Mathematics of Digitalization: Case Study

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Second lecture
What is the meaning of the Fourier transform? The Fourier transform represents the frequency content of a function/signal. It gives us which are the important oscillating building blocks of a signal and their distinctive frequency of oscillation.

Figure 1: Sinusoidal signal affected by noise: there are two fundamental frequency components
The fortune of Fourier analysis stays essentially in the fact that it is able to describe one of the most frequent behavior in the nature: the wave phenomena, most of which are governed by overlapping laws of the type

\[ y_{\alpha,s_0}(t) = \begin{cases} e^{-\alpha t} e^{2\pi i s_0 t}, & t > 0 \\ 0, & t < 0. \end{cases} \]

\( y_{\alpha,s_0}(t) \) is called Lorenzian function.
When, for example, some molecules are excited by electromagnetic radiation, damped oscillations are induced and described by the law \( y_{\alpha,s_0}(t) \).

Every building block of the molecule has its own unique special oscillations. The Lorenzians are called in this case the \textit{molecular spectrum}.

From here comes the idea which “costed” the Nobel prize to Richard Ernst for chemistry (1991) for the development of a powerful instrument to determine the structure of large complex organic molecules.
We want now to discuss how in practice we can use the tools of the Fourier analysis.

The exercises we showed on simple functions show us that it is not so trivial to compute Fourier series or transforms.

For relatively simple functions such as \( f(x) = \frac{1}{a^2 + x^2} \) the computation of the Fourier transform is far from being trivial. In this case we have \( \mathcal{F} f(w) = \frac{\pi}{a} e^{-|w|a} \).
A holomorphic complex function $f(z)$ (i.e., differentiable) on a domain $\Omega$ except for a point $a$, called singularity, can be expanded in a power series, called the Laurent series,

$$f(z) = \sum_{n \in \mathbb{Z}} c_n (z - a)^n.$$  

The specific coefficient $c_{-1}$ is called residual and we write $Res(f, a) = c_{-1}$. One can show that if $f$ is holomorphic on a domain $\Omega$ which contains the halfplane $\{\text{Im}(z) \geq 0\}$ except for a finite number of non real singularities and if the limit $\lim_{|z| \to \infty, \text{Im}(z) \geq 0} f(z) = 0$ then we have for every $\alpha > 0$,

$$\lim_{r \to \infty} \int_{-r}^{r} f(x)e^{i\alpha x} \, dx = 2\pi i \sum \{Res(f(\cdot)e^{i\alpha \cdot}, a), \text{Im}(a) > 0\}.$$

Analogous result holds for $\alpha < 0$ and for $\{\text{Im}(z) < 0\}$, but changing the sign of the integral.
It is certainly more reasonable to search for an approximation on a discrete domain and a proper definition of discrete Fourier transform, which, in some sense, can approximate well the behavior of the continuous Fourier transform. Moreover, we need to make sure that the computation of the discrete Fourier transform (DFT) can be executed in reasonable time: otherwise it does not pay off.
Let’s proceed as usual in the language of the Hilbert spaces. We consider now the cyclic group \( \mathbb{Z}_n = \frac{\mathbb{Z}}{n\mathbb{Z}}, n \in \mathbb{N}\setminus\{0\} \) of the rest classes modulo \( n \). In practice we can write

\[
\mathbb{Z}_n = \{0, 1, \ldots, n - 1\},
\]

reminding us that such elements represent the rest of the integer number with respect to the division by \( n \). For example, in \( \mathbb{Z}_3 \) the element \( h = 4 \) corresponds to 1 as \( 3 = 3 \cdot 1 + 1 \). Hence, in particular, \( 2 + 2 = 1 \) in \( \mathbb{Z}_3 \). Moreover, \( \ell^2(\mathbb{Z}_n) = \{(c_0, \ldots, c_{n-1})| \sum_{k=0}^{n-1} |c_i|^2 < \infty \} = \mathbb{C}^n \).
One can show by induction (exercise!)

\[ 1 + z + z^2 + \cdots + z^{n-1} = \begin{cases} n, & z = 1 \\ \frac{z^n - 1}{z - 1}, & \text{otherwise.} \end{cases} \quad (1) \]

But then it is not difficult to show (exercise!) that for 
\[ z = e^{2\pi i (k-l)/n} \] we have

\[
\langle \frac{1}{\sqrt{n}} (e^{2\pi i kl}/n)_{l \in \mathbb{Z}_n}, \frac{1}{\sqrt{n}} (e^{2\pi i lm}/n)_{m \in \mathbb{Z}_n} \rangle = \sum_{m=0}^{n-1} e^{2\pi im(k-l)/n} = \delta_{k,l}.
\]

Hence \( \{ \frac{1}{\sqrt{n}} (e^{2\pi i kl}/n)_{l \in \mathbb{Z}_n} \}_{k \in \mathbb{Z}_n} \) is an orthonormal basis for the finite dimensional (Hilbert) space \( \ell^2(\mathbb{Z}_n) \).
Every discrete signal $f$ of length $n$ can be written as:

$$f = \frac{1}{n} \sum_{k=0}^{n-1} \langle f, (e^{2\pi i kl/n})_{l \in \mathbb{Z}_n} \rangle (e^{2\pi i kl/n})_{l \in \mathbb{Z}_n}. \tag{2}$$

Moreover we have that

$$\mathcal{F}f(k) = \frac{1}{\sqrt{n}} \langle f, (e^{2\pi i kl/n})_{l \in \mathbb{Z}_n} \rangle = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} f(l) e^{-2\pi i kl/n}, \tag{3}$$

is again a signal of length $n$. The operator $\mathcal{F} : \ell^2(\mathbb{Z}_n) \rightarrow \ell^2(\mathbb{Z}_n)$ is called discrete Fourier transform or DFT and it realizes all the properties of the Fourier transforms already defined before for periodic functions and functions on the real line. For example, prove by exercise that

$$\|f\|_{\ell^2} = \|\mathcal{F}f\|_{\ell^2}.$$
If we assume that one operation is either a sum or a multiplication, then the computational cost of a DFT is of $2n$ sums and multiplications times $n$, i.e., $2n^2$. As one complex operation costs twice a single real, the total costs is $C(DFT)(n) = 4n^2$. If we had a PC able to compute $10^7$ operations per second and we are able to compute a DFT of a signal of length $n = 1000$ in

$$4 \cdot 1000^2 \cdot \frac{1}{10^7} = 0.4 \text{ sec}.$$ 

But already for a signal of length $n = 16384 = 2^{14}$ the cost is of $107 \text{ sec}$. A simple audio signal of few seconds, sufficiently densely sampled, can easily reach length $n = 2^{14}$!.