Fallstudien der Mathematischen Modellbildung

Exercise Sheet 2

October 14, 2015

Consider the following classes of operators on $L^2(\mathbb{R})$:

Translation by $a \in \mathbb{R}$: $T_a f(x) = f(x-a)$;

Modulation by $a \in \mathbb{R}$: $M_a f(x) = e^{2\pi iax} f(x)$;

Dilation by $a \in \mathbb{R}\backslash\{0\}$: $D_a f(x) = \frac{1}{\sqrt{|a|}} f\left(\frac{x}{a}\right)$.

Moreover, define for every $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ the **Fourier transform of** f by

$$\mathcal{F}f(\gamma) = \int_{-\infty}^{\infty} M_{-\gamma} f(x)\, dx.$$

Problem 1. Prove the following properties:

1. $T_a M_b f(x) = e^{-2\pi iba} M_b T_a f(x)$;
2. $T_a D_b f(x) = D_b T_{\frac{a}{b}} f(x)$;
3. $D_a M_b f(x) = M_{\frac{b}{a}} D_a f(x)$;
4. $\mathcal{F} T_a f(x) = M_{-a} \mathcal{F} f(x)$;
5. $\mathcal{F} M_a f(x) = T_a \mathcal{F} f(x)$;
6. $\mathcal{F} D_a f(x) = D_{\frac{1}{a}} \mathcal{F} f(x)$;
7. for each $f \in L^2(\mathbb{R})$, $a \mapsto T_a f$ is continuous from $\mathbb{R}$ to $L^2(R)$;
8. for each $f \in L^2(\mathbb{R})$, $a \mapsto M_a f$ is continuous from $\mathbb{R}$ to $L^2(R)$;
9. for each $f \in L^2(\mathbb{R})$, $a \mapsto D_a f$ is continuous from $\mathbb{R}\backslash\{0\}$ to $L^2(R)$;
10. the *adjoint* of an operator $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the unique operator $U^* : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ satisfying

    $$\langle f, Ug \rangle_{L^2(\mathbb{R})} = \langle U^* f, g \rangle_{L^2(\mathbb{R})}.$$

    U is *unitary* whenever $UU^* = U^*U = I$. Prove that the operators T_a, M_a, and D_a are unitary.

Problem 2. Prove that the following sets are orthonormal systems for $L^2(\mathbb{R})$:

1. $\{M_j T_k g\}_{j,k\in\mathbb{N}}$, where $g = \chi_{[0,1]}$;

2. $\{D^j_{\frac{1}{2}} T_k \psi\}_{j,k\in\mathbb{N}}$, where $D^j_{\frac{1}{2}} = \overbrace{D_{\frac{1}{2}} \cdot D_{\frac{1}{2}}}^{j \text{ times}}$ and

$$\psi(x) = \begin{cases} 1 & \text{if } x \in \left[0, \frac{1}{2}\right), \\ -1 & \text{if } x \in \left[\frac{1}{2}, 1\right), \\ 0 & \text{otherwise.} \end{cases}$$

Problem 3. Write a code in Matlab implementing the algorithm for the Discrete Fourier Transform (DFT) and the Fast Fourier Transform (FFT), and compare the cost of each algorithm as a function of the length of the input n.