Fallstudien der Mathematischen Modellbildung

Exercise Sheet 5

November 2, 2015

Definition 1. Given $g \in L^2(\mathbb{R})$ and $a, b > 0$ we say that the triple (g, a, b) is a *Gabor frame for* $L^2(\mathbb{R})$ if the set $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$.

Problem 1. Prove that if $g \in L^2(\mathbb{R})$ and $a, b > 0$ satisfies:

1. there exist constants $A, B > 0$ such that $A \leq \left|\sum_{n\in\mathbb{Z}} g(x - na)\right|^2 \leq B$ holds almost everywhere in $\mathbb{R}$,
2. there exists $c \in \mathbb{R}$ such that $\mathrm{supp}(g) \subset \left[c, c + \frac{1}{b}\right]$.

Then (g, a, b) is a Gabor frame for $L^2(\mathbb{R})$ with frame bounds $\frac{A}{b}, \frac{B}{b}$.

Problem 2. Let $g \in L^2(\mathbb{R})$ be the function

$$g(x) = \begin{cases} 1 + x & \text{if } x \in (0, 1] \\ \frac{1}{2}x & \text{if } x \in (1, 2] \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Using the result in Problem 1, show that $\left(g, 1, \frac{1}{2}\right)$ is a Gabor frame for $L^2(\mathbb{R})$ with frame bounds $\frac{5}{2}, 10$.