B \mid \mathcal{H}_1,$$

$$d_1'' \vdash \sim D < E \mid C \leq D \mid A \leq B \mid \mathcal{H}_1,$$

$$d_2' \vdash 0 < E \mid D < C \mid B < A \mid \mathcal{H}_1,$$

$$d_2'' \vdash \sim D < E \mid C \leq D \mid B < A \mid \mathcal{H}_1,$$

such that $\rho(d_i'), \rho(d_i'') \leq \rho(d)$, for $i = 1, 2$. Now, if we consider the trees

$$\dfrac{\begin{matrix} d_1' \\ 0 < E \mid D < C \mid A \leq B \mid \mathcal{H}_1 \end{matrix} \qquad \begin{matrix} d_2' \\ 0 < E \mid D < C \mid B < A \mid \mathcal{H}_1 \end{matrix}}{0 < E \mid D < C \mid \mathcal{H}_1} \ (\text{cut}_{\leq\backslash>})$$

$$\dfrac{\begin{matrix} d_1'' \\ \sim D < E \mid C \leq D \mid A \leq B \mid \mathcal{H}_1 \end{matrix} \qquad \begin{matrix} d_2'' \\ \sim D < E \mid C \leq D \mid B < A \mid \mathcal{H}_1 \end{matrix}}{\sim D < E \mid C \leq D \mid \mathcal{H}_1} \ (\text{cut}_{\leq\backslash>})$$

then their ending sequents contain a number of connectives strictly smaller than $\mathcal{H}$, so, by inductive hypothesis we can find derivations

$$\delta_1 \vdash 0 < E \mid D < C \mid \mathcal{H}_1,$$

$$\delta_2 \vdash \sim D < E \mid C \leq D \mid \mathcal{H}_1,$$

for which the thesis hold: in particular, $\rho(\delta_i) \leq \rho(d)$, for $i = 1, 2$. So, d' can be obtained from δ_1 and δ_2 reintroducing the connective:

$$\dfrac{\begin{matrix} \delta_1 \\ 0 < E \mid D < C \mid \mathcal{H}_1 \end{matrix} \qquad \begin{matrix} \delta_2 \\ \sim D < E \mid C \leq D \mid \mathcal{H}_1 \end{matrix}}{\mathcal{H}} \ (\sim\to, <, l)$$

The other connectives are dealt in the same way. □

Lemma 5.5. *A cut between two axioms always yields a sequent containing an axiom.*

Proof. The following table indicates what kind of type of axiom we get (c stands for chain, 0 for 0-chain, 1 for 1-chain and 0-1 for 0-1-chain) if we have a cut between two axioms. The "-" in the bottom-right corner means that no cut can occur between two 0-1-chains.

$S_1 \backslash S_2$	c	0	1	0-1
c	c	0	1	0, 1 or 0-1
0	0	0	0-1	0-1
1	1	0-1	1	0-1
0-1	0, 1 or 0-1	0-1	0-1	-

The proof of this result is the same as the one presented for the analogous theorem in [4], so what we do here is the inspection of the cut between a chain and a 0-1-chain, left to the reader there: we separate the cases where the cut formula of the chain is a $A_{i+1} \leq A_i$ and where it is $A_2 \leq A_1$:

$$A_1 \lhd_n A_n \mid \ldots \mid \underline{A_{i+1} \leq A_i} \mid \ldots \mid \underline{A_2 \leq A_1}.$$

Let

$$0 < B_1 \mid \ldots \mid B_j < B_{j+1} \mid \ldots \mid B_n < 1$$

be the 0-1-chain. If the cut formula of the chain is $A_{i+1} \leq A_i$, then the cut formula of the 0-1-chain could be

1. $0 < B_1$, so we have $A_{i+1} = B_1$ and $A_i = 0$. The sequent we obtain is

   $$A_1 \lhd_n A_n \mid \ldots \mid A_{i+2} \lhd_{i+1} B_1 \mid 0 \lhd_{i-1} A_{i-1} \mid \ldots \mid A_2 \leq A_1 \mid B_1 < B_2 \mid \ldots \mid B_n < 1$$

   and notice that $0 \lhd_{i-1} A_{i-1} \mid \ldots \mid A_2 \leq A_1$ is a 0-chain;

2. $B_j < B_{j+1}$, and in this case $A_{i+1} = B_{j+1}$ and $A_i = B_i$. Then, as one can easily check, the sequent obtained from the cut has the subsequent

   $$0 < B_1 \mid \ldots \mid B_{i-1} < B_i \mid B_i \lhd_{j-1} A_{j-1} \mid \ldots \mid A_2 \leq A_1,$$

   which is a 0-chain;

3. $B_n < 1$, then $A_{i+1} = 1$ and $A_i = B_n$. Again, the sequent obtained contains the 0-chain

   $$0 < B_1 \mid \ldots \mid B_{n-1} < B_n \mid B_n \lhd_{j-1} A_{j-1} \mid \ldots \mid A_2 \leq A_1.$$

Instead, if the cut formula of the chain is $A_2 \leq A_1$ and the cut formula of the 0-1-chain is

1. $0 < B_1$, then $A_2 = B_1$ and $A_1 = 0$ and we obtain the sequent

$$0 \lhd_n A_n \mid \ldots \mid A_3 \lhd_2 A_2 \mid B_1 < B_2 \mid \ldots \mid B_n < 1.$$

   In the case that $\lhd_j =<$ for all $j = 2, \ldots n$, then this is a 0-1-chain; instead, if there is a j such that $\lhd_j =\leq$, the above sequent contains both a 0-chain and a 1-chain;

2. $B_j < B_{j+1}$, so $A_2 = B_{j+1}$ and $A_1 = B_j$, and reordering the sequent we have

$$0 < B_1 | \ldots | B_{i-1} < B_i | B_i \lhd_n A_n | \ldots | A_3 \lhd_2 B_{i+1} | B_{i+1} < B_{i+2} | \ldots | B_n < 1,$$

   which is eventually a 0-1-chain if $\lhd_j =<$ for all $j = 2, \ldots, n$, or it contains a 0-chain and a 1-chain;

3. $B_n < 1$, then $A_2 = 1$ and $A_1 = B_n$, and we get the sequent (after a proper reordering)

$$0 < B_1 \mid \ldots \mid B_{n-1} < B_n \mid B_n \lhd_n A_n \mid \ldots \mid A_3 \lhd_2 1,$$

   that is a 0-1-chain if $\lhd_j =<$ for all $j = 2, \ldots, n$, or it contains a 0-chain and a 1-chain otherwise.

□

Theorem 5.6 (Cut-elimination for $\mathbf{RG}_\sim$)**.** *There is a procedure that, given a derivation of a sequent* S, *produces a cut-free derivation of* S.

Proof. Let $d \vdash$ S. As anticipated, this is the algorithm which produces a cut-free derivation of S from d:

1. use Lemma (5.1) to replace every compound axiom of d with a derivation from atomic axioms, and call the obtained derivation with d^a;

2. apply Lemma (5.3) to every subderivation d' of d^a such that $\rho(d'') < \rho(d')$ for every subderivation d'' of d', until only atomic cuts are left. Repeat this operation until only atomic cuts are left.

3. push these cuts up to atomic sequents with Lemma (5.4) to obtain a derivation $d^\sharp$;

4. now, notice that in every subderivation of $d^\sharp$ ending in a cut only the rules (EW) and $(\text{cut}_{\leq\backslash>})$ can appear: by Lemma (5.5), we can replace such subderivations of maximal length by atomic axioms (eventually followed by external weakenings). Call the derivation obtained this way $d^\star$.

By construction, $d^\star$ is a cut-free derivation of S. □

Chapter 6

Constructions preserving Semiprojectivity

In Fuzzy Logic several constructions are available which allow us to produce new fuzzy logics from given ones. It turns out that a lot of these constructions are well matched with our definition of semiprojective logics, that is they preserve the semiprojectivity of the logics involved. In this chapter we shall investigate two of them in full details, namely *rotation* and *ordinal sum*, but we will provide further examples at the end of the chapter.

6.1 Rotation

The *rotation* construction was introduced in [26] as a generalization of the concepts of Girard monoid and MV-algebra. Basically, as the name suggests, it takes an ordered algebra **A** with top element, endowed with a residuated lattice reduct, and returns a residuated bounded lattice $\mathbf{A}^{\circlearrowleft}$ whose universe is the universe of **A** plus its "mirror image" (with respect to the order in **A**). The "mirror" can be placed both *outside* the algebra, and in this case we talk about *disconnected rotation*, or exactly on the bottom element of **A** (if it has one), and we talk about *connected rotation*. From now on we will focus only on disconnected rotation.

Definition 6.1. Let $\mathcal{A} = \langle A, \leq, \cdot, \rightarrow, 1, f_1, \ldots, f_n \rangle$ be a partially ordered algebra such that $\langle A, \leq, \cdot, \rightarrow, 1 \rangle$ is a commutative residuated partially ordered semigroup with top element 1, and $f_1, \ldots, f_n$ are unary operations. We define the *disconnected rotation of* $\mathcal{A}$ to be the algebra

$$\mathcal{A}^{\circlearrowleft} = \langle A^{\circlearrowleft}, \leq^{\circlearrowleft}, \odot, \supset, 1', 1, F_1, \ldots, F_n, \overline{F}_1, \ldots, \overline{F}_n, N \rangle,$$

where we define $A^+ = A$, $A^- = \{a' \mid a \in A\}$ to be a disjoint set from A endowed with the partial ordering

$$a' \leq' b' \text{ iff } b \leq a$$

for all $a', b' \in A^-$, and we take $A^{\circlearrowleft} = A^+ \cup A^-$ and

$$x \leq^{\circlearrowleft} y \text{ iff } \begin{cases} x, y \in A^+ \text{ and } x \leq y \\ x, y \in A^- \text{ and } x \leq' y \\ x \in A^- \text{ and } y \in A^+ \end{cases}$$

N is a unary, involutive, order reversing operation defined as

$$N(x) = \begin{cases} x' & \text{if } x \in A^+ \\ y & \text{if } x \in A^- \text{ and } x = y' \end{cases}$$

and the other operations are defined as

$$x \odot y = \begin{cases} x \cdot y & \text{if } x, y \in A^+ \\ N(x \to N(y)) & \text{if } x \in A^+, y \in A^- \\ N(y \to N(x)) & \text{if } x \in A^-, y \in A^+ \\ 1' & \text{if } x, y \in A^- \end{cases}$$

$$x \supset y = \begin{cases} x \to y & \text{if } x, y \in A^+ \\ N(x \cdot N(y)) & \text{if } x \in A^+, y \in A^- \\ 1 & \text{if } x \in A^-, y \in A^+ \\ N(y) \to N(x) & \text{if } x, y \in A^- \end{cases}$$

$$F_i(x) = \begin{cases} f_i(x) & \text{if } x \in A^+ \\ f_i(N(x)) & \text{if } x \in A^- \end{cases}$$

$$\overline{F}_i(x) = \begin{cases} N(f_i(x)) & \text{if } x \in A^+ \\ N(f_i(N(x))) & \text{if } x \in A^- \end{cases}$$

We call *positive* those elements of $\mathcal{A}^{\circlearrowleft}$ which belong to A^+, and *negative* the ones which are in A^-.

Notice that, if the algebra has a lattice order with meet $\wedge$ and join $\vee$, then also the order of its rotation is a lattice one, with meet and join given by

$$x \sqcap y = \begin{cases} x \wedge y & \text{if } x, y \in A^+ \\ y & \text{if } x \in A^+, y \in A^- \\ x & \text{if } x \in A^-, y \in A^+ \\ N(x \vee y) & \text{if } x, y \in A^- \end{cases}$$

$$x \sqcup y = \begin{cases} x \vee y & \text{if } x, y \in A^+ \\ y & \text{if } x \in A^+, y \in A^- \\ x & \text{if } x \in A^-, y \in A^+ \\ N(x \wedge y) & \text{if } x, y \in A^- \end{cases}$$

Definition 6.2. Let $\mathcal{L}$ be a substructural logic with signature $\langle \wedge, \vee, \cdot, \rightarrow, 1, f_1, \ldots, f_n \rangle$, where $f_1, \ldots, f_n$ are unary connectives. The *rotation logic of* $\mathcal{L}$ is the logic $\mathcal{L}^{\circlearrowleft}$ with signature

$$\langle \sqcap, \sqcup, \odot, \supset, 1', 1, F_1, \ldots, F_n, \overline{F}_1, \ldots, \overline{F}_n, N \rangle$$

generated by the set

$$\{\mathcal{A}^{\circlearrowleft} \mid \mathcal{A} \text{ is a model of } \mathcal{L}\}.$$

Theorem 6.1. *If $\mathcal{L}$ is a semiprojective logic, then $\mathcal{L}^{\circlearrowleft}$ is semiprojective too.*

Proof. Let $\mathbf{T}$ be the semantic theory of $\mathcal{L}$. Our strategy consists in the definition of the semantic theory of $\mathcal{L}^{\circlearrowleft}$ as the theory of the models obtained by rotating the models of $\mathbf{T}$. For every predicate symbol R of $\mathbf{T}$ we include a predicate symbol $R^{\circlearrowleft}$ with the same arity in $\mathbf{T}^{\circlearrowleft}$ plus two unary predicate symbols U^+ and U^-, with the function of identifying positive and negative elements. Of course, the unary function symbols of $\mathbf{T}^{\circlearrowleft}$ shall be $F_1, \ldots, F_n, \overline{F}_1, \ldots, \overline{F}_n, N$.

Notice how, in the definition of rotation, the predicate symbol $\leq^{\circlearrowleft}$ can be defined from $\leq$ as

$$x \leq^{\circlearrowleft} y = \begin{cases} x \leq y & \text{if } U^+(x) \wedge U^+(y) \\ N(y) \leq N(x) & \text{if } U^-(x) \wedge U^-(y) \\ \top & \text{if } U^-(x) \wedge U^+(y) \\ \bot & \text{if } U^+(x) \wedge U^-(y) \end{cases}$$

that is we have a sort of *semiprojective representation of* $\leq^{\circlearrowleft}$. More formally, we shall say that a predicate symbol $R^{\circlearrowleft}$ of $\mathbf{T}^{\circlearrowleft}$ is *semiprojective in* $\mathbf{T}$ if

$$\mathbf{T}^{\circlearrowleft} \models \forall x_1 \ldots \forall x_n \left(R^{\circlearrowleft}(x_1, \ldots, x_n) \leftrightarrow \bigwedge_{1 \leq i \leq n} (S_i \to P_i) \right)$$

where $P_1, \ldots, P_n$ are $\bot$, $\top$ or atomic formulas with predicate symbols those of $\mathbf{T}$ and terms in the set $\{x, N(x) \mid x \in \{x_1, \ldots, x_n\}\}$, and $S_1, \ldots, S_n$ are conjuncts of atomic formulas of the form $T(x)$, where $T \in \{U^+, U^-\}$ and $x \in \{x_1, \ldots, x_n\}$, which are complete and mutually disjoint (in the sense of (T) of Definition (3.1)). So, a semiprojective predicate can be written as

$$R^{\circlearrowleft}(x_1, \ldots, x_n) = \begin{cases} P_1 & \text{if } S_1 \\ \vdots & \\ P_n & \text{if } S_n \end{cases}$$

We will see that, in order to reduce the decidability of the $\Pi_1^{\mathbf{T}^{\circlearrowleft}}$-formulas to the decidability of the $\Pi_1^{\mathbf{T}}$-formulas, it is enough to suppose that every predicate symbol of $\mathbf{T}^{\circlearrowleft}$ is semiprojective in $\mathbf{T}$.

Now, we have to check that the composition of every couple of unary connectives is equal to some unary connective of the signature. So suppose that $f_i \circ f_j = f_k$ in the semantic theory $\mathbf{T}$; the following table gives the result of every possible composition:

$\circ$	N	F_j	$\overline{F}_j$
N	id	$\overline{F}_j$	F_j
F_i	F_i	F_k	F_k
$\overline{F}_i$	$\overline{F}_i$	$\overline{F}_k$	$\overline{F}_k$

With $\mathbf{T}^{\circlearrowleft}$ is easy to write in semiprojective form every connective. Let's do the residuum:

$$x \supset y = \begin{cases} x \to y & \text{if } U^+(x) \wedge U^+(y) \\ N(x \cdot N(y)) & \text{if } U^+(x) \wedge U^-(y) \\ 1 & \text{if } U^-(x) \wedge U^+(y) \\ N(y) \to N(x) & \text{if } U^-(x) \wedge U^-(y) \end{cases}$$

Notice that the simple formulas are complete and mutually disjoint, since no element satisfies $U^+(x) \wedge U^-(x)$.

Let us discuss the decidability of the Π_1-formulas of $\mathbf{T}^{\circlearrowleft}$. For the sake of simplicity, we assume that the only predicate symbol different from U^+ and U^- is $\leq^{\circlearrowleft}$, with the semiprojective representation provided above; needless to say, all the results valid for it are valid for every other predicate symbol semiprojective in $\mathbf{T}$ (though the proof of it is, of course, much more intricate). The strategy is to construct, for every Π_1-formula ψ of $\mathbf{T}^{\circlearrowleft}$, a Π_1-formula $\phi(\psi)$ of $\mathbf{T}$ such that

$$\mathbf{T}^{\circlearrowleft} \models \psi \Leftrightarrow \mathbf{T} \models \phi(\psi).$$

In this way, a decision procedure for every Π_1-formula ψ of $\mathbf{T}^{\circlearrowleft}$ would be to check the validity of $\phi(\psi)$ in $\mathbf{T}$, which is a decidable task by hypothesis.

We do so by taking a conjunctive normal form formula $\bigwedge_{1\leq i\leq n} \bigvee_{1\leq j\leq m_i} L_{ij}$ equivalent to ψ: clearly ψ is valid iff every clause $\mathcal{C}_i = L_{i1} \vee \ldots \vee L_{im_i}$ is valid, and a clause $\mathcal{C}$ is valid iff for every model $\mathcal{M}$ of $\mathbf{T}^{\circlearrowleft}$ and every assignment μ of $\mathcal{M}$ there is a literal L of $\mathcal{C}$ such that $\mathcal{M}, \mu \models L$. The set of all assignments is infinite, but for every clause there is only a *finite* number of *sign assignments*, i.e. assignments of the variables of the clause to either 1 or -1 (representing the fact that the variable is assigned either a positive or a negative element). We shall see that sign assignments are all it takes to produce a Π_1-formula of $\mathbf{T}$ equivalent to ψ.

More formally, let L be a literal of $\mathbf{T}^{\circlearrowleft}$. A *sign assignment of* L is a function

$$\sigma : var(L) \longrightarrow \{1, -1\}.$$

If $\mathcal{C}$ is a clause, a sign assignment for $\mathcal{C}$ is a function as above with domain $var(\mathcal{C})$. With $S(\mathcal{C})$ we shall indicate the set of the sign assignments of $\mathcal{C}$, which is nothing but $\{1, -1\}^{var(\mathcal{C})}$.

We extended our notion of sign assignment to terms of $\mathbf{T}^{\circlearrowleft}$ in the following way:

1. $\sigma(1) = 1$;

2. $\sigma(1') = -1$;

3. $$\sigma(N(x)) = \begin{cases} -1 & \text{if } \sigma(x) = 1 \\ 1 & \text{if } \sigma(x) = -1 \end{cases}$$

4. for every $i = 1, \ldots, n$,
$$\sigma(F_i(x)) = 1$$

$$\sigma(\overline{F}_i(x)) = -1$$

Sign assignments are designed to *simplify literals*: for example, the literal $U^+(N(x))$ is true under every assignment of every model of $\mathbf{T}^{\circlearrowleft}$ which sends the variable x to a negative element; so under every sign assignment σ such that $\sigma(x) = -1$, we could write

$$\sigma^\star(U^+(N(x))) = \top,$$

because we already know that it is true.

Instead, if we take the formula $x \leq^{\circlearrowleft} N(y)$ and the sign assignment σ such that $\sigma(x) = 1$ and $\sigma(y) = -1$, we don't know if the formula is true or not, but we now that, by the semiprojective representation of $\leq^{\circlearrowleft}$, it is reducible to $\leq$, so we will write

$$\sigma^\star(x \leq^{\circlearrowleft} N(y)) = x \leq N(y);$$

instead if we have the sign assignment σ such that $\sigma(x) = -1$ and $\sigma(y) = 1$, the formula is reducible to $\leq$ but with terms reversed in their position and an N in front of them, so we write

$$\sigma^\star(x \leq^{\circlearrowleft} N(y)) = N(N(y)) \leq N(x) = y \leq N(x);$$

notice that, if instead $\sigma(x) = \sigma(y) = 1$ then clearly $\sigma^\star(x \leq^{\circlearrowleft} N(y)) = \bot$.

So, suppose we want to check if the clause

$$\mathcal{C} = U^+(N(x)) \vee (x \leq^{\circlearrowleft} N(y))$$

is valid in $\mathbf{T}^{\circlearrowleft}$. Our strategy is to write all the possible sign assignments and simplify the clause in the following way:

x	y	$\sigma^\star(U^+(N(x)))$	$\sigma^\star(x \leq^{\circlearrowleft} N(y))$	$\sigma^\star(\mathcal{C})$
1	1	$\top$	$\bot$	$\top$
1	-1	$\top$	$x \leq N(y)$	$\top$
-1	1	$\bot$	$y \leq N(x)$	$y \leq N(x)$
-1	-1	$\bot$	$\top$	$\top$

Since the clause is always true under sign assignments different from the first, in order to check if the clause is valid in $\mathbf{T}^{\circlearrowleft}$ we have only to check if

$$\mathbf{T} \models \forall y \forall z (y \leq z)$$

(we replaced $N(x)$ with another variable z because, under σ, we know that $N(x)$ is in the positive part of every model of $\mathbf{T}^{\circlearrowleft}$).

As another example, take the clause

$$\mathcal{C} = (x \leq^{\circlearrowleft} N(y)) \vee (N(y) \leq^{\circlearrowleft} x);$$

using the same procedure as above we get the table

x	y	$\sigma^\star(x \leq^{\circlearrowleft} N(y))$	$\sigma^\star(N(y) \leq^{\circlearrowleft} x)$	$\sigma^\star(\mathcal{C})$
1	1	$\bot$	$\top$	$\top$
1	-1	$x \leq N(y)$	$N(y) \leq x$	$(x \leq N(y)) \vee (N(y) \leq x)$
-1	1	$y \leq N(x)$	$N(x) \leq y$	$(y \leq N(x)) \vee (N(x) \leq y)$
-1	-1	$\top$	$\bot$	$\top$

Hence, our clause is valid in $\mathbf{T}^{\circlearrowleft}$ iff

$$\mathbf{T} \models \forall x \forall z((x \leq z) \vee (z \leq x)) \wedge \forall y \forall z((y \leq z) \vee (z \leq y)),$$

which is obviously equivalent to

$$\mathbf{T} \models \forall x \forall y((x \leq y) \vee (y \leq x)).$$

We now put everything we have said on a solid ground: if L is a literal of $\mathbf{T}^{\circlearrowleft}$ and σ a sign assignment of L, we define $\sigma^\star(L)$ to be:

1. $\sigma^\star(L) = t_1 \leq t_2$, if $L = t_1 \leq t_2$ and $\sigma(t_1) = \sigma(t_2) = 1$;

2. $\sigma^\star(L) = \neg(t_1 \leq t_2)$, if $L = \neg(t_1 \leq t_2)$ and $\sigma(t_1) = \sigma(t_2) = 1$;

3. $\sigma^\star(L) = N(t_2) \leq N(t_1)$, if $L = t_1 \leq t_2$ and $\sigma(t_1) = \sigma(t_2) = -1$;

4. $\sigma^\star(L) = \neg(N(t_2) \leq N(t_1))$, if $L = \neg(t_1 \leq t_2)$ and $\sigma(t_1) = \sigma(t_2) = -1$;

5. $\sigma^\star(L) = \top$ iff

   (a) $L = U^+(t)$ or $L = \neg U^-(t)$, and $\sigma(t) = 1$;

   (b) $L = \neg U^+(t)$ or $L = U^-(t)$, and $\sigma(t) = -1$;

   (c) $L = t_1 \leq t_2$, and $\sigma(t_1) = -1$ and $\sigma(t_2) = 1$;

   (d) $L = \neg(t_1 \leq t_2)$, and $\sigma(t_1) = 1$ and $\sigma(t_2) = -1$;

6. $\sigma^\star(L) = \bot$ iff

(a) $L = U^+(t)$ or $L = \neg U^-(t)$, and $\sigma(t) = -1$;

(b) $L = \neg U^+(t)$ or $L = U^-(t)$, and $\sigma(t) = 1$;

(c) $L = t_1 \leq t_2$, and $\sigma(t_1) = 1$ and $\sigma(t_2) = -1$;

(d) $L = \neg(t_1 \leq t_2)$, and $\sigma(t_1) = -1$ and $\sigma(t_2) = 1$;

If $\mathcal{C} = L_1 \vee \ldots \vee L_k$ is a clause of $\mathbf{T}^{\circlearrowleft}$ and σ a sign assignment of $\mathcal{C}$, we define

$$\sigma^\star(\mathcal{C}) = \sigma^\star(L_1) \vee \ldots \vee \sigma^\star(L_k).$$

The final step is made to eliminate the function symbol N from the literals: let $\mathcal{C} = L_1 \vee \ldots \vee L_k$ be a clause of $\mathbf{T}^{\circlearrowleft}$ and σ a sign assignment of $\mathcal{C}$; we define $\overline{\sigma}^\star(\mathcal{C})$ as the clause obtained from $\sigma^\star(\mathcal{C})$ by replacing every occurrence of a term like $N(x)$ with a brand-new variable (clearly, occurrences of $N(x)$ in the same atomic formula are replaced by the same variable), and every occurrence of $\overline{F}_i$ with F_i.

Notice that $\overline{\sigma}^\star(\mathcal{C})$ is now a clause of $\mathbf{T}$ (identifying F_i with f_i).

Theorem 6.2. *For every clause $\mathcal{C}$ of $\mathbf{T}^{\circlearrowleft}$,*

$$\mathbf{T}^{\circlearrowleft} \models \mathcal{C} \textit{ iff } \mathbf{T} \models \forall \vec{x} \bigwedge_{\sigma \in S(\mathcal{C})} \overline{\sigma}^\star(\mathcal{C}).$$

Proof. Any clause $\mathcal{C}$ of $\mathbf{T}^{\circlearrowleft}$ is valid iff the conjunction of the images of $\mathcal{C}$ under every sign assignment is valid in $\mathbf{T}^{\circlearrowleft}$. But, since every term like $N(x)$ is semantically equivalent to a new variable (because the remaining $N(x)$ in each $\sigma^\star(\mathcal{C})$ stand for arbitrary positive elements), then the validity of $\mathcal{C}$ in $\mathbf{T}^{\circlearrowleft}$ is equivalent to the validity of $\forall \vec{x} \bigwedge_{\sigma \in S(\mathcal{C})} \overline{\sigma}^\star(\mathcal{C})$ in $\mathbf{T}$. □

The theorem above ensures the correctness of the following procedure which decides the set $\Pi_1^{\mathbf{T}^{\circlearrowleft}}$ (the decision procedure for the Π_1-formulas of $\mathbf{T}$ is used at step 4).

INPUT: A Π_1-formula ψ of $\mathtt{T}^{\circlearrowleft}$;

OUTPUT: YES, if it is valid, NO, if it is not;

1. write ψ in CNF:

$$\psi \equiv \forall \vec{x} \bigwedge_{i=1}^{m} \bigvee_{j=1}^{n_i} L_{ij};$$

```
2. for every clause C = L_{i1} ∨ ... ∨ L_{in_i}
    3. for every sign assignment σ of C
        4. if σ(C) is not valid in T, return NO and end the
           procedure;
        5. else, go on;
6. return YES.
```

□

6.2 Ordinal Sum

The *ordinal sum construction* was first introduced by Büchi and Owens in an unpublished manuscript and later recalled in [17], and consists essentially in "piling up" several residuated partially ordered semigroups in a way that preserves their structure. Though the ordinal sum can be applied to a generic family of algebras, we shall focus only on finite ones.

Definition 6.3. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be algebras, each one with signature

$$\langle \leq_i, \cdot_i, \rightarrow_i, 0_i, 1, f_1^i, \ldots, f_{n_i}^i \rangle,$$

such that, for every $i = 1, 2$, $\langle A_i, \leq_i, \cdot_i, \rightarrow_i, 0_i, 1 \rangle$ is a commutative, bounded, residuated partially ordered semigroup, $f_1^i, \ldots, f_{n_i}^i$ are unary connectives, and suppose that $A_1 \cap A_2 = \{1\}$.

We define the *ordinal sum of* $\mathcal{A}_1$ and $\mathcal{A}_2$, indicated as $\mathcal{A}_1 \oplus \mathcal{A}_2$, to be the algebra

$$\langle A_1 \cup A_2, \leq^{\oplus}, \odot, \supset, 0_1, 1, \{F_i^1\}_{i=1}^{n_1}, \{F_i^2\}_{i=1}^{n_2} \rangle,$$

where we define, for all $a, b \in A_1 \cup A_2$

$$a \leq^{\oplus} b \text{ iff } \begin{cases} a, b \in A_1 \text{ and } a \leq_1 b \\ a, b \in A_2 \text{ and } a \leq_2 b \\ a \in A_1 \backslash \{1\} \text{ and } b \in A_2 \end{cases}$$

and the other operations are defined as follows

$$x \odot y = \begin{cases} x \cdot_i y & \text{if } x, y \in A_i \\ x & \text{if } x \in A_1 \backslash \{1\}, y \in A_2 \\ y & \text{if } y \in A_1 \backslash \{1\}, x \in A_2 \end{cases}$$

$$x \supset y = \begin{cases} 1 & \text{if } x \in A_1\backslash\{1\}, y \in A_2 \\ x \rightarrow_i y & \text{if } x, y \in A_i \\ y & \text{if } y \in A_1, x \in A_2 \end{cases}$$

$$F_j^i(x) = \begin{cases} f_j^i(x) & \text{if } x \in A_i \\ x & \text{otherwise} \end{cases}$$

As in the case of rotation, if the algebras involved in the ordinal sum have a lattice order with meets $\wedge_1$ and $\wedge_2$ and joins $\vee_1$ and $\vee_2$ respectively, then also the order of their sum is a lattice one, with meet and join given by

$$x \wedge y = \begin{cases} x \wedge_1 y & \text{if } x, y \in A_1 \\ x \wedge_2 y & \text{if } x, y \in A_2 \\ x & \text{if } x \in A_1\backslash\{1\}, y \in A_2 \\ y & \text{if } y \in A_1\backslash\{1\}, x \in A_2 \end{cases}$$

$$x \vee y = \begin{cases} x \vee_1 y & \text{if } x, y \in A_1 \\ x \vee_2 y & \text{if } x, y \in A_2 \\ x & \text{if } y \in A_1\backslash\{1\}, x \in A_2 \\ y & \text{if } x \in A_1\backslash\{1\}, y \in A_2 \end{cases}$$

Definition 6.4. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two substructural logics, each with signature $\langle \wedge_i, \vee_i, \cdot, \rightarrow, 0_i, 1, f_1^i, \ldots, f_{n_i}^i \rangle$, where $f_1^i, \ldots, f_{n_i}^i$ are unary connectives. The *ordinal sum of* $\mathcal{L}_1$ and $\mathcal{L}_2$, indicated as $\mathcal{L}_1 \oplus \mathcal{L}_2$, is the logic with signature

$$\langle \wedge, \vee, \odot, \supset, 0_1, 1, \{F_i^1\}_{i=1}^{n_1}, \{F_i^2\}_{i=1}^{n_2} \rangle$$

generated by the set

$$\{\mathcal{A}_1 \oplus \mathcal{A}_2 \mid \mathcal{A}_i \text{ model of } \mathcal{L}_i\}.$$

6.2.1 An extension of Semiprojectivity

Before we face the details of the next theorem, we want to point out that *by definition* the ordinal sum of two semiprojective logics is not *strictly* semiprojective, that is if we take the unary connectives F^1 and F^2 of the semantic theories of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, then their composition has the form

$$(F^1 \circ F^2)(x) = \begin{cases} f^1(x) & \text{if } x \in A_1\backslash\{1\} \\ f^2(x) & \text{if } x \in A_2 \text{ and } f^2(x) \in A_2\backslash\{1\} \\ f^1(1) & \text{if } x \in A_2 \text{ and } f^2(x) = 1 \end{cases}$$

so it does not shrink back to any of the unary connectives available, as was requested in Definition (3.1).

This problem can be overcome by noticing that, after all, it is sufficient for our purposes to require that the composition of two unary connectives of the logic is semiprojective, that is has the form

$$(F^1 \circ F^2)(x) = \begin{cases} t_1 & \text{if } A_1 \\ \vdots & \vdots \\ t_n & \text{if } A_n \end{cases}$$

where each t_i is either a truth constant, the variable x, or a term of the form $f_q(x)$ or $f_q(c)$, where f_q is a unary function symbol of the semantic theory and c is a truth constant. Moreover, the conditions A_i are mutually disjunctive simple formulas of the semantic theory whose only free variable is x.

The logical rules corresponding to the composition $F^1 \circ F^2$ are thus obtained in the same way of the other logical rules, gaining in terms of symmetry of the presentation of the calculi.

6.2.2 The Ordinal Sum Theorem

Since our strategy to prove the following theorem is basically the same as the one followed in proving the analogous theorem for rotation, we will require that the unary connectives of the logic satisfy

$$f(x) \in \{0,1\} \text{ iff } x \in \{0,1\}.$$

We do this to simplify the effort of extending "sign" assignments (this time called *type assignments*) to terms: we shall call *conservative connectives* those which satisfies

$$f(0) = 0 \text{ and } f(1) = 1,$$

while *reversing connectives* shall be those which satisfies

$$f(0) = 1 \text{ and } f(0) = 1.$$

Notice that we are actually *excluding* important semiprojective logics from the next result, above all WNM logics (whose negation does not satisfy our assumption) and logics with the Δ operator (though it is possible to extend the next result to them).

Theorem 6.3. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are semiprojective logics as above, then $\mathcal{L}_1 \oplus \mathcal{L}_2$ is semiprojective.*

Proof. Let $\mathbf{T}_1$ and $\mathbf{T}_2$ be the semantic theories of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, and suppose that the sets of predicate symbols of $\mathbf{T}_1$ and of $\mathbf{T}_2$ are disjoint. The semantic theory of $\mathcal{L}_1 \oplus \mathcal{L}_2$, which we indicate henceforth with $\mathbf{T}^\oplus$, shall be the theory of the ordinal sums of the models of $\mathbf{T}_1$ and $\mathbf{T}_2$. For every function symbol f^i_j of $\mathbf{T}_i$, we introduce in $\mathbf{T}^\oplus$ a function symbol F^i_j. The predicate symbols of $\mathbf{T}^\oplus$ will consist of those of $\mathbf{T}_1$ and $\mathbf{T}_2$, plus the predicate symbols necessary to identify the component of the model to which every element belongs, indicated as C_1 and C_2, and some particular elements: the zero of every component (Z_1 and Z_2) and the top element (U). We shall require that these predicate symbols are mutually disjoint and complete (i.e. every element must satisfy one and only one of such predicate symbols), and for each F^i_j we either have that

$$\mathbf{T}^\oplus \models \forall x(U(F^i_j(x)) \leftrightarrow U(x))$$
$$\mathbf{T}^\oplus \models \forall x(Z_i(F^i_j(x)) \leftrightarrow Z_i(x))$$

or that

$$\mathbf{T}^\oplus \models \forall x(U(F^i_j(x)) \leftrightarrow Z_i(x))$$
$$\mathbf{T}^\oplus \models \forall x(Z_i(F^i_j(x)) \leftrightarrow U(x))$$

depending on if f^i_j was conservative or reversing. We will call a function symbol F^i_j *conservative* (resp., *reversing*) if f^i_j was.

With this semantic theory, the product, for example, shall have the following semiprojective representation

$$x \odot y = \begin{cases} x \cdot_1 y & \text{if } (Z_1(x) \vee C_1(x) \vee U(x)) \wedge (Z_1(y) \vee C_1(y) \vee U(y)) \\ x \cdot_2 y & \text{if } (Z_1(x) \vee C_2(x) \vee U(x)) \wedge (Z_2(y) \vee C_2(y) \vee U(y)) \\ x & \text{if } (Z_1(x) \vee C_1(x)) \wedge (C_2(y) \vee U(y)) \\ y & \text{if } (Z_1(y) \vee C_1(y)) \wedge (C_2(x) \vee U(x)) \end{cases}$$

Much like we did in the case of rotation, in order to reduce the validity of the Π_1-formulas of $\mathbf{T}^\oplus$ to the Π_1-formulas of $\mathbf{T}_1$ and $\mathbf{T}_2$, we must require that every predicate symbol of $\mathbf{T}^\oplus$ has a *semiprojective representation* in $\mathbf{T}_1$ and $\mathbf{T}_2$, a notion easily extended from the one of the section before. Notice that, for example, that is true for $\leq^{\oplus}$, since

$$x \leq^{\oplus} y = \begin{cases} x \leq_1 y & \text{if } (Z_1(x) \vee C_1(x) \vee U(y)) \wedge (Z_1(y) \vee C_1(y) \vee U(y)) \\ x \leq_2 y & \text{if } (Z_2(x) \vee C_2(x) \vee U(y)) \wedge (Z_2(y) \vee C_2(y) \vee U(y)) \\ \top & \text{if } (Z_1(x) \vee C_1(x)) \wedge (Z_2(y) \vee C_2(y) \vee U(y)) \\ \bot & \text{if } (Z_2(x) \vee C_2(x) \vee U(x)) \wedge (Z_1(y) \vee C_1(y)) \end{cases}$$

In this particular example, we ought to require that $1 \leq_1 1 \iff 1 \leq_2 1$, otherwise we will have problems of consistency for $\mathbf{T}^{\oplus}$. Special arrangements may be required whenever predicate symbols of $\mathbf{T}_1$ and $\mathbf{T}_2$ overlap in a semiprojective representation of a predicate symbol of $\mathbf{T}^{\oplus}$.

So, let L be a literal of $\mathbf{T}^{\circlearrowleft}$. A *type assignment of* L is a function

$$\sigma : var(L) \longrightarrow \{\Delta_1, \Delta_2, 1, 2, \nabla\}.$$

If $\mathcal{C}$ is a clause, a type assignment for $\mathcal{C}$ is a function as above with domain $var(\mathcal{C})$. With $T(\mathcal{C})$ we shall indicate the set of the type assignments of $\mathcal{C}$, that is $\{\Delta_1, \Delta_2, 1, 2, \nabla\}^{var(\mathcal{C})}$. The interpretation of the types is the following: a term shall have type Δ_1 (resp. type Δ_2) if it could be *reduced* (we will specify later the meaning of this term) to 0_1 (resp. 0_2), type ∇ if it could be reduced to 1 and type 1 (resp. type 2) if it refers to elements of the first (resp. second) component different from bottom element.

Another important remark is about terms of $\mathbf{T}^{\oplus}$: even if they are still all unary, it is not true that the terms are of the form $G(x)$, where G is a unary function symbol, since any composition like $F^1_j \circ F^2_k$ does not shrink back to a unary function symbol. So we will need to extend *inductively* our definition of type assignment on terms.

1. $\sigma(1) = \nabla$;
2. $\sigma(0_1) = \Delta_1$;
3. $\sigma(0_2) = \Delta_2$;
4. for every unary connective F^1_j,

$$\sigma(F^1_j(x)) = \begin{cases} \nabla & \text{if } \sigma(x) = \nabla \text{ and } F^1_j \text{ is conservative} \\ \Delta_1 & \text{if } \sigma(x) = \Delta_1 \text{ and } F^1_j \text{ is conservative} \\ \Delta_1 & \text{if } \sigma(x) = \nabla \text{ and } F^1_j \text{ is reversing} \\ \nabla & \text{if } \sigma(x) = \Delta_1 \text{ and } F^1_j \text{ is reversing} \\ 1 & \text{if } \sigma(x) = 1 \\ 2 & \text{if } \sigma(x) = 2 \\ \Delta_2 & \text{if } \sigma(x) = \Delta_2 \end{cases}$$

5. for every unary connective F^2_j,

$$\sigma(F^2_j(x)) = \begin{cases} \nabla & \text{if } \sigma(x) = \nabla \text{ and } F^2_j \text{ is conservative} \\ \Delta_2 & \text{if } \sigma(x) = \Delta_2 \text{ and } F^2_j \text{ is conservative} \\ \Delta_2 & \text{if } \sigma(x) = \nabla \text{ and } F^2_j \text{ is reversing} \\ \nabla & \text{if } \sigma(x) = \Delta_2 \text{ and } F^2_j \text{ is reversing} \\ 2 & \text{if } \sigma(x) = 2 \\ 1 & \text{if } \sigma(x) = 1 \\ \Delta_1 & \text{if } \sigma(x) = \Delta_1 \end{cases}$$

6. for every composition $F^1_i \circ F^1_j$ such that $f^1_i \circ f^1_j = f^1_k$, and for every term t,
$$\sigma((F^1_i \circ F^1_j)(t)) = \sigma(F^1_k(t));$$

7. for every composition $F^2_i \circ F^2_j$ such that $f^2_i \circ f^2_j = f^2_k$, and for every term t,
$$\sigma((F^2_i \circ F^2_j)(t)) = \sigma(F^2_k(t));$$

8. for every composition $F^1_i \circ F^2_j$ and for every term t,

$$\sigma((F^1_i \circ F^2_j)(t)) = \begin{cases} \sigma(F^1_i(1)) & \text{if } \sigma(t) = \nabla \text{ and } F^2_j \text{ is conservative} \\ \sigma(0_2) & \text{if } \sigma(t) = \nabla \text{ and } F^2_j \text{ is reversing} \\ \sigma(0_2) & \text{if } \sigma(t) = \Delta_2 \text{ and } F^2_j \text{ is conservative} \\ \sigma(F^1_i(1)) & \text{if } \sigma(t) = \Delta_2 \text{ and } F^2_j \text{ is reversing} \\ \sigma(F^1_i(0_1)) & \text{if } \sigma(t) = \Delta_1 \\ \sigma(F^2_j(t)) & \text{if } \sigma(t) = 2 \\ \sigma(F^1_i(t)) & \text{if } \sigma(t) = 1 \end{cases}$$

9. for every composition $F^2_i \circ F^1_j$ and for every term t,

$$\sigma((F^2_i \circ F^1_j)(t)) = \begin{cases} \sigma(F^2_i(1)) & \text{if } \sigma(t) = \nabla \text{ and } F^1_j \text{ is conservative} \\ \sigma(0_1) & \text{if } \sigma(t) = \nabla \text{ and } F^1_j \text{ is reversing} \\ \sigma(0_1) & \text{if } \sigma(t) = \Delta_1 \text{ and } F^1_j \text{ is conservative} \\ \sigma(F^2_i(1)) & \text{if } \sigma(t) = \Delta_1 \text{ and } F^1_j \text{ is reversing} \\ \sigma(F^2_i(0_2)) & \text{if } \sigma(t) = \Delta_2 \\ \sigma(F^1_j(t)) & \text{if } \sigma(t) = 1 \\ \sigma(F^2_i(t)) & \text{if } \sigma(t) = 2 \end{cases}$$

The good thing is that, under every type assignment, every term is of the form c or $G(x)$, where c is a constant and G is a unary function symbol. In the following, given two terms t and t' of $\mathbf{T}^{\oplus}$ and a type assignment σ, we will write $t \rightsquigarrow_\sigma t'$ if $\sigma(t) = \sigma(t')$ and $t(x) = t'(x)$ for every x.

Lemma 6.4. *Let σ be a type assignment and t a term of $\mathbf{T}^{\oplus}$. Then:*

1. *for every $i \in \{1, 2\}$, if $t = F^i_j(t')$, $var(t) = x$ and $\sigma(t) = i$ then there is an F^i_k such that $t \rightsquigarrow_\sigma F^i_k(x)$;*

2. *if $t = F^1_j(t')$, $var(t) = x$ and $\sigma(t) = 2$ then either $t \rightsquigarrow_\sigma x$ or there is an F^2_k such that $t \rightsquigarrow_\sigma F^2_k(x)$;*

3. *if $t = F^2_j(t')$, $var(t) = x$ and $\sigma(t) = 1$ then either $t \rightsquigarrow_\sigma x$ or there is an F^1_k such that $t \rightsquigarrow_\sigma F^1_k(x)$;*

4. *if $\sigma(t) = \nabla$ (resp., Δ_1, Δ_2) then $t \rightsquigarrow_\sigma 1$ (resp., $0_1, 0_2$).*

Proof. Follows by an easy induction on the definition of $\sigma(t)$. □

If t is a term of $\mathbf{T}^{\oplus}$ and σ is a type assignment, with $\bar{t}^\sigma$ we shall indicate a term such that $t \rightsquigarrow_\sigma \bar{t}^\sigma$ (its existence follows from the lemma above).

Now, suppose we want to check if the clause

$$\mathcal{C} = F^1_i(F^2_j(F^1_k(x))) \leq^{\circlearrowleft} F^2_j(F^1_i(x))$$

is valid in $\mathbf{T}^{\oplus}$, where F^1_i and F^1_k are conservative unary function symbols of $\mathbf{T}_1$ such that $F^1_i \circ F^1_k = F^1_\ell$, and F^2_j is a reversing unary function symbol of $\mathbf{T}_2$. Basically, what we do is what we have done for the rotation construction, that is reduce every literal under every possible type assignment, though there are two main differences: the first is that now we have five assignments instead of two, so the reduction is a much more longer task; the second is that, when we apply a type assignment σ to a literal, we have to replace every term t of the literal with $\bar{t}^\sigma$ (in order to obtain a literal of $\mathbf{T}_1$ or $\mathbf{T}_2$).

x	$\sigma^\star(\mathcal{C})$
Δ_1	$0_1 \leq_1 0_1$
Δ_2	$1 \leq_1 1$
1	$F^1_\ell(x) \leq_1 F^1_i(x)$
2	$F^2_j(x) \leq_2 F^2_j(x)$
∇	$0_2 \leq_2 0_2$

Let us analyze the second row: under the type assignment σ which sends x to Δ_2, we have that

$$\begin{aligned}\sigma(F_i^1(F_j^2(F_k^1(x)))) &= \sigma(F_i^1(1)) \text{ (since } \sigma(F_k^1(x)) = \Delta_2 \text{ and } F_j^2 \text{ is reversing)} \\ &= \nabla \qquad \text{(since } F_i^1 \text{ is conservative)}\end{aligned}$$

so $\overline{F_i^1(F_j^2(F_k^1(x)))}^{\sigma} = 1$. For the second term, the analysis is similar,

$$\begin{aligned}\sigma(F_j^2(F_i^1(x))) &= \sigma(F_j^2(0_2)) \text{ (since does not compute over } 0_2) \\ &= \nabla \qquad \text{(since } F_j^2 \text{ is reversing)}\end{aligned}$$

so, again, $\overline{F_j^2(F_i^1(x))}^{\sigma} = 1$, and we get $\sigma^\star(\mathcal{C}) = 1 \leq_1 1$.

The procedure get clumsy, as anticipated, when the number of variables increases. If, for example, we would like to check if the clause

$$\mathcal{C} = C_1(F^2(x)) \vee (F^1(x) \leq^{\circlearrowleft} y)$$

is valid in $\mathbf{T}^{\oplus}$, where F^1 and F^2 are conservative unary function symbols of $\mathbf{T}_1$ and $\mathbf{T}_2$ respectively, we obtain the following table.

x	y	$\sigma^\star(C_1(F^2(x)))$	$\sigma^\star(F^1(x) \leq^{\circlearrowleft} y)$	$\sigma^\star(\mathcal{C})$
1	1	$\top$	$F^1(x) \leq_1 y$	$F^1(x) \leq_1 y$
1	2	$\top$	$\top$	$\top$
2	1	$\bot$	$\bot$	$\bot$
2	2	$\bot$	$x \leq_2 y$	$x \leq_2 y$
1	Δ_1	$\top$	$F^1(x) \leq_1 0_1$	$\top$
Δ_1	1	$\top$	$0_1 \leq_1 y$	$\top$
2	Δ_1	$\bot$	$x \leq_1 0_1$	$x \leq_1 0_1$
Δ_1	2	$\top$	$\top$	$\top$
1	Δ_2	$\top$	$\top$	$\top$
Δ_2	1	$\bot$	$\bot$	$\bot$
2	Δ_2	$\bot$	$x \leq_2 0_2$	$x \leq_2 0_2$
Δ_2	2	$\bot$	$0_2 \leq_2 y$	$0_2 \leq_2 y$
Δ_1	Δ_2	$\top$	$\top$	$\top$
Δ_2	Δ_1	$\bot$	$\bot$	$\bot$
Δ_1	Δ_1	$\top$	$0_1 \leq_1 0_1$	$\top$
Δ_2	Δ_2	$\bot$	$0_2 \leq_2 0_2$	$0_2 \leq_2 0_2$
1	∇	$\top$	$F^1(x) \leq_1 1$	$\top$
∇	1	$\bot$	$1 \leq_1 y$	$1 \leq_1 y$
2	∇	$\bot$	$x \leq_2 1$	$x \leq_2 1$
∇	2	$\bot$	$1 \leq_2 y$	$1 \leq_2 y$
∇	Δ_1	$\bot$	$1 \leq_1 0_1$	$1 \leq_1 0_1$
Δ_1	∇	$\top$	$0_1 \leq_1 1$	$\top$
∇	Δ_2	$\bot$	$1 \leq_2 0_2$	$1 \leq_2 0_2$
Δ_2	∇	$\bot$	$0_2 \leq_2 1$	$0_2 \leq_2 1$
∇	∇	$\bot$	$1 \leq_1 1$	$1 \leq_1 1$

So, since there are type assignments in which the clause fails, we can deduce that $\mathcal{C}$ is not true in $\mathbf{T}^{\oplus}$.

If L is a literal of $\mathbf{T}^{\oplus}$ and σ a type assignment of L, we define $\sigma^\star(L)$ to be:

1. $\sigma^\star(L) = \overline{s}^{\sigma} \leq_1 \overline{t}^{\sigma}$, if $L = s \leq t$ and $\sigma(s), \sigma(t) \in \{\Delta_1, 1, \nabla\}$;

2. $\sigma^\star(L) = \neg(\overline{s}^\sigma \leq_1 \overline{t}^\sigma)$, if $L = \neg(s \leq t)$ and $\sigma(s), \sigma(t) \in \{\Delta_1, 1, \nabla\}$;
3. $\sigma^\star(L) = \overline{s}^\sigma \leq_2 \overline{t}^\sigma$, if $L = s \leq t$ and $\sigma(s), \sigma(t) \in \{\Delta_2, 2, \nabla\}$;
4. $\sigma^\star(L) = \neg(\overline{s}^\sigma \leq_2 \overline{t}^\sigma)$, if $L = \neg(s \leq t)$ and $\sigma(s), \sigma(t) \in \{\Delta_2, 2, \nabla\}$;
5. $\sigma^\star(L) = \top$ iff
   (a) $L = C_1(t)$ and $\sigma(t) = 1$;
   (b) $L = \neg C_1(t)$ and $\sigma(t) \neq 1$;
   (c) $L = C_2(t)$ and $\sigma(t) = 2$;
   (d) $L = \neg C_2(t)$ and $\sigma(t) \neq 2$;
   (e) $L = Z_1(t)$ and $\sigma(t) = \Delta_1$;
   (f) $L = \neg Z_1(t)$ and $\sigma(t) \neq \Delta_1$;
   (g) $L = Z_2(t)$ and $\sigma(t) = \Delta_2$;
   (h) $L = \neg Z_2(t)$ and $\sigma(t) \neq \Delta_1$;
   (i) $L = U(t)$ and $\sigma(t) = \nabla$;
   (j) $L = \neg U(t)$ and $\sigma(t) \neq \nabla$;
   (k) $L = s \leq t$, and $\sigma(s) \in \{\Delta_1, 1\}$ and $\sigma(t) \in \{\Delta_2, 2\}$;
   (l) $L = \neg(s \leq t)$, and $\sigma(s) \in \{\Delta_2, 2\}$ and $\sigma(t) \in \{\Delta_1, 1\}$;
6. $\sigma^\star(L) = \bot$ iff
   (a) $L = C_1(t)$ and $\sigma(t) \neq 1$;
   (b) $L = \neg C_1(t)$ and $\sigma(t) = 1$;
   (c) $L = C_2(t)$ and $\sigma(t) \neq 2$;
   (d) $L = \neg C_2(t)$ and $\sigma(t) = 2$;
   (e) $L = Z_1(t)$ and $\sigma(t) \neq \Delta_1$;
   (f) $L = \neg Z_1(t)$ and $\sigma(t) = \Delta_1$;
   (g) $L = Z_2(t)$ and $\sigma(t) \neq \Delta_2$;
   (h) $L = \neg Z_2(t)$ and $\sigma(t) = \Delta_1$;
   (i) $L = U(t)$ and $\sigma(t) \neq \nabla$;
   (j) $L = \neg U(t)$ and $\sigma(t) = \nabla$;

(k) $L = s \leq t$, and $\sigma(s) \in \{\Delta_2, 2\}$ and $\sigma(t) \in \{\Delta_1, 1\}$;
(l) $L = \neg(s \leq t)$, and $\sigma(s) \in \{\Delta_1, 1\}$ and $\sigma(t) \in \{\Delta_2, 2\}$;

If $\mathcal{C} = L_1 \vee \ldots \vee L_k$ is a clause of $\mathbf{T}^{\oplus}$ and σ a type assignment of $\mathcal{C}$, we define

$$\sigma^\star(\mathcal{C}) = \sigma^\star(L_1) \vee \ldots \vee \sigma^\star(L_k).$$

Notice that each literal of $\sigma^\star(\mathcal{C})$ is either a literal of $\mathbf{T}_1$ or a literal of $\mathbf{T}_2$: we call with $\mathcal{C}_1^\sigma$ (resp. $\mathcal{C}_2^\sigma$) the clause made of all the literals of $\sigma^\star(\mathcal{C})$ which are literals of $\mathbf{T}_1$ (resp. $\mathbf{T}_2$).

Theorem 6.5. *For every clause $\mathcal{C}$ of $\mathbf{T}^{\oplus}$, $\mathbf{T}^{\oplus} \models \mathcal{C}$ iff, for every $\sigma \in T(\mathcal{C})$,*

$$\mathbf{T}_1 \models \forall \vec{x} \mathcal{C}_1^\sigma \text{ and } \mathbf{T}_2 \models \forall \vec{x} \mathcal{C}_2^\sigma.$$

Proof. A clause $\mathcal{C}$ of $\mathbf{T}^{\oplus}$ is valid iff the image of $\mathcal{C}$ under every type assignment is a valid formula of $\mathbf{T}^{\oplus}$. But, since under every type assignment σ, each term t is equivalent to $\bar{t}^\sigma$, $\mathcal{C}$ is valid in $\mathbf{T}^{\oplus}$ iff $\sigma^\star(\mathcal{C})$ is valid in $\mathbf{T}^{\oplus}$. The literals of $\sigma^\star(\mathcal{C})$ can be divided between those of $\mathbf{T}_1$ and those of $\mathbf{T}_2$; so the validity of $\mathcal{C}$ in $\mathbf{T}^{\oplus}$ is equivalent to the validity of $\mathcal{C}_1^\sigma$ in $\mathbf{T}_1$ and of $\mathcal{C}_2^\sigma$ in $\mathbf{T}_2$, for every type assignment σ. □

What follows is an algorithm which decides the Π_1-formulas of $\mathbf{T}^{\oplus}$ (the decision procedures for the Π_1-formulas of $\mathbf{T}_1$ and $\mathbf{T}_2$ are used at step 4).

INPUT: A Π_1-formula ψ of $\mathrm{T}^{\oplus}$;

OUTPUT: YES, if it is valid, NO, if it is not;

1. write ψ in CNF:

$$\psi \equiv \forall \vec{x} \bigwedge_{i=1}^{m} \bigvee_{j=1}^{n_i} L_{ij};$$

2. for every clause $\mathcal{C} = L_{i1} \vee \ldots \vee L_{in_i}$
    3. for every type assignment σ of $\mathcal{C}$
        4. if $\mathcal{C}_1^\sigma$ is not valid in T_1 or $\mathcal{C}_2^\sigma$ is not valid in T_2, return NO and end the procedure;
        5. else, go on;
6. return YES.

□

6.3 Other Constructions

In [18] a very interesting generalization of the ordinal sum construction was introduced. The *generalized ordinal sum* can be performed between a commutative residuated lattice $\mathbf{K} = \langle K, \vee_{\mathbf{K}}, \wedge_{\mathbf{K}}, \cdot_{\mathbf{K}}, \rightarrow_{\mathbf{K}}, 1_{\mathbf{K}} \rangle$ and a commutative integral residuated lattice $\mathbf{L} = \langle L, \vee_{\mathbf{L}}, \wedge_{\mathbf{L}}, \cdot_{\mathbf{L}}, \rightarrow_{\mathbf{L}}, 1_{\mathbf{L}} \rangle$ (we assume commutativity and integrality for the sake of brevity) and consists basically in placing $\mathbf{L}$ where the unit element $1_{\mathbf{K}}$ of $\mathbf{K}$ is, treating every element of $\mathbf{L}$ as a unit for the elements of $\mathbf{K}$. The new algebra, indicated as $\mathbf{K}[\mathbf{L}]$, has universe $(K \backslash \{1_{\mathbf{K}}\}) \cup L$ and its connectives are defined as (in the following, $\star \in \{\vee, \wedge, \cdot\}$):

$$x \star y = \begin{cases} x \star_{\mathbf{K}} y & \text{if } x, y \in K \backslash \{1_{\mathbf{K}}\} \\ x \star_{\mathbf{L}} y & \text{if } x, y \in L \\ x \star_{\mathbf{K}} 1_{\mathbf{K}} & \text{if } x \in K \backslash \{1_{\mathbf{K}}\}, y \in L \text{ and } x \star_{\mathbf{K}} 1_{\mathbf{K}} \neq 1_{\mathbf{K}} \\ y & \text{if } x \in K \backslash \{1_{\mathbf{K}}\}, y \in L \text{ and } x \star_{\mathbf{K}} 1_{\mathbf{K}} = 1_{\mathbf{K}} \end{cases}$$

$$x \rightarrow y = \begin{cases} x \rightarrow_{\mathbf{K}} y & \text{if } x, y \in K \backslash \{1_{\mathbf{K}}\} \\ x \rightarrow_{\mathbf{L}} y & \text{if } x, y \in L \\ x \rightarrow_{\mathbf{K}} 1_{\mathbf{K}} & \text{if } x \in K \backslash \{1_{\mathbf{K}}\}, y \in L \text{ and } x \rightarrow_{\mathbf{K}} 1_{\mathbf{K}} \neq 1_{\mathbf{K}} \\ 1_{\mathbf{L}} & \text{if } x \in K \backslash \{1_{\mathbf{K}}\}, y \in L \text{ and } x \rightarrow_{\mathbf{K}} 1_{\mathbf{K}} = 1_{\mathbf{K}} \end{cases}$$

In the paper it is shown that $\mathbf{K}[\mathbf{L}] = \langle (K \backslash \{1_{\mathbf{K}}\}) \cup L, \vee, \wedge, \cdot, \rightarrow, 1_{\mathbf{L}} \rangle$ is still a commutative residuated lattice. But, the construction preserves also semiprojectivity (at least in the format of the truth function of the connectives) since, if we take the semantic theory which extends both the semantic theories of the logics involved in the construction, and we add to it three unary predicate symbols

$$\begin{aligned} C_{\mathbf{K}}(x) &\text{ iff } x \in K \backslash \{1_{\mathbf{K}}\}, \\ C_{\mathbf{L}}(x) &\text{ iff } x \in L, \\ U_{\mathbf{K}}(x) &\text{ iff } x = 1_{\mathbf{K}}, \end{aligned}$$

then the connectives can be represented semiprojectively as

$$x \star y = \begin{cases} x \star_{\mathbf{K}} y & \text{if } C_{\mathbf{K}}(x) \wedge C_{\mathbf{K}}(y) \\ x \star_{\mathbf{L}} y & \text{if } C_{\mathbf{L}}(x) \wedge C_{\mathbf{L}}(y) \\ x \star_{\mathbf{K}} 1_{\mathbf{K}} & \text{if } C_{\mathbf{K}}(x) \wedge C_{\mathbf{L}}(y) \wedge \neg C_{\mathbf{K}}(x \star_{\mathbf{K}} 1_{\mathbf{K}}) \\ y & \text{if } C_{\mathbf{K}}(x) \wedge C_{\mathbf{L}}(y) \wedge C_{\mathbf{K}}(x \star_{\mathbf{K}} 1_{\mathbf{K}}) \end{cases}$$

Chapter 7

Beyond Semiprojective Logics

In this chapter we shall develop an extension of the concept of semiprojectivity which will allow us to automatically introduce analytic calculi for logics like Łukasiewicz and Product, which could not be treated in the framework of semiprojective logics. We will show how this new notion arises, the new kind of calculi associated to this class of logics, examples of logics captured by this new framework but uncatchable by the semiprojective one, and a modification of the calculus which shall allow us to recover the Co-NP complexity.

7.1 Hyperprojective Logics

The most relevant limitation of the expressive power of semiprojective logics relies in Proposition (3.1), which has, as corollaries, the consequence that BL logic, Łukasiewicz logic and Product logic are not semiprojective: in fact, all these logics have, as equivalent algebraic semantics, a variety which is not locally finite. So, in order to capture those logics (and many more) we have to modify radically the concepts we have introduced.

One way to do so is to change the object of our calculi: sequents of relations were finite multisets of the form

$$R_{i_1}(\varphi^1_1, \ldots, \varphi^1_{r_1}) \mid \ldots \mid R_{i_k}(\varphi^k_1, \ldots, \varphi^k_{r_k})$$

where for all $1 \leq j \leq k$: $i_j \in \{1, \ldots, n\}$, R_{i_ℓ} is a relation of a semantic theory for the logic, r_ℓ is the arity of R_{i_ℓ}, and all φ^i_j are formulas of the logic. What could happen if we use *multisets* of formulas instead of formulas?

Relational hypersequents were introduced in [13] and were objects of the form

$$A_1^1, \ldots, A_{n_1}^1 \lhd_1 B_1^1, \ldots, B_{m_1}^1 \mid \ldots \mid A_1^k, \ldots, A_{n_k}^k \lhd_k B_1^k, \ldots, B_{m_k}^k$$

where A_j^i and B_j^i are formulas and $\lhd_i$ is either $\leq$ or $<$. Using relational hypersequents, analytic calculi for Gödel, Łukasiewicz and Product logic have been defined, while the same framework was introduced in [42] to supply analytic calculi for two logics characterized by finite ordinal sums of Łukasiewicz and Product t-norms, one of which is a conservative extension of BL logic. In these successful attempts relies the motivation of the introduction of *hypersequents of relations.*

Definition 7.1. An *hypersequent of relations* of the first order classical theory $\mathbf{T}$, semantic theory of the logic $\mathcal{L}$, is an object of the form

$$R_{i_1}(\Gamma_1^1, \ldots, \Gamma_{r_{i_1}}^1) \mid \ldots \mid R_{i_s}(\Gamma_1^s, \ldots, \Gamma_{r_{i_s}}^s)$$

where for all $1 \leq j \leq k$: $i_j \in \{1, \ldots, n\}$, R_{i_ℓ} is a relation of $\mathbf{T}$, r_ℓ is the arity of R_{i_ℓ}, and all Γ_j^i are *multisets* of formulas of $\mathcal{L}$.

This object has one advantage: the comma "," which separates formulas in the multisets, can be interpreted as the product of the logic. This is why we shall allow the semantic theories to include a binary operation $\oplus$ and a constant ε satisfying the axioms of commutative monoids (eventually dropping commutativity when dealing with non-commutative logics). This allow us to represent multisets (in the non-commutative case, sequences) of formulas as special terms of the semantic theories, namely *comma terms.* More formally, let $\mathbf{T}$ be a first order classical theory with truth constants $c_1, \ldots, c_n, \varepsilon$, unary function symbols $u_1, \ldots, u_m$ and a designated binary function symbol $\oplus$, which at least satisfies the associativity law and has ε as identity element. A *comma term* of $\mathbf{T}$ shall be ε or a term $t(p_1, \ldots, p_n)$, where

$$p_i \in \{x, u_1(x), \ldots, u_m(x) \mid x \text{ a variable or a truth constant}\}$$

and $t(x_1, \ldots, x_n)$ is obtained by further compositions (possibly none) of $\oplus$ only. As an example,

$$u_p(x) \oplus 1 \oplus u_p(0) \oplus z \oplus u_q(y)$$

is a comma term (provided 0 and 1 are truth constants of the theory). The comma term above shall represent the family of multisets

$$u_p(\varphi), 1, u_p(0), \chi, u_q(\psi)$$

where the triple of formulas (φ, ψ, χ) substitutes the triple of variables (x, y, z). To represent this, we will extend every substitution σ of the variables with formulas to comma terms in the following way:

1. $\sigma(\varepsilon) = \emptyset$;
2. $\sigma(u_q(x)) = u_q(\sigma(x))$;
3. if $t = t_1 \oplus t_2$ is a comma term, then $\sigma(t_1 \oplus t_2) = \sigma(t_1), \sigma(t_2)$.

The set of comma terms of a theory **T** shall be indicated as $\mathrm{CT}_{\mathbf{T}}$. We will say that a formula of **T** is *weakly simple* if it has no quantifiers, no negation or implication and all its terms are comma terms.

Turning back to how to extend the definition of semiprojective logics, Łukasiewicz, Gödel and Product logics share an interesting fact relating the format of their residua: the truth function of the residuum of every logic $\mathcal{L}$ in $\{\text{Ł}, \mathbf{G}, \Pi\}$ can be written as

$$x \Rightarrow_{\mathcal{L}} y = \begin{cases} 1 & \text{if } x \leq y \\ g_{\mathcal{L}}(x, y) & \text{if } y < x \end{cases}$$

where

$$\begin{aligned} g_{\text{Ł}}(x, y) &= 1 - x + y, \\ g_{\mathbf{G}}(x, y) &= y, \\ g_{\Pi}(x, y) &= y/x. \end{aligned}$$

If we let

$$\begin{aligned} x \oplus_{\text{Ł}} y &= x + y - 1, \\ x \oplus_{\mathbf{G}} y &= \min(x, y), \\ x \oplus_{\Pi} y &= x \cdot y, \end{aligned}$$

then

$$x *_{\text{Ł}} y = \begin{cases} 0 & \text{if } x \oplus_{\text{Ł}} y \leq 0 \\ x \oplus_{\text{Ł}} y & \text{if } 0 < x \oplus_{\text{Ł}} y \end{cases}$$

$$x *_{\mathbf{G}} y = x \oplus_{\mathbf{G}} y$$

$$x *_{\Pi} y = x \oplus_{\Pi} y$$

and we have, for every $\mathcal{L} \in \{\text{Ł}, \mathbf{G}, \Pi\}$,

$$(y < x) \implies (g_{\mathcal{L}}(x,y) \lhd z \iff y \lhd x \oplus_{\mathcal{L}} z) \tag{7.1}$$

and

$$(y < x) \implies (z \lhd g_{\mathcal{L}}(x,y) \iff x \oplus_{\mathcal{L}} z \lhd y) \tag{7.2}$$

are true in $\mathcal{L}$, for every $\lhd \in \{\leq, <\}$ (it is just a local version of the law of residuation). Notice that it suffices to prove these equivalences in the standard structure $\langle [0,1], \leq, <, \oplus_{\mathcal{L}}, g_{\mathcal{L}} \rangle$ (notice that $g_{\mathcal{L}}$ and $\oplus_{\text{Ł}}$ are not total function over $[0,1]$, but we can extend them in order to be like to $\Rightarrow_{\mathcal{L}}$ and $*_{\text{Ł}}$) by standard completeness.

This remark is crucial because, if we want to define a hypersequents-of-relations based calculus where the comma between formulas is faithfully interpreted by a *single* binary function symbol $\oplus_{\mathcal{L}}$, we have to get rid of every occurrence of other undesired (but necessary) binary functions, such as $g_{\mathcal{L}}$; in these cases we can do it with the conditions (7.1) and (7.2). So, for each of these logic we obtain the formula

$$\begin{aligned}
\alpha_{(\rightarrow_{\mathcal{L}};\lhd;l)} &= ((x \leq y) \Rightarrow (1 \lhd z)) \wedge (\underline{(y < x) \Rightarrow (g_{\mathcal{L}}(x,y) \lhd z)}) \\
&= ((x \leq y) \Rightarrow (1 \lhd z)) \wedge ((y < x) \Rightarrow (y \lhd x \oplus_{\mathcal{L}} z)) \\
&= ((y < x) \vee (1 \lhd z)) \wedge ((x \leq y) \vee (y \lhd x \oplus_{\mathcal{L}} z))
\end{aligned}$$

which, using roughly the procedure available for semiprojective logics and considering also *contexts* Γ and Δ, give us the rule

$$\frac{\Gamma, 1 \lhd \Delta \mid B < A \mid \mathcal{H} \qquad \Gamma, B \lhd A, \Delta \mid A \leq B \mid \mathcal{H}}{\Gamma, A \rightarrow B \lhd \Delta \mid \mathcal{H}} \; (\rightarrow; \lhd; l)$$

Taking into account all the observations we have made so far, and mixing them with the knowledge on semiprojective logics (whose expressive power we are interested to widen), the following definition comes out naturally.

Definition 7.2. A logic $\mathcal{L}$ is said to be *hyperprojective* if there is a classical first order theory $\mathbf{T}$ such that:

1. $\mathbf{T}$ has a binary associative function symbol $\oplus$, binary function symbols $g_1, \ldots, g_n$, unary function symbols $u_1, \ldots, u_m$ corresponding to (some of) the homonym unary connectives of $\mathcal{L}$, constant symbols corresponding to the homonym truth constants of $\mathcal{L}$ plus a constant ε acting as the identity for $\oplus$, and predicate symbols $R_1, \ldots, R_\ell$ with arity $r_1, \ldots, r_\ell$, respectively;

2. the set of theorems of $\mathbf{T}$ that are universal closure of weakly simple formulas is decidable;

3. the truth function of every

   $$\Box \in \{\circ, u_p(\circ), u_p(u_q) \mid \circ \text{ binary connective of } \mathcal{L}, u_\ell \text{ unary function of } \mathbf{T}\}$$

   is of the form

   $$\tilde{\Box}(x_1, \ldots, x_k) = \begin{cases} t_1 & \text{if } P^1_\Box(s_1, \ldots, s_j) \\ \vdots & \vdots \\ t_h & \text{if } P^h_\Box(s_1, \ldots, s_j) \end{cases}$$

   where $P^1_\Box, \ldots, P^h_\Box$ are simple formulas of $\mathbf{T}$ such that for all $i \neq j$

   $$\mathbf{T} \models \forall \vec{x} \bigvee_{i=1}^{n} P^i_\Box \text{ and } \mathbf{T} \models \forall \vec{x} \neg (P^i_\Box \wedge P^j_\Box),$$

   $t_1, \ldots, t_h \in \mathrm{CT}_\mathbf{T} \cup \{g_i(x, y) \mid x, y \in \{x_1, \ldots, x_n\}\}$, and $s_1, \ldots, s_j \in \mathrm{CT}_\mathbf{T}$;

4. there is a simple formula of $\mathbf{T}$ with exactly one free variable $\mathrm{Des}(x)$ such that

   $$\mathcal{L} = \{\varphi \mid \mathcal{M} \models \mathrm{Des}(\varphi^{\mathcal{M}^\star, \sigma}) \text{ for all } \mathcal{M}, \sigma \text{ of } \mathbf{T}\},$$

   where $\mathcal{M}^\star$ is the model obtained from $\mathcal{M}$ enriching it by the interpretations of every $\Box \in \{\circ, u_p(\circ), u_p(u_q)\}$;

5. let $P^h_\Box(s_1, \ldots, s_j)$ be a simple formula of the truth function of $\Box$ associated to a term $g_k(x, y)$. Then for every predicate symbol R_i *with arity* $r_i > 1$ and for every $1 \leq p \leq r_i$ there exists a $1 \leq q \leq r_i$, and a comma term $t \in \mathrm{CT}_\mathbf{T}$ whose only free variables are x and z, such that

   $$\begin{aligned} \mathbf{T} \models \forall \vec{x} (P^h_\Box(s_1, \ldots, s_j) \quad &\Rightarrow \quad (R_i(x_1, \ldots, x_{r_i})[g_k(x, y)/x_p, z/x_q] \Leftrightarrow \\ &\Leftrightarrow \quad R_i(x_1, \ldots, x_{r_i})[y/x_p, t/x_q])); \end{aligned}$$

6. let $P^h_\Box(s_1, \ldots, s_j)$ be a simple formula of the truth function of $\Box$ associated to a term $g_k(x, y)$. Then for every predicate symbol R_i *with arity* $r_i = 1$ there exists a comma term $t \in \mathrm{CT}_\mathbf{T}$ whose only free variables are x and y, such that

   $$\mathbf{T} \models \forall \vec{x} (P^h_\Box(s_1, \ldots, s_j) \quad \Rightarrow \quad (R_i(g_k(x, y)) \Leftrightarrow R_i(t))).$$

If $\mathcal{L} \in \{\text{Ł}, \mathbf{G}, \Pi\}$, taking $\mathbf{T}_{\mathcal{L}}$ to be the first order classical theory generated by the model

$$\langle [0,1], \leq, <, \oplus_{\mathcal{L}}, g_{\mathcal{L}} \rangle,$$

we have that $\mathcal{L}$ and $\mathbf{T}_{\mathcal{L}}$ satisfies conditions (1), (3), (4) and (5) (and (6), because no unary predicate symbols are present) of the above definitions by the previous remarks.

Remark 6. What we want to require with condition (5) and (6) is that the extra binary functions $g_1, \ldots, g_n$ can be replaced by a comma term with free variables x and z "somewhere in the predicate symbol", hence by a (series of) comma(s). Since in a unary predicate symbol there is only one spot available for this replacement, condition (6) needs to be as it is.

7.1.1 The axioms

The calculus we shall develop for an hyperprojective logic $\mathcal{L}$ with semantic theory $\mathbf{T}$ shall be indicated as $\mathbf{HR}\mathcal{L}_{\mathbf{T}}$. In the following, we introduce the axioms of such calculi, motivating their format with examples.

Definition 7.3 (Axiom sequents)**.** Let $\mathbf{T} \models \forall \vec{x} \bigvee_{1 \leq j \leq n} B_j$ where the B_j are atomic formulas whose terms are comma terms. Let σ be any substitution of the variables $\vec{x}$ with formulas of $\mathcal{L}$. Then

$$\sigma(B_1) \mid \ldots \mid \sigma(B_n)$$

is an axiom of $\mathbf{HR}\mathcal{L}_{\mathbf{T}}$

Remark 7. Condition (2) of Definition (7.2) implies that the set of axioms of $\mathbf{HR}\mathcal{L}_{\mathbf{T}}$ is decidable, since every axiom comes from a universal closure of a weakly simple formula.

We can show that this definition, in the case of $\mathcal{L} \in \{\text{Ł}, \mathbf{G}, \Pi\}$, encompasses the characterization of the axioms of the calculi given in [13] for the three fundamental fuzzy logics. An atomic hypersequent of relations of these calculi is an object of the form

$$\Gamma_1 \lhd_1 \Delta_1 \mid \ldots \mid \Gamma_n \lhd_n \Delta_n,$$

where each Γ_i, Δ_i is a multiset of formulas. If we consider the comma terms $t_{1_1}, t_{1_2}, \ldots, t_{n_1}, t_{n_2}$ and the assignment σ of formulas of the logics to the variables such that

$$\sigma(t_{i_1}) = \Gamma_i \text{ and } \sigma(t_{i_2}) = \Delta_i,$$

then, letting, for all $i = 1 \ldots, n$

$$\sigma(t_{i_1} \lhd_i t_{i_2}) = \sigma(t_{i_1}) \lhd_i \sigma(t_{i_2}),$$

we have, clearly, that the relational hypersequent above is equivalent to the hypersequent of relations

$$\sigma(t_{1_1} \lhd_1 t_{1_2}) \mid \ldots \mid \sigma(t_{n_1} \lhd_n t_{n_2}),$$

which is an axiom of the calculi if and only if

$$\langle [0,1], \leq, <, \oplus_{\mathcal{L}}, g_{\mathcal{L}} \rangle \models \forall \vec{x} \bigvee_{1 \leq i \leq n} (t_{i_1} \lhd_i t_{i_2}),$$

or, equivalently, if

$$\langle [0,1], \leq, <, \oplus_{\mathcal{L}}, g_{\mathcal{L}} \rangle \models \exists \vec{x} \bigwedge_{1 \leq i \leq n} \neg (t_{i_1} \lhd_i t_{i_2}).$$

So, the atomic hypersequent above is an axiom of the calculi iff there is an assignment $\mu_{\mathcal{L}}$ of real values in $[0,1]$ to the variables such that, for all $i = 1, \ldots, n$,

$$\langle [0,1], \leq, <, \oplus_{\mathcal{L}}, g_{\mathcal{L}} \rangle, \mu_{\mathcal{L}} \models \neg (t_{i_1} \lhd_i t_{i_2}),$$

that is no inequality in the set $\{\mu_{\mathcal{L}}(t_{i_1}) \ntriangleleft_i \mu_{\mathcal{L}}(t_{i_2}) \mid i = 1, \ldots, n\}$ is true in the structure $\langle [0,1], \leq, <, \oplus_{\mathcal{L}}, g_{\mathcal{L}} \rangle$, where $\nleq$ is $>$ and $\nless$ is $\geq$. We let $\mu_{\mathcal{L}}(\varepsilon) = 1$ and, depending on $\mathcal{L}$, $\mu_{\mathcal{L}}(t)$ is

$$\begin{aligned}
\mu_{\text{Ł}}(t) &= \max \left\{ 0, 1 + \sum_{x \in var(t) \cup \{0,1\}} (\mu_{\text{Ł}}(x) - 1) \right\} \\
\mu_{\mathbf{G}}(t) &= \min\{\mu_{\mathbf{G}}(x) \mid x \in var(t) \cup \{0,1\}\} \\
\mu_{\Pi}(t) &= \prod_{x \in var(t) \cup \{0,1\}} \mu_{\Pi}(x)
\end{aligned}$$

where, as already said, each $\mu_{\mathcal{L}}(x) \in [0,1]$, $\mu_{\mathcal{L}}(0) = 0$ and $\mu_{\mathcal{L}}(1) = 1$.

Instead in [13], every relational hypersequent H was associated to the following set of inequalities

$$\Sigma_{\text{H}} = \{*_{\mathcal{L}}(\Gamma_1) \ntriangleleft_1 *_{\mathcal{L}} (\Delta_1), \ldots, *_{\mathcal{L}}(\Gamma_n) \ntriangleleft_n *_{\mathcal{L}} (\Delta_n)\},$$

where $\not\leq$ is $>$, $\not<$ is $\geq$, $*_{\mathcal{L}}(\emptyset) = 1$ and

$$\begin{aligned} *_{\text{Ł}}(\Gamma) &= \max\left\{0, 1 + \sum_{q\in\Gamma}(x_q - 1)\right\} \\ *_G(\Gamma) &= \min\{x_q \mid q \in \Gamma\} \\ *_{\Pi}(\Gamma) &= \prod_{q\in\Gamma} x_q \end{aligned}$$

where x_q is a real-valued variable for all propositional variables q, $x_0 = 0$ and $x_1 = 1$. In that paper was stated that a relational hypersequent H is an axiom of the calculus of the logic $\mathcal{L} \in \{\text{Ł}, \mathbf{G}, \Pi\}$ iff Σ_{H} is inconsistent over $\langle[0,1], *_{\mathcal{L}}\rangle$, i.e. there is an assignment μ of the variables in Σ_{H} to the real values in $[0,1]$ that does not satisfy one of its inequalities, an assertion equivalent to the one we obtained studying the particular semantic theories of the three fundamental fuzzy logics.

7.1.2 The rules and the properties of the calculi

The following definition is a little cumbersome, but necessary for the introduction of *contexts* in the rules of our calculi; first, we write $\neg_Z p$ if $p \in \text{CT}_{\mathbf{T}}$ is a term not containing the variables in the set $Z = \{z_1, \ldots, z_r\}$. For every function

$$\tau : Z \longrightarrow \text{CT}_{\mathbf{T}},$$

for every predicate symbol $R(z_1, \ldots, z_r)$ and terms $t_1, \ldots, t_r \in \text{CT}_{\mathbf{T}}$, we define

$$\tau(R(t_1, \ldots, t_r)) = R(\tau(t_1), \ldots, \tau(t_r)),$$

where

$$\tau(t_j) = \begin{cases} \tau(z_j) & \text{if } t_j = z_j \\ \tau(z_j) \oplus p & \text{if } t_j = p \text{ and } \neg_Z p \\ \tau(z_j) \oplus p & \text{if } t_j = z_j \oplus p \text{ or } t_j = p \oplus z_j \text{ and } \neg_Z p \\ \tau(z_j) \oplus p \oplus q & \text{if } t_j = p \oplus q \text{ and } \neg_Z p \text{ and } \neg_Z q \end{cases}$$

Since a comma term is interpreted as a multiset of formulas of $\mathcal{L}$, applying τ to an atomic formula of $\mathbf{T}$ coincides with transforming it into a hypersequent of relations. For example, the relation $1 \oplus \neg x_i < z_2$ first become the relation

$$t_1 \oplus 1 \oplus \neg x_i < t_2,$$

and then the hypersequent of relations

$$\Gamma, 1, \neg A_i < \Delta,$$

after assigning a formula of $\mathcal{L}$ to every variable. This is because, for every variable z_i and every assignment of formulas to variables σ, $\sigma(\tau(z_i))$ is a multiset of formulas.

The above definitions are central in our effort of extending our methodology for sequents-of-relations calculi to calculi whose objects are hypersequents of relations.

Definition 7.4 (Logical Rules)**.** For every connective

$$\Box \in \{\circ, u_p(\circ), u_p(u_q) \mid \circ \text{ binary connective of } \mathcal{L}, u_\ell \text{ unary function of } \mathbf{T}\},$$

for every predicate symbol R with arity $r > 1$ and each p', $1 \leq p' \leq r$, we have a rule $(\Box : R : p')$ introducing $\Box(x_1, \ldots, x_k)$ at position p' into an R-component of a hypersequent of relations. If

$$\tilde{\Box}(x_1, \ldots, x_k) = \begin{cases} t_1 & \text{if } P^1_\Box(s_1, \ldots, s_j) \\ \vdots & \vdots \\ t_h & \text{if } P^h_\Box(s_1, \ldots, s_j) \end{cases}$$

to define $(\Box; R; p')$ we start from the $\mathbf{T}$-formula

$$\alpha_{(\Box;R;p')} = \forall \vec{x} \bigvee_{1 \leq \ell \leq h} P^\ell_\Box(s_1, \ldots, s_j) \wedge R(z_1, \ldots, z_r)[t_\ell / z_{p'}],$$

and we remove each occurrence of any g_k from each R using condition (5) of Definition (7.2) (we call Q^ℓ the atomic formula of the ℓ-th component of the disjunction obtained from R after this process). We fix a function

$$\tau : \{z_1, \ldots, z_r\} \longrightarrow \mathrm{CT}_\mathbf{T}$$

and we consider the formula

$$\alpha'_{(\Box;R;p')} = \forall \vec{x} \bigvee_{1 \leq \ell \leq h} P^\ell_\Box(s_1, \ldots, s_j) \wedge \tau(Q^\ell),$$

and then we take any conjunction of disjunction of atomic formulas of $\mathbf{T}$

$$\bigwedge_{1 \leq j \leq s} \bigvee_{1 \leq k \leq w_s} B_{j,k}$$

which is equivalent to $\alpha'_{(\Box;R;p')}$. Then we have the rule

$$\frac{\sigma(B_{1,1}) \mid \ldots \mid \sigma(B_{1,w_1}) \mid \mathcal{H} \qquad \ldots \qquad \sigma(B_{s,1}) \mid \ldots \mid \sigma(B_{s,w_s}) \mid \mathcal{H}}{\sigma(\tau(R(z_1, \ldots, z_r)\,[\Box(x_1, \ldots, x_k)/z_{p'}])) \mid \mathcal{H}}\ (\Box : R : p')$$

where σ substitutes formulas of $\mathcal{L}$ for the free variables in $\alpha'_{(\Box;R;p')}$, and $\mathcal{H}$ is a side sequent.

Remark 8. Notice that each rule depends on two "parameters": τ and σ. This is because their joint application produces multisets of formulas which, in our plans, have to replace single formulas; so, in the end, every rule depends only on the choice of multisets of formulas.

For example, take the rule for introducing $\rightarrow_{\mathcal{L}}$ on the right side of $<$, where $\mathcal{L}$ is either Ł, $\mathbf{G}$ or Π: from

$$\alpha_{(\rightarrow;<;r)} \quad = \quad ((y < x) \vee (z < 1)) \wedge ((x \leq y) \vee (z \oplus x < y))$$

we obtain the formula

$$\alpha'_{(\rightarrow;<;r)} \quad = \quad ((y < x) \vee (t_1 < t_2 \oplus 1)) \wedge ((x \leq y) \vee (t_1 \oplus x < t_2 \oplus y))$$

where t_1 and t_2 are arbitrary comma terms, and so are interpreted as arbitrary multisets Γ and Δ. Hence, an arbitrary substitution of formulas for the free variables of $\alpha'_{(\rightarrow;<;r)}$ gives us the rule

$$\frac{B < A \mid \Gamma < 1, \Delta \mid \mathcal{H} \qquad A \leq B \mid \Gamma, A < B, \Delta \mid \mathcal{H}}{\Gamma < A \rightarrow B, \Delta \mid \mathcal{H}}\ (\rightarrow; <; r)$$

It is worth noticing how, for both $\mathbf{G}$ and Π, the rules for introducing & are

$$\frac{\Gamma, A, B \lhd \Delta \mid \mathcal{H}}{\Gamma, A\&B \lhd \Delta \mid \mathcal{H}}\ (\&; \lhd; l) \qquad \frac{\Gamma \lhd A, B, \Delta \mid \mathcal{H}}{\Gamma \lhd A\&B, \Delta \mid \mathcal{H}}\ (\&; \lhd; r)$$

so the product *is* the comma, while this is not true for Ł, since the rules are

$$\frac{\Gamma, A, B \lhd \Delta \mid 0 < A, B \mid \mathcal{H} \qquad \Gamma, 0 \lhd \Delta \mid A, B \leq 0 \mid \mathcal{H}}{\Gamma, A\&B \lhd \Delta \mid \mathcal{H}}\ (\&; \lhd; l)$$

$$\frac{\Gamma \lhd A, B, \Delta \mid 0 < A, B \mid \mathcal{H} \qquad \Gamma, 0 \lhd \Delta \mid A, B \leq 0 \mid \mathcal{H}}{\Gamma \lhd A\&B, \Delta \mid \mathcal{H}}\ (\&; \lhd; r)$$

This is because, as already pointed out, if $x \oplus_{\text{Ł}} y = x + y - 1$, Łukasiewicz product has truth function

$$x *_{\text{Ł}} y = \begin{cases} 0 & \text{if } x \oplus_{\text{Ł}} y \leq 0 \\ x \oplus_{\text{Ł}} y & \text{if } 0 < x \oplus_{\text{Ł}} y \end{cases}$$

We will now show how the methodology proposed introduces conservative (i.e. sound and invertible) logical rules.

Definition 7.5. Let H be a hypersequent of relations of the form

$$R_1(\Gamma_{1,1}, \dots, \Gamma_{1,r_1}) \mid \dots \mid R_n(\Gamma_{n,1}, \dots, \Gamma_{n,r_n}),$$

$\mathcal{M}$ a model of $\mathbf{T}$ and σ a valuation on $\mathcal{M}$. Then, *the formula associated to* H *under* σ is

$$\psi^{\sigma}_{\mathrm{H}} = \forall \vec{x} \bigvee_{1 \le i \le n} R_i(\oplus^{\mathcal{M}}_{\sigma}(\Gamma_{i,1}), \dots, \oplus^{\mathcal{M}}_{\sigma}(\Gamma_{i,r_i})),$$

where

$$\oplus^{\mathcal{M}}_{\sigma}(\varphi_1, \dots, \varphi_n) = \varphi_1^{\mathcal{M}^{\star},\sigma} \oplus \dots \oplus \varphi_n^{\mathcal{M}^{\star},\sigma}.$$

We write $\mathcal{M}, \sigma \models \mathrm{H}$ to mean $\mathcal{M}, \sigma \models \psi^{\sigma}_{\mathrm{H}}$.

Proposition 7.1. *(Soundness and Invertibility of the rules) Let*

$$\frac{\mathrm{H}_1 \mid \mathcal{H} \qquad \dots \qquad \mathrm{H}_n \mid \mathcal{H}}{\mathrm{H} \mid \mathcal{H}}$$

be any logical rule of $\mathbf{HR}\mathcal{L}_{\mathbf{T}}$*. Then for each model* $\mathcal{M}$ *of* $\mathbf{T}$ *and each valuation* σ *on* $\mathcal{M}$*,*

$$\mathcal{M}, \sigma \models \mathrm{H} \quad \textit{iff} \quad \mathcal{M}, \sigma \models \mathrm{H}_i \textit{ for all } i = 1, \dots, k.$$

We will show it with an example. It is easy to see that we can drop side hypersequents with a slightly modified version of Lemma (3.2). Now, take a trinary predicate symbol $R(z_1, z_2, z_3)$ and a connective $\Box$ whose truth function is

$$\tilde{\Box}(x_1, \dots, x_k) = \begin{cases} t_1 & \text{if } P^1_{\Box}(s_1, \dots, s_j) \\ t_2 & \text{if } P^2_{\Box}(s_1, \dots, s_j) \end{cases}$$

The bottom hypersequent H of the rule $(\Box; R; 2)$ is

$$\sigma(\tau(R(z_1, z_2, z_3)[\Box(x_1, \dots, x_k)/z_2])),$$

where σ substitutes formulas for variables, and τ substitutes the variables z_1, z_2, z_3 with comma terms. So, let $\sigma(\tau(z_i)) = \Gamma_i$ and $\sigma(x_i) = A_i$; then

$$\psi^{\sigma}_{\mathrm{H}} = R(\oplus^{\mathcal{M}}_{\sigma}(\Gamma_1), \oplus^{\mathcal{M}}_{\sigma}(\Box(A_1, \dots, A_k), \Gamma_2), \oplus^{\mathcal{M}}_{\sigma}(\Gamma_3)),$$

which, by the format of the truth function, is equivalent to

$$\begin{aligned}&(\sigma(P^1_\Box(s_1,\ldots,s_j))\wedge R(\oplus^{\mathcal{M}}_\sigma(\Gamma_1),\oplus^{\mathcal{M}}_\sigma(\sigma(t_1),\Gamma_2),\oplus^{\mathcal{M}}_\sigma(\Gamma_3)))\\ \vee&(\sigma(P^2_\Box(s_1,\ldots,s_j))\wedge R(\oplus^{\mathcal{M}}_\sigma(\Gamma_1),\oplus^{\mathcal{M}}_\sigma(\sigma(t_2),\Gamma_2),\oplus^{\mathcal{M}}_\sigma(\Gamma_3)))\end{aligned}$$

(notice that τ does not act on $t_1, t_2, s_1, \ldots, s_j$ since their variables are just among $x_1, \ldots, x_k$). Since for $i = 1, 2$,

$$R(\oplus^{\mathcal{M}}_\sigma(\Gamma_1),\oplus^{\mathcal{M}}_\sigma(\sigma(t_i),\Gamma_2),\oplus^{\mathcal{M}}_\sigma(\Gamma_3)) = \sigma(\tau(R(z_1,z_2,z_3)[t_i/z_2])),$$

then ψ^σ_{H} is equivalent to

$$\begin{aligned}&\sigma((P^1_\Box(s_1,\ldots,s_j)\wedge\tau(R(z_1,z_2,z_3)[t_1/z_2]))\\ \vee&(P^2_\Box(s_1,\ldots,s_j)\wedge\tau(R(z_1,z_2,z_3)[t_2/z_2]))),\end{aligned}$$

which is nothing but $\sigma(\alpha'_{(\Box;R;2)})$. This one, in turn, is clearly equivalent to the conjunction $\bigwedge_{1\le i\le n}\psi^\sigma_{\mathrm{H}_i}$, so the equivalence is proven.

Remark 9. From this result follows, similarly to Theorem (3.5), the soundness and completeness of $\mathbf{HR}\mathcal{L}$ with respect to $\mathcal{L}$.

7.2 New captured logics

As already seen in the previous sections, hyperprojective logics provide a wide framework, able to cover semiprojective logics (since the semantic theory of a semiprojective logic completely lacks of binary function symbols) and, moreover, all the three fundamental fuzzy logic, introducing very similar calculi to those reported in [13]. But that is not all.

In [42] is considered a fuzzy logic based on finite ordinal multiple of the Łukasiewicz t-norm, called **MŁ**: the ordinal sum of Łukasiewicz t-norm generate already the variety of BL-algebras (as seen in [36]), but here the language is enriched with a unary connective ∇, interpreted by the function which maps a truth value to the greatest product-idempotent element below it. The introduction of this connective (already studied in [35]) is essential; each continuous t-norm is build up, using the ordinal sum construction, from the Łukasiewicz, the Gödel and the Product t-norm (see [23]), and the introduction of ∇ serves to exclude all those continuous t-norms in whose construction the product t-norm is involved. As long as only the conjunction, the implication and the constant 0 are involved, **MŁ** is BL, so it is a conservative extension of BL. Below we report the truth function of its connectives.

$$x * y = \begin{cases} \nabla x & \text{if } (\nabla x \leq \nabla y) \wedge (\nabla y \leq \nabla x) \wedge (x \oplus_{\text{Ł}} y \leq \nabla x) \\ x \oplus_{\text{Ł}} y & \text{if } (\nabla x \leq \nabla y) \wedge (\nabla y \leq \nabla x) \wedge (\nabla x < x \oplus_{\text{Ł}} y) \\ x & \text{if } \nabla x < \nabla y \\ y & \text{if } \nabla y < \nabla x \end{cases}$$

$$x \Rightarrow y = \begin{cases} 1 & \text{if } (\nabla x < \nabla y) \vee ((\nabla x \leq \nabla y) \wedge (\nabla y \leq \nabla x) \wedge (x \leq y)) \\ g_{\text{Ł}}(x,y) & \text{if } (\nabla x \leq \nabla y) \wedge (\nabla y \leq \nabla x) \wedge (y < x) \\ y & \text{if } (\nabla y < \nabla x) \end{cases}$$

$$\nabla(x * y) = \begin{cases} \nabla x & \text{if } \nabla x \leq \nabla y \\ \nabla y & \text{if } \nabla y < \nabla x \end{cases}$$

$$\nabla(x \Rightarrow y) = \begin{cases} 0 & \text{if } x \leq y \\ \nabla y & \text{if } y < x \end{cases}$$

$$\nabla\nabla x = \nabla x$$

Hence, interpreting the comma with $\oplus_{\text{Ł}}$, this logic is hyperprojective (notice how condition (5) of Definition (7.2) is true, because within every component the product is Łukasiewicz's), and hence BL can be treated with the calculi under development.

7.3 Co-NP calculi

We are now going to study the efficiency of the calculi we have introduced. We do so from a very general point of view, emphasizing that what follows stands for all the hypersequents based calculi sharing the mentioned properties.

A *finite* multiset on a set S is a map μ from a finite subset S_0 of S into $\mathbb{N}\backslash\{0\}$. S_0 is called the *support* of μ and is denoted by $S(\mu)$. Moreover, for every $s \in S(\mu)$, $\mu(s)$ is called the *multiplicity* of s. Define, for all $s \in S$, $\mu'(s) = \mu(s)$ if $s \in S(\mu)$ and $\mu'(s) = 0$ otherwise. The *sum* $\mu_1 \oplus \ldots \oplus \mu_n$ of $\mu_1, \ldots, \mu_n$ is defined as follows:

1. $S(\mu_1 \oplus \ldots \oplus \mu_n) = S(\mu_1) \cup \ldots S(\mu_n)$

2. for $s \in S(\mu_1 \oplus \ldots \oplus \mu_n)$, $(\mu_1 \oplus \ldots \oplus \mu_n)(s) = \mu'_1(s) + \ldots + \mu'_n(s)$.

We will identify any finite set X with the multiset μ_X such that $S(\mu_X) = X$ and $\mu_X(x) = 1$ for all $x \in X$. If μ and ν are finite multiset, we write $\mu \subseteq \nu$ to mean $S(\mu) \subseteq S(\nu)$ and, for all $x \in S(\mu)$, $\mu(x) \leq \nu(x)$. Moreover, we write $\mu \subseteq X$, where X is a set, to mean $\mu \subseteq \mu_X$ (and so, for all $x \in S(\mu)$, $\mu(x) = 1$).

Take any hyperprojective logic $\mathcal{L}$ with semantic theory $\mathbf{T}$, with designated binary function symbol $\oplus$, unary function symbols $u_1, \ldots, u_m$ and truth constants $c_1, \ldots, c_s$. We are in the case that, for each n-ary predicate symbol P, for each k-ary connective $\Box$ of $\mathcal{L}$ belonging to

$$\{\circ, u_p(\circ), u_p(u_q) \mid \circ \text{ binary connective of } \mathcal{L}, u_\ell \text{ unary function symbol of } \mathbf{T}\},$$

and for multisets $\mu_1, \ldots, \mu_n$ such that

$$S(\mu_j) \subseteq \{x, u_1(x), \ldots, u_m(x), c_1, \ldots, c_s \mid x \in \{x_1, \ldots, x_k\}\},$$

there are multisets $\nu_1, \ldots, \nu_n$ with

$$S(\nu_j) \subseteq \{x, u_1(x), \ldots, u_m(x), c_1, \ldots, c_s \mid x \in \{x_1, \ldots, x_k\}\},$$

such that, for every choice of $\psi_1, \ldots, \psi_k$,

$$P(\mu_1, \ldots, \mu_i \oplus \Box(x_1, \ldots, x_k), \ldots, \mu_n)[\psi_1/x_1, \ldots, \psi_k/x_k]$$

is equivalent in $\mathbf{T}$ to

$$Q(\mu_1 \oplus \nu_1, \ldots, \mu_n \oplus \nu_n)[\psi_1/x_1, \ldots, \psi_k/x_k],$$

where $Q(x_1, \ldots, x_n)$ is a simple formula. Since Q can be written as a conjunction of disjunctions $Q_1, \ldots, Q_t$ of atomic formulas, letting

$$\mu'_j = \mu_j \oplus \nu_j[\psi_1/x_1, \ldots, \psi_k/x_k]$$

we have the rule $(P; \Box; i)$

$$\frac{Q_1(\mu'_1, \ldots, \mu'_n) \mid \mathcal{H} \qquad \ldots \qquad Q_t(\mu'_1, \ldots, \mu'_n) \mid \mathcal{H}}{P(\mu_1, \ldots, \mu_i \oplus \Box(x_1, \ldots, x_k), \ldots, \mu_n)[\psi_1/x_1, \ldots, \psi_k/x_k] \mid \mathcal{H}} \, (P; \Box; i)$$

where $\mathcal{H}$ is any side hypersequent.

Read upward, each rule eliminates just one occurrence of a formula and may produce several occurrences of proper subformulas. So, it is clear that the procedure terminates with leaf hypersequents which are disjunctions of atomic formulas of the form $P(\mu_1, \ldots, \mu_n)$, where $\mu_1, \ldots, \mu_n$ are multisets consisting of atomic formulas of $\mathcal{L}$, eventually preceded by the unary connectives $u_1, \ldots, u_m$, because the subformula property is "up to" those special unary function symbols. Termination is sure to happen in the case that, at every step, we reduce the most complex formula, as the following theorem shows.

Theorem 7.2. *Any hyperprojective logic is decidable.*

Proof. Let $\mathcal{L}$ be a hyperprojective logic with semantic theory $\mathbf{T}$. Let us say that a formula ϕ is *weakly atomic* if it is either atomic or of the form $f_j(p)$ with p atomic and f_j a unary function symbol of $\mathbf{T}$. Let us say that ϕ is a *weak proper subformula* of ψ if it either is a proper subformula of ψ or it has the form $f_j(\gamma)$ for some proper subformula γ of ψ. We interpret rules as rewriting rules, and hence we read them bottom-up. Hence, each rule eliminates just one occurrence of a formula and may produce several occurrences of weak proper subformulas. Supposing that (as it is the case for most logics) if $\mathbf{T}$ has unary function symbols $f_1, ..., f_h$, then for $i, j = 1, ..., h$ it has an axiom of the form $f_i(f_j(x)) = f_r(x)$ for some r, or $f_i(f_j(x)) = x$, we can assume that no consecutive occurrences of unary functions occur in our formulas.

Let $b(\phi)$ denote the number of function symbols in ϕ with arity > 1 and $u(\phi)$ be the number of unary function symbols in ϕ, and set $c(\phi) = 2b(\phi)+u(\phi)$. We call $c(\phi)$ the *complexity of* ϕ. Now let, for each hypersequent H, $c(\mathrm{H})$ denote the maximum complexity of formulas in H and $k(\mathrm{H})$ denote the number of occurrences of formulas of maximum complexity. Then by induction on the pair $(c(\mathrm{H}), k(\mathrm{H}))$ ordered lexicographically, we see that if H' is a premise of a rule and H is the conclusion of the same rule, that $(c(\mathrm{H}'), k(\mathrm{H}')) < (c(\mathrm{H}), k(\mathrm{H}))$. It follows that every path of the reduction tree terminates with a hypersequent whose formulas are weakly atomic.

These hypersequents are weakly simple formulas (atomic formulas whose terms are comma terms) and hence they are decidable by our assumptions on $\mathbf{T}$. It follows that $\mathcal{L}$ is in turn decidable: ϕ is a theorem of $\mathcal{L}$ iff all the leaves of the reduction tree of $\mathrm{Des}(\phi)$ are weakly simple formulas provable in $\mathbf{T}$. □

So, in the end, since the leaves of every proof tree are weakly simple formulas, which are decidable in $\mathbf{T}$, then $\mathcal{L}$ is decidable, but even when an efficient algorithm is available to recognize them, the complexity of the derivation may be multiexponential.

We point out that the inefficiency of the calculi relies on the fact that, after a step upward in the tree, the complexity of the hypersequents (intented as the maximum number of connectives in a hypersequent) is reduced strictly, but several less complicated formulas may be added, with ominous results on the time complexity. The underlying idea to speed up the procedure is, for some choice of $\psi_1, \ldots, \psi_k$, to reduced in one step *all* formulas of the form $\Box(\psi_1, \ldots, \psi_k)$.

Before going further, we define the substitution $\mu(\nu/a)$, in a multiset μ of an element $a \in S(\mu)$ by a multiset ν, as follows

1. $S(\mu(\nu/a)) = (S(\mu)\backslash\{a\}) \cup S(\nu)$;
2. if $b \in S(\nu)$ and $b \neq a$, then $\mu(\nu/a)(b) = \mu'(b) + \mu(a)\nu(b)$; this means that the number of occurrences of b is determined as follows:
   (a) compute the number of times it occurs in μ;
   (b) for each occurence of a in μ introduce $\nu(b)$ occurrences of b.
3. if $b \in S(\mu)\backslash S(\nu)$ and $b \neq a$, then $\mu(\nu/a)(b) = \mu(b)$;
4. if $b = a$ and $a \notin S(\nu)$, then $b \notin S(\mu(\nu/a))$ and $\mu(\nu/a)(b)$ is not defined.

In a similar way we can define $\mu(\nu_1/a_1, \ldots, \nu_k/a_k)$. Notice that we differentiate it from the substitution in the formula φ of the variable x by the formula ψ, which is written $\varphi[\psi/x]$.

What we will require from semantic theories is that for each n-ary connective $\Box$ there are mutually incompatible formulas

$$C_1(x_1, \ldots, x_n), \ldots, C_k(x_1, \ldots, x_n)$$

and multisets

$$\mu_1, \ldots, \mu_n \subseteq \{x, u_1(x), \ldots, u_m(x), c_1, \ldots, c_s \mid x \in \{x_1, \ldots, x_k\}\}$$

such that for $i = 1, \ldots, k$, and for every hypersequent of relations H

$$\mathbf{T} \models \forall \vec{x}(C_i(x_1, \ldots, x_n) \Longrightarrow (\mathrm{H} \iff \mathrm{H}(\mu_i/\Box(x_1, \ldots, x_n)))).$$

Hence, we assume that:

1. in the multisets $\mu_1, \ldots, \mu_n$ which are added in the reduction, all elements have multiplicity 1, and hence subformulas are added only once;

2. the reduction only depends on the *context* C_i and on the connective $\Box$, and not on the position of $\Box$;

3. formulas not involving $\Box$ or the context C_i remain unchanged.

When these conditions hold, we say that reductions are *context independent*. The negation of each C_i can be written in CNF, but for simplicity we assume that there is a predicate symbol C_i^* of **T** such that the negation of C_i is equivalent to C_i^* (if there is no such symbol, we add one). Let σ the substitution of the formulas $\psi_1, \ldots, \psi_n$ to the variables $x_1, \ldots, x_n$; to any context independent reduction we can associate the rule

$$\frac{\sigma(C_1^*) \mid \mathrm{H}(\sigma(\mu_1)/\Box(\psi_1, \ldots, \psi_n)) \qquad \ldots \qquad \sigma(C_k^*) \mid \mathrm{H}(\sigma(\mu_k)/\Box(\psi_1, \ldots, \psi_n))}{\mathrm{H}}$$

which eliminates *all* occurrences of $\Box(\psi_1, \ldots, \psi_n)$ from H, because every occurrence of it is substituted by a multiset consisting of simpler formulas. In each rule, components $\psi_1, \ldots, \psi_n$ also occur in the contexts C_i^*, but if we replace H by

$$\sigma(C_i^*) \mid \mathrm{H}(\sigma(\mu_i)/\Box(\psi_1, \ldots, \psi_n))$$

the size increases by a constant. Indeed, if the root of our tree is $\mathrm{Des}(\varphi)$, then $\psi_1, \ldots, \psi_n$ are subformulas of φ and hence the complexity of $C_i^*(\psi_1, \ldots, \psi_n)$ does not exceed $M \cdot \mathit{size}(\mathrm{Des}(\varphi))$ for some constant M. It follows:

Theorem 7.3. *If there is a P-time procedure to decide the universal closures of weakly simple formulas of* **T***, then* $\mathcal{L}$ *is in Co-NP.*

As an example of a calculus with context independent reductions, just take **RG** and the rule $(\to; <; r)$:

$$\frac{A \leq B \mid C < B \qquad B < A \mid C < 1}{C < A \to B}$$

In our language, the atomic formulas $A \leq B$ and $B < A$ are the *contexts* of our reduction, so, as a matter of fact, we can modify our calculi in order to reduce *in one step* all the occurrences of $A \to B$ from a sequent:

$$\frac{A \leq B \mid C < B \mid D < B \qquad B < A \mid C < 1 \mid D < 1}{C < A \to B \mid D < A \to B}$$

Hence, **RG** may enjoy the modified procedure we have already introduced.

But, in general, even the hypersequents-of-relations calculi we have introduced in the section before have context independent rules. This means that, letting every rule eliminate (upward) a connective from the entire hypersequent makes each calculi Co-NP, matching the time complexity we have for semiprojective logics.

7.4 Conclusions

Our aim in this work was that to summarize the state-of-the-art of semiprojective logics and sequents-of-relations calculi. We focused on proof-theoretic aspects as well as on model-theoretic one, showing how they turn out to be entwined, both contributing crucially to the validity of the presented framework. Upon this solid ground, we tried to go further, generalizing the notions reported in the first six chapters and establishing new syntactic methods to deal with fuzzy logics.

Further developments would go in the direction of trying to encompass (most of) substructural logics, in particular those which do not enjoy prelinearity, for which a satisfactory framework is missing. On the other hand, improving the efficiency of the calculi is very difficult because substructural logics which enjoy the disjunction property are shown to be Co-NP Hard in [25], making this path of research less attractive.

However, a lot can be said and done in the *classification* of proof systems for fuzzy logics, that is the identification of the characteristics of proof systems that guarantee properties like decidability or Co-NP complexity, briefly tackled in the last section; in this way our approach can provide a great amount of information, since it unveils so much insights about how a proof system is built, showing, for example, how axioms and rules are built from a semantic point of view. Focusing the attention on semantic theories is, without any doubt, a turning point in the context of proof theory for fuzzy logics, and promises plentiful rewards to those who will pursue this strategy.

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