Invisible Control of Self-Organizing Agents Leaving Unknown Environments

(joint work with G. Albi, E. Cristiani, and D. Kalise)
Pedestrian dynamics: multiscale models and control

Rising interest towards **modeling and control of large groups of individuals**:

- **Models for pedestrian dynamics**: Hughes ’02, Maury-Chupin-Santambrogio ’11, Di Francesco-Markowich-Pietschmann-Wolfram ’11;
- **Control of multiagent systems via external agents**: B.-B.-Buttazzo ’16 (theoretical), Albi-Pareschi-Zanella ’14, Pinna-Totzeck-Tse-Burger ’16 (numerical), Butail-Bartolini-Porfiri ’13 (practical);
- **Multiscale modeling and mean-field optimal control**: Fornasier-Piccoli-Rossi ’14, B.-Fornasier-Rossi-Solombrino ’15, Cristiani-Piccoli-Tosin ’11;

List is far from exhaustive, but shows two central issues:

- **dimensionality reduction** of micro models via mesoscopic ones when number of agents explodes;
- influencing the behavior of the crowd via **controlled external agents**.
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The evacuation problem

The crowd find its way to the exit thanks to the two fuchsia leaders.

- Our goal is to evacuate a crowd of individuals from an environment they don’t know under limited visibility.
- We show that invisible sparse strategies (i.e., by means of few, unrecognized, external agents) influence the crowd effectively and improve evacuation.
- Through Boltzmann equation we propose a mesoscopic description of this dynamics when the number of pedestrians is large.
- Develop a set of numerical techniques for computing optimal exit strategies in both cases.
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Model guidelines: followers (i.e., evacuees)

The non-controlled agents of the crowd are called followers, and are subject to a second-order dynamics with

- an isotropic metric short-range repulsion force;
- a relaxation term toward a given characteristic speed;
- if the exit is not visible
  - an isotropic topological alignment force, i.e., given $\mathcal{N} \in \mathbb{N}$, the $i$-th agent aligns with those inside $\mathcal{B}_{\mathcal{N}}(x_i, t)$, the smallest ball at time $t$ containing at least $\mathcal{N}$ agents.
  - a random walk, in order to explore the unknown environment;
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Model guidelines: leaders (i.e., stewards)

The controlled agents of the crowd are called leaders. They are less than followers ($N^L \ll N^F$) and evolve according to a first-order dynamics with

- an isotropic metric short-range repulsion force;
- an optimal force which is the result of an offline optimization procedure, minimizing some cost functional.

First vs. second-order model: for followers a second-order model is necessary since they must perceive velocities to align. The bigger inertia is compensated by stronger forces w.r.t. the ones in leaders’ dynamics.

Metric vs. topological interaction: alignment is topological since empirical evidence suggests that only close neighbors play a role, (see Ballerini et al. ’08).
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**Microscopic model**

For $i = 1, \ldots, N^F$ and $k = 1, \ldots, N^L$

\[
\begin{align*}
\dot{x}_i &= v_i, \\
\dot{v}_i &= A(x_i, v_i) + \sum_{j=1}^{N^F} H(x_i, v_i, x_j, v_j) + \sum_{\ell=1}^{N^L} H(x_i, v_i, y_\ell, w_\ell) \\
\dot{y}_k &= w_k = \sum_{j=1}^{N^F} R_{\zeta,r}(y_k, x_j) + \sum_{\ell=1}^{N^L} R_{\zeta,r}(y_k, y_\ell) + u_k,
\end{align*}
\]

Let $\theta(x)$ represents the characteristic function of the target’s visibility zone and

\[A(x, v) := (1 - \theta(x))C^z(z - v) + \theta(x)C^{i0} \left( \frac{v^0 - x}{|x^0 - x|} - v \right) + C^v (\alpha^2 - |v|^2) v,\]

where $z \sim N(0, \sigma^2)$, $\alpha$ is the characteristic speed.

\[H(x, v, y, w) := -C^n R_{\gamma,r}(x, y) + (1 - \theta(x)) \frac{C^\lambda}{N^s} (w - v) \chi_{B_r(x)}(y)\]

\[R_{\gamma,r}(x, y) = \begin{cases} 
\exp (-|y-x|^\gamma) \frac{y-x}{|y-x|} & \text{if } y \in B_r(x) \setminus \{x\}, \\
0 & \text{otherwise};
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In the dynamics of $y_k$, $\zeta \neq \gamma$. 
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- In the dynamics of $y_k$, $\zeta \neq \gamma$. 
Dynamics of followers

Followers dynamics without leaders
Dynamics of followers

Followers dynamics without leaders

If not influenced, the random term + topological alignment splits the followers

Influence of external agents with preferred direction: when removed, the group splits
Control strategy: “dumb”

The control $u_k : [0, T] \rightarrow \mathbb{R}^d, \quad k = 1, \ldots, N^L$ as “dumb” strategy:

$$u_k(t) = \left( \frac{x^D - y_k(t)}{|x^D - y_k(t)|} - y_k(t) \right)$$

(Left) Dynamics with “dumb” strategy and (Right) occupancy of the exit’s visibility zone
Smart strategy: Modified Compass Search

The control $u_k : [0, T] \rightarrow \mathbb{R}^d, \ k = 1, \ldots, N^L$, $u_k$ minimizes the functional
\[ J(u) = \int_0^T \left( P(t) + \nu \sum_{k=1}^{N^L} \|u_k(t)\|^2 \right) dt, \]

Where $P(t)$ is the number of followers outside exit at time $t$.

We proceed via Modified Compass Search$^1$:

- leaders’ trajectories are piecewise constant;
- starting from an initial guess, at each iteration we modify the current best strategy with small random variations;
- we keep the variation if the evaluated cost is smaller than before;
- the method generates a sequence converging to a local minimum.

$^1$Audet-Dang-Orban ’14
Clog up effect around exit

Dynamics with “dumb” strategy

Dynamics with “smart” strategy
Clog up effect around exit

Dynamics with “dumb” strategy

Dynamics with “smart” strategy

Occupancy of the exit’s visibility zone

with “dumb” strategy

Occupancy of the exit’s visibility zone

with “smart” strategy

Good strategies avoid exit’s clog up, hence congestion drop.
Mean-field approximation

- The number of interacting agents can be rather large, thus we need to solve a very large system of ODEs, which can constitute a serious difficulty.
- A natural way to tackle this problem is to approximate the problem with a kinetic equation.
- The problem of rigorously derive a mean-field equation from a system of interacting particles in the case of swarming and flocking models is well studied\(^2\) also for an optimal control problem\(^3\).
- We consider here binary interaction Boltzmann-type and mean-field approximation of the microscopic dynamic\(^4\).
- This approach is related with the development of Boltzmann collision algorithms in plasma physics\(^5\)

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\(^2\)Canizo-Carrillo-Rosado ’10, Carrillo-Y. P. Choi-Hauray-Salem ’15
\(^3\)Fornasier-Solombrino ’13, Fornasier-Piccoli-Rossi ’14, B.-Fornasier-Rossi-Solombrino ’15
\(^4\)Carrillo-Fornasier-Toscani-Vecil ’10, Pareschi-Toscani ’13
\(^5\)Bird ’94, Bobylev-Nanbu ’00, Caflisch-Pareschi-Dimarco ’10
Boltzmann approximation

- Dynamic of $N$ interacting agents
- $N \gg 1$
- Mean-Field Equation
- Fast numerical solution via Binary Interaction algorithms
- Binary Interaction
- Boltzmann-like equation
- Grazing limit
Mean-field controlled dynamic

The “mean-field limit” of the microscopic model describes the evolution of the continuous density of followers $f = f(t, x, v)$ coupled with the evolution of the microscopic leaders

\[
\begin{cases}
\partial_t f + v \cdot \nabla_x f = -\nabla_v \cdot ((S + \mathcal{H}[f] + \mathcal{H}[g]) f) + \frac{1}{2} \sigma^2 (\theta C^z)^2 \Delta_v f, \\
\dot{y}_k = \int_{\mathbb{R}^{2d}} R_{\zeta, r}(y_k, x) f(x, v) \, dx \, dv + \sum_{\ell=1}^{N^L} R_{\zeta, r}(y_k, y_\ell) + u_k, \\
k = 1, \ldots, N^L
\end{cases}
\]

Where the non-linear interactions terms are
\[
\mathcal{H}[f](x, v) = \int_{\mathbb{R}^{2d}} H(x, \hat{x}, v, \hat{v}) f(\hat{x}, \hat{v}) \, d\hat{x} \, d\hat{v},
\]
\[
S(x, v) = -\theta(x) C^z v + (1 - \theta(x)) C^D \left( \frac{x^D - x}{|x^D - x|} - v \right) + C^V (\alpha^2 - |v|^2) v.
\]

Density $g = g(t, x, v) = \sum_{k=1}^{N^L} \delta_{y_k(t), w_k(t)}(x, v)$, represents the empirical measure of leaders and $u_k$ is given by the solution of an optimal control problem.
Mean-field controlled dynamic

The “mean-field limit” of the microscopic model describes the evolution of the continuous density of followers \( f = f(t, x, v) \) coupled with the evolution of the microscopic leaders

\[
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Kinetic approximation

We assume that $f, g$ satisfy the following Boltzmann-Povzner dynamics\(^6\) + the ODEs of leaders

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f &= \lambda^F Q^F(f, f) + \lambda^L Q^L(f, g) \\
\dot{y}_k &= \int_{\mathbb{R}^{2d}} R_{\zeta, r}(y_k, x) f(x, v) \, dx \, dv + \sum_{\ell=1}^{N^L} R_{\zeta, r}(y_k, y_{\ell}) + u_k, \quad (BP)
\end{align*}
\]

where $\lambda^F$ and $\lambda^L$ are the interaction frequencies and

\[
Q^F(f, f) = \mathbb{E} \left[ \int_{\mathbb{R}^{2d}} \left( \frac{1}{J_{FF}} f(x, v_*) f(\hat{x}, \hat{v}_*) - f(x, v) f(\hat{x}, \hat{v}) \right) \, d\hat{x} \, d\hat{v} \right],
\]

\[
Q^L(f, g) = \mathbb{E} \left[ \int_{\mathbb{R}^{2d}} \left( \frac{1}{J_{FL}} f(x, v_{**}) g(\tilde{x}, \tilde{v}_{**}) - f(x, v) g(\tilde{x}, \tilde{v}) \right) \, d\tilde{x} \, d\tilde{v} \right].
\]

with $v_*, v_{**}$ indicates the pre-interaction velocities and $\mathbb{E}[\cdot]$ is the expected value w.r.t. $\xi$.

\(^6\) Povzner ’62
Kinetic approximation

We assume that $f, g$ satisfy the following Boltzmann-Povzner dynamics\(^6\) + the ODEs of leaders

\[
\begin{cases}
\partial_t f + v \cdot \nabla_x f = \lambda^F Q^F(f, f) + \lambda^L Q^L(f, g) \\
\dot{y}_k = \int_{\mathbb{R}^2} R_{\zeta, r}(y_k, x) f(x, v) \, dx \, dv + \sum_{\ell=1}^{N^L} R_{\zeta, r}(y_k, y_\ell) + u_k, \quad (BP)
\end{cases}
\]

where $\lambda^F$ and $\lambda^L$ are the interaction frequencies and

\[
Q^F(f, f) = \mathbb{E} \left[ \int_{\mathbb{R}^2} \left( \frac{1}{J_{FF}} f(x, v_*) f(\hat{x}, \hat{v}_*) - f(x, v) f(\hat{x}, \hat{v}) \right) \, d\hat{x} \, d\hat{v} \right],
\]

\[
Q^L(f, g) = \mathbb{E} \left[ \int_{\mathbb{R}^2} \left( \frac{1}{J_{FL}} f(x, v_{**}) g(\tilde{x}, \tilde{v}_{**}) - f(x, v) g(\tilde{x}, \tilde{v}) \right) \, d\tilde{x} \, d\tilde{v} \right].
\]

with $v_*, v_{**}$ indicates the pre-interaction velocities and $\mathbb{E}[\cdot]$ is the expected value w.r.t. $\xi$.

\(^6\)Povzner ’62
Binary interactions

When a follower \((x, v)\) interacts with another follower \((\hat{x}, \hat{v})\), they update their state variables according to

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
\dot{v}^* = v + \eta^F \left[ \theta(x) C^z \xi + S(x, v) + N^F H(x, v, \hat{x}, \hat{v}) \right] \\
\dot{\hat{v}}^* = \hat{v} + \eta^F \left[ \theta(\hat{x}) C^z \xi + S(\hat{x}, \hat{v}) + N^F H(\hat{x}, \hat{v}, x, v) \right]
\end{array}
\right.
\end{aligned}
\]

where \(\eta^F\) is the strength of the interaction follower-follower, and \(\xi \sim \mathcal{N}(0, \varsigma^2)\).

When a follower \((x, v)\) interacts with a leader \((\tilde{x}, \tilde{v})\),

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
\dot{v}^{**} = v + \eta^L N^L H(x, v, \tilde{x}, \tilde{v}) \\
\dot{\tilde{v}}^{**} = \tilde{v}
\end{array}
\right.
\end{aligned}
\]

where \(\eta^L\) is the strength of the interaction follower-leader.

\(^7\)Cercignani-Illner-Pulvirenti '94
Theorem: Grazing collision limit

Suppose that for some \( \delta \in (0, 1) \)

- \( \mathbb{E} \left( \| \xi \|^{2+\delta} \right) \) is finite,
- the functions \( S \) and \( H \) are in \( L^p_{\text{loc}} \) for \( p = 2, 2 + \delta \).

Fix the control \( u \). Let consider the following scaling

\[
\eta^F = \eta^L = \varepsilon, \quad \lambda^F = \lambda^L = \frac{1}{\varepsilon}, \quad \varsigma^2 = \frac{\sigma^2}{\varepsilon}
\]

and define \((f^\varepsilon, y^\varepsilon)\) be a solution of (BP). Then, as \( \varepsilon \to 0 \), \((f^\varepsilon, y^\varepsilon)\) converges pointwise to a solution of (MF), the “mean-field limit” of the microscopic model.
Idea of the proof

- Let $\mathcal{T}$ be a function space with suitably smooth test functions. Then for $\varphi \in \mathcal{T}_\delta$, the weak formulation of (BP) reads

$$\lambda \langle Q(f,f), \varphi \rangle = \lambda \mathbb{E} \left( \int_{\mathbb{R}^{4d}} (\varphi(x,v^*) - \varphi(x,v)) f(x,v) f(\hat{x},\hat{v}) \, dx dv d\hat{x} d\hat{v} \right).$$

- We derive the Taylor expansion around $v^* - v$ up to the second order of the weak interaction operator, obtaining

$$\lambda \langle Q(f,f), \varphi \rangle = \lambda \mathbb{E} \left( \int_{\mathbb{R}^{4d}} \nabla_v \varphi(x,v) \cdot (v^* - v) f(x,v) f(\hat{x},\hat{v}) \, dx dv d\hat{x} d\hat{v} + \right.$$

$$+ \frac{\lambda}{2} \int_{\mathbb{R}^{4d}} \left[ \sum_{i,j=1}^{d} \partial_{v}^{(i,j)} \varphi(x,v) (v^* - v)_i (v^* - v)_j \right] f(x,v) f(\hat{x},\hat{v}) \, dx dv d\hat{x} d\hat{v} \right) + \lambda R_\varphi$$

- Thanks to the boundedness of $S$ and $H$ we have that the remainder $R_\varphi$ is bounded.

- Therefore using the grazing collision scaling, and taking the limit for $\varepsilon \to 0$ for any $\varphi \in \mathcal{T}_\delta$ we obtain the mean-field equation (MF).

\textsuperscript{8}Toscani '06
Binary Interaction algorithm

The main idea is to simulate the *binary interaction dynamic* for small values of \( \varepsilon \) in order to approximate the Fokker-Planck equation model\(^9\).

- We use a *splitting method*, for transport and collisional part of the scaled Boltzmann equation (BP).

\[
\begin{align*}
\partial_t f &= -v \cdot \nabla_x f \quad &\text{(T)} \\
\partial_t f &= \frac{1}{\varepsilon} Q^F_{\varepsilon}(f,f) + \frac{1}{\varepsilon} Q^L_{\varepsilon}(f,g) \quad &\text{(C)}
\end{align*}
\]

- In order to simulate the evolution of (C) we use a *Monte-Carlo method*.

- The algorithms *cost is linear*, \( O(N_s) \), w.r.t. to the number of sample particles \( N_s \) used to reconstruct the density \( f \).

- The resulting algorithm is fully *meshless* since the binary interactions are non local.

\(^{9}\) Albi-Pareschi '13
Uncontrolled case

Uncontrolled evolution of followers’ density.
Control $u : [0, T] \rightarrow \mathbb{R}^d$ as a “dumb” strategy: $u_k(t) = \left( \frac{x^D - y_k(t)}{|x^D - y_k(t)|} - y_k(t) \right)$.

Evolution of followers’ density under microscopic leaders’ fixed strategy.
Smart strategy

Control $u : [0, T] \rightarrow \mathbb{R}^d$ as a \textquote{smart} strategy such that

$$\min_u J(u) = \int_0^T \left( P(t) + \nu \sum_{k=1}^{N^L} |u_k(t)|^2 \right) dt,$$

where $P(t)$ represents the mass of followers outside exit at time $t$.

Evolution of followers’ density under microscopic leaders’ smart strategy.
A few info

WWW: http://www-m15.ma.tum.de/Allgemeines/MattiaBongini

References: