Sparse Stabilization of Dynamical Systems driven by Attraction and Avoidance Forces

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Introduction

Large particle systems arise in many modern applications:

- **Large Facebook “friendship” network**
- **Image halftoning via variational dithering**
- **Dynamical data analysis: R. palustris protein-protein interaction network**
- **Computational chemistry: molecule simulation**
Split coherence in homophilious societies: government?

- A society is said to be *homophilious* whenever its agents are sharply more influenced by near agents than far ones;

- In homophilious societies, global self-organization can be expected as soon as enough initial coherence is reached (Cucker and Smale 2007 – consensus emergence);

- However, it is common experience that coherence in a homophilious society can be lost, leading sometimes to dramatic consequences, questioning strongly the role and the effectiveness of governments.

Question: can a government endowed with limited resources rescue/stabilize a society by minimal interventions? Which ones?
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A framework for consensus emergence

The Cucker-Smale model is one of the most famous models for social dynamics.

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\begin{align*}
\dot{x}_i &= v_i \in \mathbb{R}^d \\
\dot{v}_i &= \frac{1}{N} \sum_{j=1}^{N} a\left(\|x_i - x_j\|^2\right) (v_j - v_i) \in \mathbb{R}^d
\end{align*}
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where \( a(t) := a_\beta(t) = \frac{1}{(1+t^2)\beta} , \beta > 0 \) models the exchange of information between agents.
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- \( \beta \leq \frac{1}{2} \) heterophilious society \( \Rightarrow \) unconditional consensus,
- \( \beta > \frac{1}{2} \) homophilious society \( \Rightarrow \) consensus conditional to initial coherence.

Homophilious societies are sparsely stabilizable

- The work Caponigro-Fornasier-Piccoli-Trélat shows that, in the regime of homophilious society ($\beta > \frac{1}{2}$) the Cucker-Smale system

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\dot{x}_i &= v_i \\
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can be stabilized to consensus by using only sparse controls, i.e. controls which are zero for almost every agent.
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If, on the one side, the homophilious character of a society plays against its coherence, on the other side, it plays at its advantage if we allow for sparse external intervention.
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If, on the one side, the homophilious character of a society plays against its coherence, on the other side, it plays at its advantage if we allow for sparse external intervention.

- Explains the effectiveness of parsimonious interventions of governments in societies.
Dynamical systems driven by attraction and repulsion forces

The Cucker-Dong model: for every $1 \leq i \leq N$

$$\begin{cases} 
\dot{x}_i = v_i \in \mathbb{R}^d \\
\dot{v}_i = -b_i v_i + \sum_{j=1}^{N} a \left( \|x_i - x_j\|^2 \right) (x_j - x_i) + \sum_{\substack{j=1 \\ j \neq i}}^{N} f \left( \|x_i - x_j\|^2 \right) (x_i - x_j) \in \mathbb{R}^d 
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where

- $b_i : [0, +\infty) \to [0, \Lambda]$ is the friction acting on the system,
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- $a : [0, +\infty) \rightarrow [0, +\infty)$ is the rate of communication,
- $f : (0, +\infty) \rightarrow (0, +\infty)$ such that

$$\int_{\delta}^{+\infty} f(r) \, dr < \infty \text{ for every } \delta > 0, \quad \int_{0}^{+\infty} f(r) \, dr = +\infty$$

models the repulsion between agents.
Example: Lennard-Jones potential

- It is the potential of the Van der Waals force.
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- It can be seen as a Cucker-Dong system with

\[ a(r) = \frac{\sigma_a}{r^7} \quad \text{and} \quad f(r) = \frac{\sigma_f}{r^{13}}. \]
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Total Energy of Cucker-Dong Systems

We introduce

- the kinetic energy \( K(t) := \frac{1}{2} \sum_{i=1}^{N} \|v_i(t)\|^2 \),

- the potential energy \( P(t) := \frac{1}{2} \sum_{i,j=1}^{N} \int_0^a \|x_i(t) - x_j(t)\|^2 0 dr + \frac{1}{2} \sum_{i,j=1}^{N} \int_a^\infty \|x_i(t) - x_j(t)\|^2 f(r) dr \),

- the total energy \( E(t) := K(t) + P(t) \).

Proposition

If the system is frictionless (\( b_i \equiv 0 \)) then for every \( t \geq 0 \),
\[ \frac{d}{dt} E(t) = 0. \]
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*If the system is frictionless \( b_i \equiv 0 \) then for every \( t \geq 0 \), \( \frac{d}{dt} E(t) = 0 \).*
Conditional consensus emergence

Theorem (Cucker - Dong)

Consider a population of $N$ agents modeled by a Cucker-Dong system with $a(t) := a_\beta(t) = \frac{1}{(1+t^2)^\beta}$, $\beta > 0$

\[ \|x_i(0) - x_j(0)\| > 0 \text{ for all } i \neq j. \]

Then there exists a unique solution $(x(t), v(t))$ of the system with initial state $(x(0), v(0))$. Moreover if one of the two following hypotheses holds:
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1. $\beta \leq 1$

2. $\beta > 1$ and $E(0) < \vartheta := (N - 1) \int_0^\infty a(r) \, dr$ then the population is cohesive and collision-avoiding, i.e., there exist two constants $B_0$ and $b_0 > 0$ such that, for every $t \geq 0$ $b_0 \leq \|x_i(t) - x_j(t)\| \leq B_0$ for all $1 \leq i \neq j \leq N$. 

Mattia Bongini Sparse Stabilization of Dynamical Systems driven by Attraction and Avoidance Forces 9 of 24
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Example: a system satisfying the hypotheses
Non-consensus events are possible

- We call the conclusion of the Cucker-Dong Theorem the consensus state for the system.
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- Indeed, the condition \( E(0) < \vartheta \) can be violated in three cases:
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- Indeed, the condition $E(0) < \vartheta$ can be violated in three cases:
  - the agents have too high initial speed $\Rightarrow K$ explodes;
  - there are two or more very near agents $\Rightarrow P$ explodes.
  - a big majority of the agents are very far from each other.
Example: a system not satisfying the hypotheses
Non-consensus events need intervention

- Assume we are in the case \( \beta > 1 \) and \( E(0) \geq \vartheta \). Can we again stabilize the society by external parsimonious intervention?
Non-consensus events need intervention

- Assume we are in the case $\beta > 1$ and $E(0) \geq \vartheta$. Can we again stabilize the society by external parsimonious intervention?
- We introduce a control term inside the model

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\dot{v}_i &= -b_i v_i + \sum_{j=1}^{N} a \left( \|x_i - x_j\|^2 \right) (x_j - x_i) + \sum_{j=1, j \neq i}^{N} f \left( \|x_i - x_j\|^2 \right) (x_i - x_j) + u_i
\end{align*}
$$

where $u_1, \ldots, u_N : [0, +\infty) \to (\mathbb{R}^d)^N$ are measurable functions satisfying

$$
\sum_{i=1}^{N} \|u_i(t)\| \leq M
$$

for every $t \geq 0$, for a given constant $M > 0$. 
Consequences of the introduction of control

Proposition

Assume $b_i \equiv 0$. The total energy is no more a conserved quantity. In particular

$$\frac{d}{dt} E(t) = 2 \langle u(t), v(t) \rangle .$$
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This form of the energy dissipation suggests controls only acting on the kinetic part of the energy:

$$u_i(t) = -\alpha_i \frac{v_i(t)}{\|v_i(t)\|}, \quad \alpha_i \geq 0.$$
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- The \( \ell^N_1 - \ell^d_2 \) constraint maximizes the sparsity of \( u_i \), i.e. \( \alpha_i = 0 \) for almost every \( i \).
Introducing the sparse control

Definition
Let $0 \leq \varepsilon \leq \frac{M}{E(0)}$ and $t \geq 0$. We define the sparse feedback control with strength $\varepsilon$ $u(t) \in (\mathbb{R}^d)^N$ as

$$u_i(t) = \begin{cases} -\varepsilon E(t) \frac{v_i(t)}{\|v_i(t)\|} & \text{if } i = \hat{i}(t) \\ 0 & \text{if } i \neq \hat{i}(t) \end{cases}$$

where $\hat{i}(t) \in \{1, \ldots, N\}$ is the minimum index such that

$$\|v_{\hat{i}}(t)(t)\| = \max_{j=1,\ldots,N} \|v_j(t)\|.$$
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\|v_{\hat{i}(t)}(t)\| = \max_{j=1,\ldots,N} \|v_j(t)\|.
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Hence the control acts on the most “stubborn” agent at every time. We may call this control the “shepherd dog strategy”.

Aims of the work

We want to show that

- if $E(0) > \vartheta$ and $E(0) \approx \vartheta$ \implies there is \textit{sampled} sparse strategy as before which steers the system to consensus in finite time,
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We want to show that

- if $E(0) > \vartheta$ and $E(0) \approx \vartheta \implies$ there is sampled sparse strategy as before which steers the system to consensus in finite time,
- the sparse control minimizes $\frac{d}{dt}E(t)$ in a very large set $U$ of controls satisfying the $\ell_1^N - \ell_2^d$ constraint,
- if $u \in U$, there exists a solution of the system

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\end{align*}$$
Sampling and hold

**Strategy of proof:** we will follow a *sampling-and-hold* approach as in Caponigro-Fornasier-Piccoli-Trélat.

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\[\text{such that the control satisfies } \tilde{u}_i(t) = u_i(k\tau) \text{ for every } t \in [k\tau, (k+1)\tau), \ k \in \mathbb{N}, \text{ where } u \text{ is the sparse control;}
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- if \(\tau\) is sufficiently small we avoid chattering phenomena;
- if the control is sufficiently strong (i.e. the parameter \(\varepsilon\) is sufficiently large) the system is steered to satisfy \(E(t) < \vartheta\) in finite time.
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\dot{v}_i &= -b_i v_i + \sum_{j=1}^{N} a \left( \|x_i - x_j\|^2 \right) (x_j - x_i) + \sum_{j=1 \atop j \neq i}^{N} f \left( \|x_i - x_j\|^2 \right) (x_i - x_j) + \tilde{u}_i
\end{align*}
\]

such that the control satisfies $\tilde{u}_i(t) = u_i(k\tau)$ for every $t \in [k\tau, (k + 1)\tau]$, $k \in \mathbb{N}$, where $u$ is the sparse control;

- if $\tau$ is sufficiently small we avoid chattering phenomena;

- if the control is sufficiently strong (i.e. the parameter $\varepsilon$ is sufficiently large) the system is steered to satisfy $E(t) < \vartheta$ in finite time.
Sampled sparse strategies drives the system to consensus

Main Theorem (B. - Fornasier)

Fix $M > 0$. Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ be such that the following hold:

1. $\|x_{0i} - x_{0j}\| > 0$ for all $i \neq j$,
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3. $E(0) \geq \vartheta > E(0) \exp \left( -\frac{2\sqrt{3}}{9} \frac{M\|\frac{1}{N} \sum_{i=1}^{N} v_i(0)\|^3}{E(0) \sqrt{E(0)}} \right)$.

Then there exist $\tau_0 > 0$, $L > 0$ and $T > 0$ such that the sampling solution of the Cucker-Dong system associated with the sparse control $u$ with strength $\varepsilon \geq L$, the sampling time $\tau \leq \tau_0$ and initial datum $(x_0, v_0)$ reaches the consensus region in finite time $T$. 
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Enlarging the set of admissible controls

The above result cannot be used to prove directly the existence of a solution for controlled Cucker-Dong systems, because if we let $\tau$ in the Main Theorem go to 0 we usually do not obtain a sparse control.

Define for every $t > 0$ the set $K(t) := \{ u \in \mathbb{R}^d \mid \sum_{i=1}^N \|u_i\| \leq M \cdot E(t) E(0) \}$, and for every $t > 0$ and $q > 0$ the functional $J_{t,q}(u) = \langle v(t), u \rangle + \|1\| \sum_{i=1}^N v_i(0) \|u_i\|$. 

Mattia Bongini Sparse Stabilization of Dynamical Systems driven by Attraction and Avoidance Forces
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and for every $t > 0$ and $q > 0$ the functional $\mathcal{J}_{t,q} : (\mathbb{R}^d)^N \to \mathbb{R}$

$$\mathcal{J}_{t,q}(u) = \langle v(t), u \rangle + \frac{\frac{1}{N} \sum_{i=1}^N v_i(0)}{q} \sum_{i=1}^N \|u_i\|. $$
Existence of solutions

Theorem (B. - Fornasier)

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Theorem (B. - Fornasier)

If the hypotheses of the Main Theorem are satisfied, then there exist $T > 0$ and $q > 0$ such that

- the sparse feedback control belongs to the set $\text{argmin}_{u \in K(t)} J_{t,q}(u)$ for every $t \leq T$;
Existence of solutions

Theorem (B. - Fornasier)

If the hypotheses of the Main Theorem are satisfied, then there exist $T > 0$ and $q > 0$ such that

- the sparse feedback control belongs to the set $\arg\min_{u \in K(t)} \mathcal{J}_{t,q}(u)$ for every $t \leq T$;
- there exists a solution of the system

\[
\begin{cases}
\dot{x}_i = v_i \\
\dot{v}_i = -b_i v_i + \sum_{j=1}^{N} a \left( \|x_i - x_j\|^2 \right) (x_j - x_i) + \sum_{\substack{j=1 \\ \text{瘠} \neq i}}^{N} f \left( \|x_i - x_j\|^2 \right) (x_i - x_j) + \tilde{u}_i
\end{cases}
\]

associated to a control $\tilde{u} \in \arg\min_{u \in K(t)} \mathcal{J}_{t,q}(u)$ for every $t \leq T$. 

Exponential decay rate of the energy

**Theorem (B. - Fornasier)**

Suppose we are under the assumptions of the Main Theorem. The sparse feedback control is then an instantaneous minimizer of the functional

\[ D(t, u) = \frac{d}{dt} E(t) \]

over all possible feedback controls in \( \arg\min_{u \in K(t)} J_{t, q}(u) \).
Exponential decay rate of the energy

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$$\mathcal{D}(t, u) = \frac{d}{dt} E(t)$$

over all possible feedback controls in $\arg\min_{u \in K(t)} J_{t,q}(u)$.

Moreover for the sparse feedback control strategy we have for every $t \geq 0$,

$$E(t) \leq E(0) e^{-\frac{2\left\| \frac{1}{N} \sum_{i=1}^{N} v_i(0) \right\|_{E(0)}}{Mt}}.$$

Summing up our results

- Again we have proven that an homophilious society can be stabilized by \textit{parsimonious intervention};
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- the sparse control strategy is the most efficient: we pay our attention solely to the “most stubborn” agent while leaving the other free to adjust themselves;
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Summing up our results

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- The sparse control strategy is the most efficient: we pay our attention solely to the “most stubborn” agent while leaving the other free to adjust themselves;
- In contrast to what happen with the Cucker-Smale model, our result is conditional (it depends on the initial conditions of the system)
  ⇒ we don’t know if the conditions are necessary.
A numerical experiment

Consider a frictionless Cucker-Dong system with 8 agents, \( d = 2 \), \( \beta = 1.02 \), and \( f(r) = 1/r^{1.1} \).
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Sparse control with $M = 1$
A few info

- **WWW:** http://www-m15.ma.tum.de/

- **References:**
  - M. Bongini and M. Fornasier, *Sparse stabilization of dynamical systems driven by attraction and avoidance forces*, to appear in Networks and Heterogeneous Media, pp. 32