Invisible Sparse Control of Self-Organizing Agents Leaving Unknown Environments

(joint work with G. Albi, E. Cristiani, and D. Kalise)

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The evacuation problem

Our problem: evacuating a crowd of individuals from an environment they don’t know under limited visibility by means of unrecognized informed agents

Our objectives:
- validate a microscopic model for agents leaving an unknown area;
- show that invisible sparse strategies (i.e., by means of few, unrecognized agents) influence the crowd effectively;
- provide numerical techniques for optimal exit strategies;
- propose a mesoscopic description of this dynamics when the number of pedestrian is large.

The crowd find its way to the exit thanks to the two fuchsia leaders
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Model guidelines: followers

The non-informed agents of the crowd are called followers, and are subject to a second-order dynamics with

- an isotropic metric short-range repulsion force;
- a relaxation term toward a given characteristic speed;
- if the exit is not visible
  - an isotropic topological alignment force, i.e., given $\mathcal{N} \in \mathbb{N}$, the $i$-th agent aligns with those inside $\mathcal{B}_\mathcal{N}(x_i, x)$, the smallest ball containing at least $\mathcal{N}$ agents with positions $x = (x_1, \ldots, x_N)$;
  - a random walk, in order to explore the unknown environment;
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Dynamics of followers

Followers dynamics without leaders
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If not influenced, the random term + topological alignment splits the followers

Influence of external agents with preferred direction: when removed, the group splits
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The informed agents of the crowd are called leaders. They are less than followers \((N^L \ll N^F)\) and evolve according to a first-order dynamics with

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- an optimal force which is the result of an offline optimization procedure, minimizing some cost functional.

First vs. second-order model: for followers a second-order model is necessary since they must perceive velocities to align. The bigger inertia is compensated by stronger forces w.r.t. the ones in leaders’ dynamics.

Metric vs. topological interaction: alignment is topological since empirical evidence suggests that only close neighbors play a role.
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The microscopic model

For $i = 1, \ldots, N^F$ and $k = 1, \ldots, N^L$

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\begin{align*}
\dot{x}_i &= v_i, \\
\dot{v}_i &= A(x_i, v_i) + \sum_{j=1}^{N^F} H(x_i, v_i, x_j, v_j; \mathbf{x}, \mathbf{y}) \\
&\quad + \sum_{\ell=1}^{N^L} H(x_i, v_i, y_\ell, w_\ell; \mathbf{x}, \mathbf{y}), \\
\dot{y}_k &= w_k = \sum_{j=1}^{N^F} R_{\zeta,r}(y_k, x_j) + \sum_{\ell=1}^{N^L} R_{\zeta,r}(y_k, y_\ell) + u_k,
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where:

- $A(x, v) := (1 - \theta(x)) C^2(z - v) + \theta(x) C^D \left( \frac{x - x|}{|x - x|} - v \right) + C^V (\alpha^2 - |v|^2) v$, where $z \sim N(0, \sigma^2)$, $\alpha$ is the characteristic speed and $\theta$ is the characteristic function of the target’s visibility zone;
- $H := -C^p R_{\gamma,r}(x, y) + (1 - \theta(x)) \frac{C^\alpha}{N^*} (w - v) \chi_{B_N(x; \mathbf{x}, \mathbf{y})}(y)$, for
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- In the dynamics of $y_k$, $\zeta \neq \gamma$. 

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The control

The control \( u : [0, T] \rightarrow \mathbb{R}^d \) is chosen in two different ways:

- **“dumb” strategy:** \( u_k(t) = \left( \frac{x^d - y_k(t)}{|x^d - y_k(t)|} - y_k(t) \right) \);
- **“smart” strategy:** \( u \) minimizes the functional

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J(u) = \int_0^T \left( P(t) + \nu \sum_{k=1}^{N^L} |u_k(t)|^2 \right) dt,
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\( P(t) = \) number of followers outside exit at time \( t \).

Numerically, minimization via Modified Compass Search:

- leaders’ trajectories are piecewise constant;
- starting from an initial guess, at each iteration we modify the current best strategy with small random variations;
- we keep the variation if the evaluated cost is smaller than before;
- the method generates a sequence converging to a local minimum.
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Clog up effect around exit

Dynamics with “dumb” strategy

Dynamics with “smart” strategy

Good strategies avoid exit’s clog up, hence congestion drop.
Clog up effect around exit

Dynamics with “dumb” strategy

Dynamics with “smart” strategy

Occupancy of the exit’s visibility zone with “dumb” strategy

Occupancy of the exit’s visibility zone with “smart” strategy

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Binary interactions

- When $N^F$ is large, the simulation of the micro model is no more feasible and we need a mesoscopic approximation.

- Fix a control $u$ and let $f(t,x,v)$ be the density of followers and $g(t,x,v) = \sum_{k=1}^{N^L} \delta_{yk(t),w_k(t)}(x,v)$.

- When a follower $(x,v)$ interacts with another follower $(\hat{x},\hat{v})$ or a leader $(\tilde{x},\tilde{v})$, they update their state variables according to

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\begin{align*}
    v^* &= v + \varepsilon \left[ \theta(x)C^Z \xi + S(x,v) + N^F H(x,v,\hat{x},\hat{v};\pi_1 f, \pi_1 g) \right] \\
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where $\varepsilon$ is the interaction strength and $\xi \sim \mathcal{N}(0,\varsigma^2)$ ($\varsigma \neq \sigma$!!!).
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- When a follower $(x, v)$ interacts with another follower $(\hat{x}, \hat{v})$ or a leader $(\tilde{x}, \tilde{v})$, they update their state variables according to

\[
\begin{align*}
\text{(FF)} & \quad v^* = v + \varepsilon \left[ \theta(x) C^z \xi + S(x, v) + N^F H(x, v, \hat{x}, \hat{v}; \pi_1 f, \pi_1 g) \right] \\
& \quad \hat{v}^* = \hat{v} + \varepsilon \left[ \theta(\hat{x}) C^z \xi + S(\hat{x}, \hat{v}) + N^F H(\hat{x}, \hat{v}, x, v; \pi_1 f, \pi_1 g) \right] \\
\text{(FL)} & \quad v^{**} = v + \varepsilon N^L H(x, v, \tilde{x}, \tilde{v}; \pi_1 f, \pi_1 g) \\
& \quad \tilde{v}^* = \tilde{v}
\end{align*}
\]

where $\varepsilon$ is the interaction strength and $\xi \sim \mathcal{N}(0, \varsigma^2)$ ($\varsigma \neq \sigma$!!!).
Boltzmann-Povzner dynamics

We obtain a Boltzmann-Povzner dynamics + the ODEs of leaders

\begin{equation}
\begin{aligned}
\partial_t f + v \cdot \nabla_x f &= \lambda^F Q(f, f) + \lambda^L Q(f, g), \\
\dot{y}_k &= \int_{\mathbb{R}^d} R_{\zeta, r}(y_k, x) f(x, v) \, dx \, dv + \sum_{\ell=1}^{N_L} R_{\zeta, r}(y_k, y_\ell) + u_k,
\end{aligned}
\end{equation}

where $\lambda^F$ and $\lambda^L$ are the interaction frequencies and

\[
Q(f, f) = \mathbb{E}\left[\int_{\mathbb{R}^{2d}} \left( \frac{1}{J_{FF}} f(x_*, v_*) f(\hat{x}_*, \hat{v}_*) - f(x, v) f(\hat{x}, \hat{v}) \right) \, d\hat{x} \, d\hat{v} \right],
\]

\[
Q(f, g) = \mathbb{E}\left[\int_{\mathbb{R}^{2d}} \left( \frac{1}{J_{FL}} f(x**, v**) g(\tilde{x}_*, \tilde{v}_*) - f(x, v) g(\tilde{x}, \tilde{v}) \right) \, d\tilde{x} \, d\tilde{v} \right].
\]

- Meshless Monte-Carlo method!
- How does it relate with the micro model (1)?
We obtain a Boltzmann-Povzner dynamics + the ODEs of leaders

\[
\begin{align*}
\frac{\partial}{\partial t} f + v \cdot \nabla_x f &= \lambda^F Q(f, f) + \lambda^L Q(f, g), \\
\dot{y}_k &= \int_{\mathbb{R}^2} R_{\zeta, r}(y_k, x) f(x, v) dx \, dv + \sum_{\ell=1}^{N_L} R_{\zeta, r}(y_k, y_\ell) + u_k,
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\]

- Meshless Monte-Carlo method!
- How does it relate with the micro model (1)?
Grazing collision limit

Theorem

Fix the control $u$. Let $\lambda^F = 1/\varepsilon N^F$, $\lambda^L = 1/\varepsilon N^L$, $\zeta^2 = \sigma^2/\varepsilon$ and $(f^\varepsilon, y^\varepsilon)$ be a solution of (2). Then, as $\varepsilon \to 0$, $(f^\varepsilon, y^\varepsilon)$ converges pointwise to a solution of the Fokker-Planck-type equation

$$
\begin{align*}
&\begin{cases}
\partial_t f + v \cdot \nabla_x f = -\nabla_v \cdot ((S + \mathcal{H}[f] + \mathcal{H}[g])f) + \frac{1}{2} \sigma^2 (\theta C^z)^2 \Delta_v f,
\end{cases}
\end{align*}
$$

$$
\begin{align*}
&\begin{cases}
\dot{y}_k = \int_{\mathbb{R}^2} R_{\zeta,r}(y_k, x) f(x, v) \, dx \, dv + \sum_{\ell=1}^{N^L} R_{\zeta,r}(y_k, y_\ell) + u_k,
\end{cases}
\end{align*}
$$

which is the “mean-field limit” of (1), where

$$
\begin{align*}
\mathcal{H}[f] = \int_{\mathbb{R}^{2d}} H f(\hat{x}, \hat{v}) \, d\hat{x} \, d\hat{v}, \quad \mathcal{H}[g] = \int_{\mathbb{R}^{2d}} H g(\tilde{x}, \tilde{v}) \, d\tilde{x} \, d\tilde{v}.
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\dot{y}_k = \int_{\mathbb{R}^2d} R_{\zeta,r}(y_k, x) f(x, v) \, dx \, dv + \sum_{\ell=1}^{N^L} R_{\zeta,r}(y_k, y_\ell) + u_k,
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Grazing collision limit

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Fix the control $u$. Let $\lambda^F = 1/\varepsilon N^F, \lambda^L = 1/\varepsilon N^L, \zeta^2 = \sigma^2/\varepsilon$ and $(f^\varepsilon, y^\varepsilon)$ be a solution of (2). Then, as $\varepsilon \to 0$, $(f^\varepsilon, y^\varepsilon)$ converges pointwise to a solution of the Fokker-Planck-type equation

$$\begin{align*}
\partial_t f + v \cdot \nabla_x f &= -\nabla_v \cdot ((S + \mathcal{H}[f] + \mathcal{H}[g])f) + \frac{1}{2} \sigma^2 (\theta C^z)^2 \Delta_v f, \\
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\end{align*}$$

which is the “mean-field limit” of (1), where

$$\begin{align*}
\mathcal{H}[f] &= \int_{\mathbb{R}^{2d}} H f(\tilde{x}, \tilde{v}) \, d\tilde{x} \, d\tilde{v}, \quad \mathcal{H}[g] = \int_{\mathbb{R}^{2d}} H g(\tilde{x}, \tilde{v}) \, d\tilde{x} \, d\tilde{v}.
\end{align*}$$
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Fix the control $u$. Let $\lambda^F = 1/\varepsilon N^F$, $\lambda^L = 1/\varepsilon N^L$, $\varsigma^2 = \sigma^2/\varepsilon$ and $(f^\varepsilon, y^\varepsilon)$ be a solution of (2). Then, as $\varepsilon \to 0$, $(f^\varepsilon, y^\varepsilon)$ converges pointwise to a solution of the Fokker-Planck-type equation

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    \partial_t f + v \cdot \nabla_x f &= -\nabla_v \cdot ((S + \mathcal{H}[f] + \mathcal{H}[g]) f) + \frac{1}{2} \sigma^2 (\theta C^Z)^2 \Delta_v f, \\
    \dot{y}_k &= \int_{\mathbb{R}^d} R_{\zeta, r}(y_k, x) f(x, v) \, dx \, dv + \sum_{\ell=1}^{N^L} R_{\zeta, r}(y_k, y_\ell) + u_k,
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$$

For $\varepsilon$ small we recover the micro model!
A few info

- **WWW:** http://www-m15.ma.tum.de/Allgemeines/MattiaBongini

- **References:**
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