Fallstudien der Mathematischen Modellbildung

Exercise Sheet 1

October 2, 2015

Problem 1. Let $\mathcal{H}$ be a complex vector space and $\langle\cdot,\cdot\rangle$ be a scalar product on $\mathcal{H}$. Prove the *Cauchy-Schwarz identity*, i.e., for every $u, v \in \mathcal{H}$ it holds

$$|\langle u, v\rangle| \leq \|u\| \, \|v\| \, , \tag{1}$$

where $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$.

Problem 2. Let $\mathcal{H}$ be an Hilbert space. Let $\{u_\alpha\}_{\alpha\in A}$ be an *orthonormal system*, i.e., $\{u_\alpha\}_{\alpha\in A}$ satisfies

$$\langle u_\alpha, u_\beta\rangle = \delta_{\alpha\beta} \text{ for every } \alpha, \beta \in A \tag{2}$$

(where $\delta_{\alpha\beta}$ is the Kroenecker's Delta function). Then prove that the following are equivalent:

- (i) (Completeness) if for every $x \in \mathcal{H}$, $\langle x, u_\alpha\rangle = 0$ for every $\alpha \in A$, then $x = 0$.
- (ii) (Fourier expansion) for every $x \in \mathcal{H}$, $x = \sum_{\alpha\in A} \langle x, u_\alpha\rangle \, u_\alpha$;
- (iii) (Parseval's identity) for every $x, y \in \mathcal{H}$, $\langle x, y\rangle = \sum_{\alpha\in A} \langle x, u_\alpha\rangle \, \overline{\langle y, u_\alpha\rangle}$;
- (iv) (Bessel's equality) for every $x \in \mathcal{H}$, $\|x\|^2 = \sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2$.

Problem 3. Let $\mathcal{H}$ be an Hilbert space and let $U = \{u_\alpha\}_{\alpha\in A}$ be an orthonormal system. Recall the definitions of digitalization and recovery maps associated to $\{u_\alpha\}_{\alpha\in A}$:

$$\begin{aligned} \mathcal{C}_U &: \mathcal{H} \to \ell^2(A), \qquad C_U(x) = \{\langle x, u_\alpha\rangle\}_{\alpha\in A}, \\ \mathcal{R}_U &: \ell^2(A) \to \mathcal{H}, \qquad R_U(\{c_\alpha\}_{\alpha\in A}) = \sum_{\alpha\in A} c_\alpha u_\alpha. \end{aligned} \tag{3}$$

Show that if $\{u_\alpha\}_{\alpha\in A}$ is an *orthonormal basis*, i.e. an orthonormal system satisfying (i), then $\mathcal{C}_U$ is the only map $\mathcal{C} : \mathcal{H} \to \ell^2(A)$ such that

$$\mathcal{R}_U \circ \mathcal{C} = Id_{\mathcal{H}}, \tag{4}$$

where $Id_{\mathcal{H}}$ is the identity map on $\mathcal{H}$.
Add now to the orthonormal basis $\{u_\alpha\}_{\alpha\in A}$ a new element $u_* \in \mathcal{H}$, and consider the set $U' = \{u_\alpha\}_{\alpha\in A\cup\{*\}}$. Show that there exists more than one map $\mathcal{C} : \mathcal{H} \to \ell^2(A \cup \{*\})$ such that

$$\mathcal{R}_{U'} \circ \mathcal{C} = Id_{\mathcal{H}}. \tag{5}$$

Furthermore, for every such a map $\mathcal{C}$, find an estimate for the constants $A, B > 0$ such that

$$A \, \|x\|_{\mathcal{H}} \leq \|\mathcal{C}(x)\|_{\ell^2(A\cup\{*\})} \leq B \, \|x\|_{\mathcal{H}} \, . \tag{6}$$