Fallstudien der Mathematischen Modellbildung

Exercise Sheet 4

October 26, 2015

Definition 1. Let $\mathcal{H}$ be a Hilbert space.

(i) A sequence $\{x_n\} \subseteq \mathcal{H}$ is *a frame for* $\mathcal{H}$ if there exist two constants $A, B > 0$, called the *frame bounds*, such that for every $x \in \mathcal{H}$

$$A \|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \leq B \|x\|^2 . \tag{1}$$

(ii) if $\{x_n\} \subseteq \mathcal{H}$ is a frame for $\mathcal{H}$ we define

(a) the *coefficient map for* $\{x_n\}$ to be the function $\mathcal{C} : \mathcal{H} \to \ell^2$ such that

$$\mathcal{C}(x) = \{\langle x, x_n \rangle\}; \tag{2}$$

(b) the *synthesis map for* $\{x_n\}$ to be the function $\mathcal{R} : \ell^2 \to \mathcal{H}$ such that

$$\mathcal{R}(\{c_n\}) = \sum_{n=1}^{\infty} c_n x_n; \tag{3}$$

(c) the *frame operator for* $\{x_n\}$ to be the function $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ such that

$$\mathcal{S}(x) = (R \circ \mathcal{C})(x) = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n; \tag{4}$$

Problem 1. Let $\mathcal{H}$ be a Hilbert space and $\{x_n\} \subseteq \mathcal{H}$ be a frame for $\mathcal{H}$ with frame bounds A, B. Then prove then following statements:

(i) the frame operator $\mathcal{S}$ for $\{x_n\}$ is a homeomorphism of $\mathcal{H}$ onto itself. Moreover, its inverse $\mathcal{S}^{-1}$ satisfies

$$\langle B^{-1}x, x \rangle \leq \langle \mathcal{S}^{-1}(x), x \rangle \leq \langle A^{-1}x, x \rangle ; \tag{5}$$

(ii) $\{\mathcal{S}^{-1}(x_n)\}$ is a frame for $\mathcal{H}$ with frame bounds B^{-1}, A^{-1} called the *dual frame of* $\{x_n\}$;

(iii) the following series converge for each $x \in \mathcal{H}$ and we have

$$x = \sum_{n=1}^{\infty} \langle x, \mathcal{S}^{-1}(x_n) \rangle x_n = \sum_{n=1}^{\infty} \langle x, x_n \rangle \mathcal{S}^{-1}(x_n); \tag{6}$$

(*iv*) if the frame $\{x_n\}$ is *tight*, i.e. $A = B$, then $\mathcal{S} \equiv AI$, $\mathcal{S}^{-1} \equiv A^{-1}I$ and for every $x \in \mathcal{H}$

$$x = A^{-1} \sum_{n=1}^{\infty} \langle x, x_n \rangle \, x_n. \tag{7}$$

Problem 2. Let $\mathcal{H}$ be a Hilbert space, $\{x_n\} \subseteq \mathcal{H}$ be a frame for $\mathcal{H}$ and $\mathcal{S}$ its frame operator. Prove that for every $x \in \mathcal{H}$, if $x = \sum_{n=1}^{\infty} c_n x_n$ for some sequence of scalars $\{c_n\}$, then

$$\sum_{n=1}^{\infty} |c_n|^2 = \sum_{n=1}^{\infty} |\left\langle x, \mathcal{S}^{-1}(x_n) \right\rangle|^2 + \sum_{n=1}^{\infty} |\left\langle x, \mathcal{S}^{-1}(x_n) \right\rangle - c_n|^2. \tag{8}$$

In particular, the sequence $\{\left\langle x, \mathcal{S}^{-1}(x_n) \right\rangle\}$ has the minimal ℓ^2-norm among all such sequences $\{c_n\}_{n=1}^{\infty}$.