Sparse Control of Force Field Dynamics
(joint work with M. Fornasier, F. Frölich, and L. Hagverdi)

Mattia Bongini
Technische Universität München,
Department of Mathematics,
Chair of Applied Numerical Analysis
mattia.bongini@ma.tum.de

NetGCoop 2014
Trento
October 29-31, 2014
A framework for social dynamics

We consider large particle systems of the following form:

\[
\begin{align*}
\dot{x}_i &= v_i, \\
\dot{v}_i &= (H \ast \mu_N)(x_i, v_i), \quad i = 1, \ldots N,
\end{align*}
\]

where \( \mu_N = \frac{1}{N} \sum_{j=1}^{N} \delta(x_i, v_i) \).

Several "physical" and "social" forces can be encoded in the interaction kernel \( H \), like

- alignment;
- repulsion-attraction;
- preference of "local" objectives...
A framework for social dynamics

We consider large particle systems of the following form:

\[
\begin{cases}
\dot{x}_i = v_i, \\
\dot{v}_i = (H \ast \mu_N)(x_i, v_i), \ i = 1, \ldots N,
\end{cases}
\]

where \( \mu_N = \frac{1}{N} \sum_{j=1}^{N} \delta(x_i, v_i) \).

Several “physical” and “social” forces can be encoded in the interaction kernel \( H \), like

- alignment;
- repulsion-attraction;
- preference of “local” objectives...

Understanding how superposition of re-iterated binary “social forces” yields global self-organization.
A society is said to be *homophilious* whenever its agents are sharply more influenced by near agents than far ones;
Split coherence in homophilious societies: government?

- A society is said to be *homophilious* whenever its agents are sharply more influenced by near agents than far ones;
- In homophilious societies, global self-organization can be expected as soon as enough initial coherence is reached (Cucker and Smale 2007 – consensus emergence);
Split coherence in homophilious societies: government?

- A society is said to be *homophilious* whenever its agents are sharply more influenced by near agents than far ones;
- In homophilious societies, global self-organization can be expected as soon as enough initial coherence is reached (Cucker and Smale 2007 – consensus emergence);
- However, it is common experience that coherence in a homophilious society can be lost, leading sometimes to dramatic consequences, questioning strongly the role and the effectiveness of governments.
Split coherence in homophilious societies: government?

- A society is said to be *homophilious* whenever its agents are sharply more influenced by near agents than far ones;
- In homophilious societies, global self-organization can be expected as soon as enough initial coherence is reached (Cucker and Smale 2007 – consensus emergence);
- However, it is common experience that coherence in a homophilious society can be lost, leading sometimes to dramatic consequences, questioning strongly the role and the effectiveness of governments.

**Question:** can a government endowed with limited resources rescue/stabilize a society by minimal interventions? Which ones?
A framework for consensus emergence

The Cucker-Smale model is obtained by the choice of the interaction kernel \( H(x, v) = a(|x|)(-v) \).

\[
\begin{align*}
\dot{x}_i &= v_i \in \mathbb{R}^d \\
\dot{v}_i &= \frac{1}{N} \sum_{j=1}^{N} a \left( \|x_i - x_j\|^2 \right) (v_j - v_i) \in \mathbb{R}^d, \text{ for } i = 1, \ldots, N,
\end{align*}
\]

where \( a(r) := a_{\beta}(r) = \frac{1}{(1+r^2)^{\beta}}, \beta > 0 \) models the exchange of information between agents.
A framework for consensus emergence

The Cucker-Smale model is obtained by the choice of the interaction kernel $H(x, v) = a(|x|)(-v)$.

\[
\begin{align*}
\dot{x}_i &= v_i \in \mathbb{R}^d \\
\dot{v}_i &= \frac{1}{N} \sum_{j=1}^{N} a \left( \|x_i - x_j\|^2 \right) (v_j - v_i) \in \mathbb{R}^d, \text{ for } i = 1, \ldots N,
\end{align*}
\]

where $a(r) := a_\beta(r) = \frac{1}{(1+r^2)^\beta}, \beta > 0$ models the exchange of information between agents.

- $\beta \leq \frac{1}{2}$ heterophilious society $\Rightarrow$ unconditional consensus;
A framework for consensus emergence

The **Cucker-Smale model** is obtained by the choice of the interaction kernel $H(x, v) = a(|x|)(-v)$.

\[
\begin{align*}
\dot{x}_i &= v_i \in \mathbb{R}^d \\
\dot{v}_i &= \frac{1}{N} \sum_{j=1}^{N} a \left( \|x_i - x_j\|^2 \right) (v_j - v_i) \in \mathbb{R}^d, \text{ for } i = 1, \ldots N,
\end{align*}
\]

where $a(r) := a_\beta(r) = \frac{1}{(1+r^2)^\beta}$, $\beta > 0$ models the exchange of information between agents.

- $\beta \leq \frac{1}{2}$ heterophilious society $\Rightarrow$ unconditional consensus;
- $\beta > \frac{1}{2}$ homophilious society $\Rightarrow$ consensus conditional to initial coherence.
Homophilious societies are sparsely stabilizable

The work Caponigro-Fornasier-Piccoli-Trélat shows that, in the regime of homophilious society \((\beta > \frac{1}{2})\) the Cucker-Smale system

\[
\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= \frac{1}{N} \sum_{j=1}^{N} a \left( \|x_i - x_j\|^2 \right) (x_j - x_i) + u_i
\end{align*}
\]

can be stabilized to consensus by using only sparse controls, i.e., controls which are zero for almost every agent.
Homophilious societies are sparsely stabilizable

The work Caponigro-Fornasier-Piccoli-Trélat shows that, in the regime of homophilious society \((\beta > \frac{1}{2})\) the Cucker-Smale system

\[
\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= \frac{1}{N} \sum_{j=1}^{N} a \left( \|x_i - x_j\|^2 \right) (x_j - x_i) + u_i
\end{align*}
\]

can be stabilized to consensus by using only sparse controls, i.e., controls which are zero for almost every agent.

If, on the one side, the homophilious character of a society plays against its coherence, on the other side, it plays at its advantage if we allow for sparse external intervention.

Explains the effectiveness of parsimonious interventions of governments in societies.
Dynamical systems driven by attraction and repulsion forces

The **Cucker-Dong model**: for every $1 \leq i \leq N$

\[
\begin{align*}
    \dot{x}_i &= v_i \in \mathbb{R}^d \\
    \dot{v}_i &= -b_i v_i + \sum_{j=1}^{N} a \left( \|x_i - x_j\|^2 \right) (x_j - x_i) + \sum_{\substack{j=1 \\ j \neq i}}^{N} f \left( \|x_i - x_j\|^2 \right) (x_i - x_j) \in \mathbb{R}^d
\end{align*}
\]
Dynamical systems driven by attraction and repulsion forces

The Cucker-Dong model: for every $1 \leq i \leq N$

\[
\begin{align*}
\dot{x}_i &= v_i \in \mathbb{R}^d \\
\dot{v}_i &= -b_i v_i + \sum_{j=1}^{N} a \left( \|x_i - x_j\|^2 \right) (x_j - x_i) + \sum_{\substack{j=1 \atop j \neq i}}^{N} f \left( \|x_i - x_j\|^2 \right) (x_i - x_j) \in \mathbb{R}^d
\end{align*}
\]

where

- $b_i : [0, +\infty) \rightarrow [0, \Lambda]$ is the friction acting on the system,
Dynamical systems driven by attraction and repulsion forces

The Cucker-Dong model: for every $1 \leq i \leq N$

\[
\begin{align*}
\dot{x}_i &= v_i \in \mathbb{R}^d \\
\dot{v}_i &= -b_i v_i + \sum_{j=1}^{N} a \left( \| x_i - x_j \| ^2 \right) (x_j - x_i) + \sum_{\substack{j=1 \\ j \neq i}}^{N} f \left( \| x_i - x_j \| ^2 \right) (x_i - x_j) \in \mathbb{R}^d
\end{align*}
\]

where

- $b_i : [0, +\infty) \rightarrow [0, \Lambda]$ is the friction acting on the system,
- $a : [0, +\infty) \rightarrow [0, +\infty)$ is the rate of communication,
Dynamical systems driven by attraction and repulsion forces

The **Cucker-Dong model**: for every $1 \leq i \leq N$

\[
\begin{align*}
\dot{x}_i &= v_i \in \mathbb{R}^d \\
\dot{v}_i &= -b_iv_i + \sum_{j=1}^{N} a (\|x_i - x_j\|^2) (x_j - x_i) + \sum_{\substack{j=1 \atop j \neq i}}^{N} f (\|x_i - x_j\|^2) (x_i - x_j) \in \mathbb{R}^d
\end{align*}
\]

where

- $b_i : [0, +\infty) \rightarrow [0, \Lambda]$ is the friction acting on the system,
- $a : [0, +\infty) \rightarrow [0, +\infty)$ is the rate of communication,
- $f : (0, +\infty) \rightarrow (0, +\infty)$ such that

\[
\int_{\delta}^{+\infty} f(r) \, dr < \infty \text{ for every } \delta > 0, \quad \int_{0}^{+\infty} f(r) \, dr = +\infty
\]

models the repulsion between agents.
Example: Lennard-Jones potential

- It is the potential of the Van der Waals force.
Example: Lennard-Jones potential

- It is the potential of the Van der Waals force.
- It can be seen as a Cucker-Dong system with

\[ a(r) = \frac{\sigma a}{r^7} \quad \text{and} \quad f(r) = \frac{\sigma f}{r^{13}}. \]
Example: Lennard-Jones potential

- It is the potential of the Van der Waals force.
- It can be seen as a Cucker-Dong system with

\[ a(r) = \frac{\sigma_a}{r^7} \quad \text{and} \quad f(r) = \frac{\sigma_f}{r^{13}}. \]

\[\text{Difference } f(r) - a(r) \text{ for Lennard-Jones potentials.}\]
Total Energy of Cucker-Dong Systems

We introduce

- the **kinetic energy** $K(t) := \frac{1}{2} \sum_{i=1}^{N} \|v_i(t)\|^2$,
Total Energy of Cucker-Dong Systems

We introduce

- the kinetic energy \( K(t) := \frac{1}{2} \sum_{i=1}^{N} \| v_i(t) \|^2 \),
- the potential energy

\[
P(t) := \frac{1}{2} \sum_{i,j=1, i \neq j}^{N} \int_{0}^{\| x_i(t) - x_j(t) \|^2} a(r) \, dr + \frac{1}{2} \sum_{i,j=1, i \neq j}^{N} \int_{\| x_i(t) - x_j(t) \|^2}^{\infty} f(r) \, dr,
\]

Proposition

If the system is frictionless \( (b_i \equiv 0) \) then for every \( t \geq 0 \),
\[
\frac{d}{dt} E(t) = 0.
\]
Total Energy of Cucker-Dong Systems

We introduce

- the kinetic energy \( K(t) := \frac{1}{2} \sum_{i=1}^{N} \|v_i(t)\|^2 \),
- the potential energy

\[
P(t) := \frac{1}{2} \sum_{\substack{i,j=1 \atop i \neq j}}^{N} \int_{0}^{\infty} \|x_i(t) - x_j(t)\|^2 \ a(r) \, dr + \frac{1}{2} \sum_{\substack{i,j=1 \atop i \neq j}}^{N} \int_{\infty}^{\infty} \|x_i(t) - x_j(t)\|^2 \ f(r) \, dr,
\]
- the total energy \( E(t) := K(t) + P(t) \).
Total Energy of Cucker-Dong Systems

We introduce

- **the kinetic energy** \( K(t) := \frac{1}{2} \sum_{i=1}^{N} \|v_i(t)\|^2 \),

- **the potential energy**

\[
P(t) := \frac{1}{2} \sum_{i,j=1}^{N} \int_{0}^{\infty} \|x_i(t) - x_j(t)\|^2 a(r) \, dr + \frac{1}{2} \sum_{i,j=1}^{N} \int_{\infty}^{\infty} \|x_i(t) - x_j(t)\|^2 f(r) \, dr,
\]

- **the total energy** \( E(t) := K(t) + P(t) \).

**Proposition**

If the system is frictionless \((b_i \equiv 0)\) then for every \( t \geq 0 \),

\[
\frac{d}{dt} E(t) = 0.
\]
Conditional consensus emergence

Theorem (Cucker - Dong)

Consider a population of \( N \) agents modeled by a Cucker-Dong system with
\[
a(t) := a_\beta(t) = \frac{1}{(1+t^2)^\beta}, \quad \beta > 0
\]

\[
\|x_i(0) - x_j(0)\| > 0 \text{ for all } i \neq j.
\]

Then there exists a unique solution \((x(t), v(t))\) of the system with initial state \((x(0), v(0))\). Moreover if one of the two following hypotheses holds:
Conditional consensus emergence

Theorem (Cucker - Dong)

Consider a population of $N$ agents modeled by a Cucker-Dong system with $a(t) := a_\beta(t) = \frac{1}{(1+t^2)^\beta}$, $\beta > 0$

$$\|x_i(0) - x_j(0)\| > 0 \text{ for all } i \neq j.$$

Then there exists a unique solution $(x(t), v(t))$ of the system with initial state $(x(0), v(0))$. Moreover if one of the two following hypotheses holds:

1. $\beta \leq 1$, 

2. $\beta > 1$ and $\mathbb{E}(0) < \vartheta := (N-1) \int_0^{\infty} a(r) \, dr$,

then the population is cohesive and collision-avoiding, i.e., there exist two constants $B_0$ and $b_0 > 0$ such that, for every $t \geq 0$

$$b_0 \leq \|x_i(t) - x_j(t)\| \leq B_0 \text{ for all } 1 \leq i \neq j \leq N.$$
Conditional consensus emergence

Theorem (Cucker - Dong)

Consider a population of $N$ agents modeled by a Cucker-Dong system with $a(t) := a_\beta(t) = \frac{1}{(1+t^2)^\beta}$, $\beta > 0$

$$\|x_i(0) - x_j(0)\| > 0 \text{ for all } i \neq j.$$  

Then there exists a unique solution $(x(t), v(t))$ of the system with initial state $(x(0), v(0))$. Moreover if one of the two following hypotheses holds:

1. $\beta \leq 1$,
2. $\beta > 1$ and $E(0) < \vartheta := (N - 1) \int_0^\infty a(r)dr$, 


Conditional consensus emergence

Theorem (Cucker - Dong)

Consider a population of \( N \) agents modeled by a Cucker-Dong system with \( a(t) := a_\beta(t) = \frac{1}{(1+t^2)^\beta}, \beta > 0 \)

\[ ||x_i(0) - x_j(0)|| > 0 \text{ for all } i \neq j. \]

Then there exists a unique solution \((x(t), v(t))\) of the system with initial state \((x(0), v(0))\). Moreover if one of the two following hypotheses holds:

1. \( \beta \leq 1 \),
2. \( \beta > 1 \) and \( E(0) < \vartheta := (N - 1) \int_0^\infty a(r)dr \),

then the population is cohesive and collision-avoiding, i.e., there exist two constants \( B_0 \) and \( b_0 > 0 \) such that, for every \( t \geq 0 \)

\[ b_0 \leq ||x_i(t) - x_j(t)|| \leq B_0 \text{ for all } 1 \leq i \neq j \leq N. \]
Non-consensus events are possible

- We call the conclusion of the Cucker-Dong Theorem the consensus state for the system.
Non-consensus events are possible

- We call the conclusion of the Cucker-Dong Theorem the \textit{consensus state} for the system.
- The theorem says that if the system is heterophilious \((\beta \leq 1)\), the consensus state naturally occurs.
Non-consensus events are possible

- We call the conclusion of the Cucker-Dong Theorem the consensus state for the system.
- The theorem says that if the system is heterophilious ($\beta \leq 1$), the consensus state naturally occurs.
- If $\beta > 1$ then the consensus state is not reached by all $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, as proved by Cucker and Dong.
Non-consensus events are possible

- We call the conclusion of the Cucker-Dong Theorem the consensus state for the system.
- The theorem says that if the system is heterophilious \((\beta \leq 1)\), the consensus state naturally occurs.
- If \(\beta > 1\) then the consensus state is not reached by all \((x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N\), as proved by Cucker and Dong.
- Indeed, the condition \(E(0) < \vartheta\) can be violated in three cases:
Non-consensus events are possible

- We call the conclusion of the Cucker-Dong Theorem the consensus state for the system.
- The theorem says that if the system is heterophilious ($\beta \leq 1$), the consensus state naturally occurs.
- If $\beta > 1$ then the consensus state is not reached by all $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, as proved by Cucker and Dong.
- Indeed, the condition $E(0) < \vartheta$ can be violated in three cases:
  - the agents have too high initial speed $\Rightarrow K$ explodes;
Non-consensus events are possible

- We call the conclusion of the Cucker-Dong Theorem the **consensus state** for the system.
- The theorem says that if the system is heterophilious ($\beta \leq 1$), the consensus state naturally occurs.
- If $\beta > 1$ then the consensus state is **not** reached by all $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, as proved by Cucker and Dong.
- Indeed, the condition $E(0) < \vartheta$ can be violated in three cases:
  - the agents have too high initial speed $\Rightarrow K$ explodes;
  - there are two or more very near agents $\Rightarrow P$ explodes;
Non-consensus events are possible

- We call the conclusion of the Cucker-Dong Theorem the consensus state for the system.
- The theorem says that if the system is heterophilious ($\beta \leq 1$), the consensus state naturally occurs.
- If $\beta > 1$ then the consensus state is not reached by all $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, as proved by Cucker and Dong.
- Indeed, the condition $E(0) < \vartheta$ can be violated in three cases:
  - the agents have too high initial speed $\Rightarrow K$ explodes;
  - there are two or more very near agents $\Rightarrow P$ explodes;
  - a big majority of the agents are very far from each other.
Non-consensus events need intervention

Assume we are in the case $\beta > 1$ and $E(0) \geq \vartheta$. Can we again stabilize the society by external parsimonious intervention?
Non-consensus events need intervention

- Assume we are in the case $\beta > 1$ and $E(0) \geq \vartheta$. Can we again stabilize the society by external parsimonious intervention?
- We introduce a control term inside the model

\[
\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= -b_i v_i + \sum_{j=1}^{N} a (\|x_i - x_j\|^2) (x_j - x_i) + \sum_{j=1, j \neq i}^{N} f (\|x_i - x_j\|^2) (x_i - x_j) + u_i
\end{align*}
\]

where $u_1, \ldots, u_N : [0, +\infty) \rightarrow (\mathbb{R}^d)^N$ are measurable functions satisfying the sparsifying constraint

\[
\sum_{i=1}^{N} \|u_i(t)\| \leq M
\]

for every $t \geq 0$, for a given constant $M > 0$. 

Mattia Bongini
Sparse Control of Force Field Dynamics
Consequences of the introduction of control

Proposition

Assume $b_i \equiv 0$. The total energy is no more a conserved quantity. In particular

$$\frac{d}{dt} E(t) = 2 \langle u(t), v(t) \rangle .$$
Consequences of the introduction of control

Proposition

Assume \( b_i \equiv 0 \). The total energy is no more a conserved quantity. In particular

\[
\frac{d}{dt} E(t) = 2 \langle u(t), v(t) \rangle.
\]

This form of the energy dissipation suggests controls only acting on the kinetic part of the energy:

\[
u_i(t) = -\alpha_i \frac{v_i(t)}{\|v_i(t)\|}, \quad \alpha_i \geq 0.\]
Consequences of the introduction of control

Proposition

Assume $b_i \equiv 0$. The total energy is no more a conserved quantity. In particular

$$\frac{d}{dt} E(t) = 2 \langle u(t), v(t) \rangle.$$ 

- This form of the energy dissipation suggests controls only acting on the kinetic part of the energy:

$$u_i(t) = -\alpha_i \frac{v_i(t)}{\|v_i(t)\|}, \quad \alpha_i \geq 0.$$ 

- The $\ell^N_1 - \ell^d_2$ constraint maximizes the sparsity of $u_i$, i.e. $\alpha_i = 0$ for almost every $i$. 
Introducing the sparse control

**Definition**
Let $0 \leq \varepsilon \leq \frac{M}{E(0)}$ and $t \geq 0$. We define the sparse feedback control with strength $\varepsilon$ to be the vector $u(t) \in (\mathbb{R}^d)^N$ satisfying

$$u_i(t) = \begin{cases} 
-\varepsilon E(t) \frac{v_i(t)}{\|v_i(t)\|} & \text{if } i = \hat{i}(t) \\
0 & \text{if } i \neq \hat{i}(t) 
\end{cases}$$

where $\hat{i}(t) \in \{1, \ldots, N\}$ is the minimum index such that

$$\|v_{\hat{i}(t)}(t)\| = \max_{j=1,\ldots,N} \|v_{j}(t)\|.$$
Introducing the sparse control

Definition
Let $0 \leq \varepsilon \leq \frac{M}{E(0)}$ and $t \geq 0$. We define the sparse feedback control with strength $\varepsilon$ to be the vector $u(t) \in (\mathbb{R}^d)^N$ satisfying

$$u_i(t) = \begin{cases} 
-\varepsilon E(t) \frac{v_i(t)}{\|v_i(t)\|} & \text{if } i = \hat{i}(t) \\
0 & \text{if } i \neq \hat{i}(t)
\end{cases}$$

where $\hat{i}(t) \in \{1, \ldots, N\}$ is the minimum index such that

$$\|v_{\hat{i}(t)}(t)\| = \max_{j=1,\ldots,N} \|v_j(t)\|.$$  

Hence the control acts on the most “stubborn” agent at every time. We may call this control the “shepherd dog strategy”.

Mattia Bongini  
Sparse Control of Force Field Dynamics 13 of 22
Aims of the work

We want to show that

- if $E(0) > \vartheta$ and $E(0) \approx \vartheta \implies$ there is a sampled sparse strategy as before which steers the system to consensus in finite time,
Aims of the work

We want to show that

- if \( E(0) > \vartheta \) and \( E(0) \approx \vartheta \) \( \implies \) there is a \textit{sampled} sparse strategy as before which steers the system to consensus in finite time,
- the sparse control is the minimizer of \( \frac{d}{dt} E(t) \) in a very large set \( U \) of controls satisfying the \( \ell_1^N - \ell_2^d \) constraint,
Aims of the work

We want to show that

- if $E(0) > \vartheta$ and $E(0) \approx \vartheta \implies$ there is a sampled sparse strategy as before which steers the system to consensus in finite time,
- the sparse control is the minimizer of $\frac{d}{dt} E(t)$ in a very large set $U$ of controls satisfying the $\ell_1^N - \ell_2^d$ constraint,
- for some $u \in U$, there exists a solution of the system

\[
\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= -b_i v_i + \sum_{j=1}^{N} a \left( \|x_i - x_j\|^2 \right) \left( x_j - x_i \right) + \sum_{\substack{j=1 \\ j \neq i}}^{N} f \left( \|x_i - x_j\|^2 \right) \left( x_i - x_j \right) + u_i.
\end{align*}
\]
Sampling and hold

Strategy of proof: we will follow a *sampling-and-hold* approach as in Caponigro-Fornasier-Piccoli-Trélat.
Sampling and hold

Strategy of proof: we will follow a *sampling-and-hold* approach as in Caponigro-Fornasier-Piccoli-Trélat.

- We will take a sampling time $\tau$ and consider the system

$$
\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= -b_i v_i + \sum_{j=1}^{N} a(\|x_i - x_j\|^2) (x_j - x_i) + \sum_{j=1 \atop j \neq i}^{N} f(\|x_i - x_j\|^2) (x_i - x_j) + \tilde{u}_i
\end{align*}
$$

such that the control satisfies $\tilde{u}_i(t) = u_i(k\tau)$ for every $t \in [k\tau, (k + 1)\tau], k \in \mathbb{N}$, where $u$ is the sparse control;
Sampling and hold

Strategy of proof: we will follow a *sampling-and-hold* approach as in Caponigro-Fornasier-Piccoli-Trélat.

- We will take a **sampling time** $\tau$ and consider the system

\[
\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= -b_i v_i + \sum_{j=1}^{N} a \left( \|x_i - x_j\|^2 \right) (x_j - x_i) + \sum_{\substack{j=1 \\ j \neq i}}^{N} f \left( \|x_i - x_j\|^2 \right) (x_i - x_j) + \tilde{u}_i
\end{align*}
\]

such that the control satisfies $\tilde{u}_i(t) = u_i(k\tau)$ for every $t \in [k\tau, (k + 1)\tau], k \in \mathbb{N}$, where $u$ is the sparse control;

- if $\tau$ is sufficiently small we avoid chattering phenomena;
Sampling and hold

**Strategy of proof:** we will follow a *sampling-and-hold* approach as in Caponigro-Fornasier-Piccoli-Trélat.

- We will take a **sampling time** $\tau$ and consider the system

\[
\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= -b_i v_i + \sum_{j=1}^{N} a(\|x_i - x_j\|^2) (x_j - x_i) + \sum_{\substack{j=1 \\ j \neq i}}^{N} f(\|x_i - x_j\|^2) (x_i - x_j) + \tilde{u}_i 
\end{align*}
\]

such that the control satisfies $\tilde{u}_i(t) = u_i(k\tau)$ for every $t \in [k\tau, (k + 1)\tau], k \in \mathbb{N}$, where $u$ is the sparse control;

- if $\tau$ is sufficiently small we avoid chattering phenomena;

- if the control is sufficiently strong (i.e., the parameter $\varepsilon$ is sufficiently large) the system is steered to satisfy $E(t) < \vartheta$ in finite time.
Sampled sparse strategies drives the system to consensus

Main Theorem (B. - Fornasier)

Fix $M > 0$. Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ be such that the following hold:

1. $\|x_{0i} - x_{0j}\| > 0$ for all $i \neq j$, 

2. $\|\sum_{i=1}^{N} v_i(0)\| > 0$,

3. $E(0) \geq \vartheta > E(0) \exp\left(-\frac{2}{3} \sqrt{\frac{9}{M}}\right)$.

Then there exist $\tau_0 > 0$, $L > 0$, and $T > 0$ such that the sampling solution of the Cucker-Dong system associated with the sparse control with strength $\varepsilon \geq L$, the sampling time $\tau \leq \tau_0$ and initial datum $(x_0, v_0)$ reaches the consensus region in finite time $T$. 

Sampled sparse strategies drives the system to consensus

Main Theorem (B. - Fornasier)

Fix \( M > 0 \). Let \( (x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \) be such that the following hold:

1. \( \|x_{0i} - x_{0j}\| > 0 \) for all \( i \neq j \),

2. \( \left\| \frac{1}{N} \sum_{i=1}^{N} v_i(0) \right\| > 0 \),
Sampled sparse strategies drives the system to consensus

Main Theorem (B. - Fornasier)

Fix $M > 0$. Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ be such that the following hold:

1. $\|x_{0i} - x_{0j}\| > 0$ for all $i \neq j$,
2. $\|\frac{1}{N} \sum_{i=1}^{N} v_i(0)\| > 0$,
3. $E(0) \geq \vartheta > E(0) \exp \left( -\frac{2\sqrt{3}}{9} \frac{M\|\frac{1}{N} \sum_{i=1}^{N} v_i(0)\|^3}{E(0)\sqrt{E(0)}(\Lambda\sqrt{E(0)} + \frac{M}{N})} \right)$.
Sampled sparse strategies drives the system to consensus

Main Theorem (B. - Fornasier)

Fix $M > 0$. Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ be such that the following hold:

1. $\|x_{0i} - x_{0j}\| > 0$ for all $i \neq j$,
2. $\|\frac{1}{N} \sum_{i=1}^N v_i(0)\| > 0$,
3. $E(0) \geq \vartheta > E(0) \exp \left( - \frac{2\sqrt{3}}{9} \frac{M \|\frac{1}{N} \sum_{i=1}^N v_i(0)\|^3}{E(0) \sqrt{E(0) \left( \Lambda \sqrt{E(0)} + \frac{M}{N} \right)}} \right)$.

Then there exist $\tau_0 > 0$, $L > 0$ and $T > 0$ such that the sampling solution of the Cucker-Dong system associated with the sparse control $u$ with strength $\varepsilon \geq L$, the sampling time $\tau \leq \tau_0$ and initial datum $(x_0, v_0)$ reaches the consensus region in finite time $T$. 
Enlarging the set of admissible controls

The above result cannot be used to prove directly the existence of a solution for controlled Cucker-Dong systems, because if we let $\tau$ in the Main Theorem go to 0 we usually do not obtain a sparse control.
Enlarging the set of admissible controls

- The above result cannot be used to prove directly the existence of a solution for controlled Cucker-Dong systems, because if we let $\tau$ in the Main Theorem go to 0 we usually do not obtain a sparse control.
- We thus need to enlarge the set of admissible control to obtain an existence result with this argument.
Enlarging the set of admissible controls

- The above result cannot be used to prove directly the existence of a solution for controlled Cucker-Dong systems, because if we let $\tau$ in the Main Theorem go to 0 we usually do not obtain a sparse control.
- We thus need to enlarge the set of admissible control to obtain an existence result with this argument.
- Define for every $t > 0$ the set

$$K(t) := \left\{ u \in (\mathbb{R}^d)^N \left| \sum_{i=1}^{N} \|u_i\| \leq M \cdot \frac{E(t)}{E(0)} \right. \right\},$$

and for every $t > 0$ and $q > 0$ the functional $\mathcal{J}_{t,q} : (\mathbb{R}^d)^N \rightarrow \mathbb{R}$

$$\mathcal{J}_{t,q}(u) = \langle v(t), u \rangle + \frac{1}{N} \sum_{i=1}^{N} v_i(0) \frac{1}{q} \sum_{i=1}^{N} \|u_i\|. $$
Existence of solutions

Theorem (B. - Fornasier)

*If the hypotheses of the Main Theorem are satisfied, then there exist* $T > 0$ *and* $q > 0$ *such that*

\[ \text{the sparse feedback control belongs to the set} \]
\[ \arg\min_{u \in K(t)} J_{t,q}(u) \quad \text{for every} \quad t \leq T; \]

\[ \text{there exists a solution of the system} \]
\[ \begin{cases} 
\dot{x}_i = v_i \\
\dot{v}_i = -b_i v_i + \sum_{j=1}^{N} a(\|x_i - x_j\|_2)(x_j - x_i) + \sum_{j=1}^{N} j \neq i f(\|x_i - x_j\|_2)(x_i - x_j) + \tilde{u}_i 
\end{cases} \]

\[ \text{associated to a control} \]
\[ \tilde{u} \in \arg\min_{u \in K(t)} J_{t,q}(u) \quad \text{for every} \quad t \leq T. \]
Existence of solutions

Theorem (B. - Fornasier)

*If the hypotheses of the Main Theorem are satisfied, then there exist $T > 0$ and $q > 0$ such that*

- the sparse feedback control belongs to the set $\arg \min_{u \in K(t)} J_{t,q}(u)$ for every $t \leq T$;
Existence of solutions

Theorem (B. - Fornasier)

If the hypotheses of the Main Theorem are satisfied, then there exist $T > 0$ and $q > 0$ such that

- the sparse feedback control belongs to the set $\arg\min_{u \in K(t)} J_{t,q}(u)$ for every $t \leq T$;
- there exists a solution of the system

$$\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= -b_i v_i + \sum_{j=1}^{N} a \left( \|x_i - x_j\|^2 \right) (x_j - x_i) + \sum_{j=1}^{N} f \left( \|x_i - x_j\|^2 \right) (x_i - x_j) + \tilde{u}_i
\end{align*}$$

associated to a control $\tilde{u} \in \arg\min_{u \in K(t)} J_{t,q}(u)$ for every $t \leq T$. 
Exponential decay rate of the energy

**Theorem (B. - Fornasier)**

Suppose we are under the assumptions of the Main Theorem. The sparse feedback control is then an instantaneous minimizer of the functional

\[ D(t, u) = \frac{d}{dt} E(t) \]

over all possible feedback controls in \( \arg\min_{u \in K(t)} \mathcal{J}_{t,q}(u) \).
Exponential decay rate of the energy

Theorem (B. - Fornasier)

Suppose we are under the assumptions of the Main Theorem. The sparse feedback control is then an instantaneous minimizer of the functional

\[ D(t, u) = \frac{d}{dt} E(t) \]

over all possible feedback controls in \( \arg\min_{u \in K(t)} J_{t, q}(u) \).

Moreover for the sparse feedback control strategy we have for every \( t \geq 0 \),

\[ E(t) \leq E(0) e^{-\frac{2\| \frac{1}{N} \sum_{i=1}^{N} v_i(0) \|}{E(0)} Mt}. \]
Summing up our results

- Again we have proven that an homophilious society can be stabilized by parsimonious intervention;
Summing up our results

- Again we have proven that an homophilious society can be stabilized by parsimonious intervention;
- the sparse control strategy is the most efficient: we pay our attention solely to the “most stubborn” agent while leaving the other free to adjust themselves;
Summing up our results

- Again we have proven that an homophilious society can be stabilized by parsimonious intervention;
- the sparse control strategy is the most efficient: we pay our attention solely to the “most stubborn” agent while leaving the other free to adjust themselves;
- in contrast to what happen with the Cucker-Smale model, our result is conditional (it depends on the initial conditions of the system).
Summing up our results

- Again we have proven that an homophilious society can be stabilized by parsimonious intervention;
- the sparse control strategy is the most efficient: we pay our attention solely to the “most stubborn” agent while leaving the other free to adjust themselves;
- in contrast to what happen with the Cucker-Smale model, our result is conditional (it depends on the initial conditions of the system).

⇒ We don’t know if the conditions are necessary.
A numerical experiment

Consider a frictionless Cucker-Dong system with 8 agents, $d = 2$, $\beta = 1.02$, and $f(r) = 1/r^{1.1}$. 
A numerical experiment

Consider a frictionless Cucker-Dong system with 8 agents, $d = 2$, $\beta = 1.02$, and $f(r) = 1/r^{1.1}$.

No control active
A numerical experiment

Consider a frictionless Cucker-Dong system with 8 agents, $d = 2$, $\beta = 1.02$, and $f(r) = 1/r^{1.1}$.

No control active

Sparse control with $M = 1$
A numerical experiment

Consider a frictionless Cucker-Dong system with 8 agents, $d = 2$, $\beta = 1.02$, and $f(r) = 1/r^{1.1}$.

Energy in function of time

No control active

Sparse control with $M = 1$
A numerical experiment

Consider a frictionless Cucker-Dong system with 8 agents, $d = 2$, $\beta = 1.02$, and $f(r) = 1/r^{1.1}$.

No control active

Sparse control with $M = 1$

Sparse control with $M = 1$
A few info

- **WWW:** http://www-m15.ma.tum.de/

- **References:**
  - M. Bongini and M. Fornasier, *Sparse stabilization of dynamical systems driven by attraction and avoidance forces*, Networks and Heterogeneous Media, Pages 1 - 31, Volume 9, Issue 1, March 2014
  - M. Bongini, D. Kalise, and M. Fornasier, *(Un)conditional consensus emergence under perturbed and decentralized feedback controls*, to appear in Discrete and Continuous Dynamical Systems, pp. 27