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# Mathematics of Digitalization: Case Study

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Eighth lecture

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# INTRODUCTION

Main topics:

1. After a short introduction on the Hilbert spaces, we describe the concepts of Fourier series, Fourier transforms and their discrete implementation, in particular the algorithm of the Fast Fourier Transform (FFT).
2. To connect the continuous Fourier transforms and the discrete one, we need to address the Shannon sampling theory (analog-to-digital conversion). For that it will be fundamental to understand the Poisson summation formula which will allow us to estimate also the errors in the analog-to-digital conversion, the so-called *aliasing* errors.
3. We shall introduce systems for robust analog-to-digital

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conversion called *frames* in Hilbert spaces. We address special cases of frames, in particular Gabor frames in their continuous and discrete versions and the implementation of their so-called canonical duals. Applications to audio signals and in image compression will be showed.

4. We address in more detail the study case of the fresco restoration and how to use the tools we considered so far to solve it.

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## THE IMPORTANCE OF HARMONIC ANALYSIS

- We can synthesize a variety of complicated functions by means of pure sinusoids in the same way we can produce a chord of C major pushing the keyboard (C, E, G) of a piano.
- It was Joseph Fourier (<http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html>) who first developed the modern methods of trigonometric series and integrals in the study of the heat conduction in the solids (*Analytical Theory of Heat*, 1815).

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## SOME OF THE MILESTONES

- The most cited paper in mathematics  
J. W. Cooley and J. W. Tukey, An algorithm for the machine computation of complex Fourier series, *Math. Comp.*, 19 (1965), 297-301.
- Almost  $3/4$  of the Nobel prizes in Physics are given on work using methods and tools from Fourier analysis.  
F. N. Magill, ed., *The Nobel Prize Winners — Physics*, Vol 1-3, Salem Press. Englewood Cliffs, NJ. 1989.
- Francis Crick, James Watson and Maurice Wilkins got the Nobel prize for the medicine in 1962 for the discovery of the molecular structure of the DNA. This was the first example of the use of Fourier methods on data coming from X ray diffraction.

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- Herbert Hauptmann (a mathematician) and Jerome Karle shared in 1985 the Nobel prize for chemistry after having shown how to use systematically Fourier analysis in order to determine the structure of large molecules from X ray diffraction data.

W. A. Hendrickson, *the 1985 Nobel Prize in Chemistry*,  
*Science* 231 (1986), 362-364.

- The starting of the time-frequency analysis and the wavelets (1960-1990).
- A large number of technological products (CD-DVD, high definition TV, digital phones, medical imaging instruments ...) are based on discrete Fourier analysis.

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## HILBERT SPACES

Let  $\mathcal{H}$  be a vector space. A scalar product  $\langle u, v \rangle$  is a map from  $\mathcal{H} \times \mathcal{H}$  to  $\mathbb{C}$  such that:

- (i)  $\langle au + bv, z \rangle = a\langle u, z \rangle + b\langle v, z \rangle$  for all  $u, v, z \in \mathcal{H}$  and  $a, b \in \mathbb{C}$ .
- (ii)  $\langle u, v \rangle = \overline{\langle v, u \rangle}$  for all  $u, v \in \mathcal{H}$ .
- (iii)  $\langle u, u \rangle \in \mathbb{R}$ ,  $\langle u, u \rangle \geq 0$  for all  $u \in \mathcal{H}$  and  $\langle u, u \rangle \neq 0$  if  $u \neq 0$ .

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We recall that a scalar product verify the Cauchy-Schwarz inequality (which will be proved in the exercises):

$$|\langle u, v \rangle| \leq \langle u, u \rangle^{1/2} \langle v, v \rangle^{1/2} \quad \forall u, v \in \mathcal{H}. \quad (1)$$

and that  $\|u\| = \langle u, u \rangle^{1/2}$  define a norm for  $\mathcal{H}$ .

A Hilbert spaces is a vector space  $\mathcal{H}$  endowed with a scalar product  $\langle u, v \rangle$  and is complete with respect to the norm  $\langle u, u \rangle^{1/2}$ .



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**Example 0.1** *Some relevant examples:*

- *Let  $\Omega \subset \mathbb{R}^n$ . The vector space of functions  $L^2(\Omega) = \{f : \Omega \rightarrow \mathbb{C} \mid \int_{\Omega} |f|^2 dx < \infty\}$  is a Hilbert spaces with the scalar product:*

$$\langle u, v \rangle = \int_{\Omega} u(x) \overline{v(x)} dx.$$

- *Let  $\Omega_d \subset \mathbb{Z}^n$ . The vector space  $\ell^2(\Omega_d) = \{f : \Omega_d \rightarrow \mathbb{C} \mid \sum_{k \in \Omega_d} |f(k)|^2 < \infty\}$  is a Hilbert spaces with the scalar product:*

$$\langle u, v \rangle = \sum_{k \in \Omega_d} u(k) \overline{v(k)}.$$

*In particular if  $|\Omega_d| = d < \infty$  then  $\ell^2(\Omega_d) = \mathbb{C}^d$ .*

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**Definition 0.2** *Let  $E$  be a topological vector space. We say that  $E'$  is the topological dual space of  $E$ , i.e.,  $E' = \{\varphi \text{ from } E \text{ to } \mathbb{C} \mid \varphi \text{ is continuous and linear}\}$ . Given two Hilbert spaces  $\mathcal{H}, \mathcal{K}$ , the set of linear operators  $T : \mathcal{H} \rightarrow \mathcal{K}$ , i.e.,  $T(au + bv) = aT(u) + bT(v)$ , which are also continuous is a vector space which, endowed with the norm  $\|T\|_{\mathcal{H} \rightarrow \mathcal{K}} \equiv \sup_{u \in \mathcal{H}, u \neq 0} \frac{\|Tu\|_{\mathcal{K}}}{\|u\|_{\mathcal{H}}}$  is a complete normed space, i.e., a Banach space, and we indicate it with  $\mathcal{L}(\mathcal{H}, \mathcal{K})$ . In particular, if  $\mathcal{H} = \mathcal{K}$  then we set  $\mathcal{L}(\mathcal{H}) \equiv \mathcal{L}(\mathcal{H}, \mathcal{H})$  and if  $\mathcal{K} = \mathbb{C}$  then we have  $\mathcal{L}(\mathcal{H}, \mathbb{C}) = \mathcal{H}'$ .*

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**Theorem 0.3 (Representation theorem of Riesz-Frechet)** *Let  $\mathcal{H}$  be a Hilbert space. For every  $\varphi \in \mathcal{H}'$  there exists a unique  $f \in \mathcal{H}$  such that*

$$\langle \varphi, v \rangle \equiv \varphi(v) = \langle v, f \rangle \quad \forall v \in \mathcal{H}.$$

*Moreover we have*

$$\|f\| = \|\varphi\|_{\mathcal{H}'} \equiv \sup_{\|v\|=1} |\langle \varphi, v \rangle|.$$

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**Proposition 0.4 (Bessel inequality)** *Let  $\{u_\alpha\}_{\alpha \in A}$  be an orthonormal system in  $\mathcal{H}$ , i.e.,  $\langle u_\alpha, u_\beta \rangle = \delta_{\alpha, \beta}$  where  $\delta_{\cdot, \cdot}$  is the Kronecker symbol. Then for  $x \in \mathcal{H}$ :*

$$\sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2 \leq \|x\|^2.$$

*In particular  $\{\alpha | \langle x, u_\alpha \rangle \neq 0\}$  is countable.*

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**Theorem 0.5 (Fourier)** *Let  $\{u_\alpha\}_{\alpha \in A}$  be a countable orthonormal system. Then the following are equivalent:*

(i)  $x = \sum_{\alpha \in A} \langle x, u_\alpha \rangle u_\alpha$  for all  $x \in \mathcal{H}$ .

(ii) (Parseval identity)  $\|x\|^2 = \sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2$  for all  $x \in \mathcal{H}$ .

(iii) (Completeness) If  $x \in \mathcal{H}$  and if  $\langle x, u_\alpha \rangle = 0$  for all  $\alpha$ , then  $x = 0$ .

The sequence  $\langle x, u_\alpha \rangle$  is such that  $\sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2 < \infty$ , then the series  $\sum_{\alpha \in A} \langle x, u_\alpha \rangle u_\alpha$  is convergent in  $\mathcal{H}$  and converges to  $x$ . A orthonormal set which verifies one of the previous conditions is called an orthonormal basis for the Hilbert space  $\mathcal{H}$ .

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**Theorem 0.6** *Every Hilbert space has an orthonormal basis.*

**Definition 0.7** *A Hilbert space  $\mathcal{H}$  is separable if and only if it has a countable orthonormal basis.*

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## FOURIER SERIES AND TRANSFORM

A trigonometric series (or Fourier series) in complex form of period  $\tau > 0$  is a function series of the type:

$$T(x) = \frac{1}{\sqrt{\tau}} \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x / \tau} \quad (2)$$

Set

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}) \quad (3)$$

we have, by the Euler formulas ( $e^{iw} = \cos(w) + i \sin(w)$ ), that

$$T(x) = \frac{1}{\sqrt{\tau}} \left( \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(2\pi k x / \tau) + b_k \sin(2\pi k x / \tau)) \right) \quad (4)$$

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We observe now that  $\{\frac{1}{\sqrt{\tau}}e^{2\pi inx/\tau}\}_{n\in\mathbb{Z}}$  and  $\{\sqrt{\frac{\tau}{2}}\cos(2\pi kx/\tau)\}_{k\in\mathbb{N}}\cup\{\sqrt{\frac{\tau}{2}}\sin(2\pi kx/\tau)\}_{k\in\mathbb{N},k\neq 0}$  are an orthonormal system in the Hilbert space  $\mathcal{H} = L^2(0, \tau)$ . For example

$$\begin{aligned}\left\langle \frac{1}{\sqrt{\tau}}e^{2\pi inx/\tau}, \frac{1}{\sqrt{\tau}}e^{2\pi imx/\tau} \right\rangle &= \frac{1}{\tau} \int_0^\tau e^{2\pi inx/\tau} \overline{e^{2\pi imx/\tau}} dx \\ &= \frac{1}{\tau} \int_0^\tau e^{2\pi i(n-m)x/\tau} dx \\ &= \int_0^1 e^{2\pi i(n-m)x} dx = \delta_{m,n}.\end{aligned}$$



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We wonder whether they constitute systems of orthonormal bases. We consider the function

$$f(x) = \frac{\pi x}{2} - \frac{x^2}{4}.$$

Note that it is symmetric and therefore the coefficients  $b_k(f) = \langle f, \sqrt{\frac{\tau}{2}} \sin(2\pi kx/\tau) \rangle$  vanish for all  $k$ . By computing (Exercise! Integrate twice by parts) instead the coefficients  $a_k(f) = \langle f, \sqrt{\frac{\tau}{2}} \cos(2\pi kx/\tau) \rangle$  we obtain the trigonometric series associated to  $f$  given by

$$T_f(x) = \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}. \quad (5)$$

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**Theorem 0.8 (Pointwise convergence)** *Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  periodic with period  $\tau > 0$  and locally integrable. Assume that in  $x_0 \in \mathbb{R}$  the limits  $f(x_0)^+ \in f(x_0^-)$  exist and are finite, and that the left and right finite difference*

$$\frac{f(x_0 + h) - f(x_0^+)}{h}, \quad \frac{f(x_0 - h) + f(x_0^-)}{h},$$

*are bounded for any  $h > 0$  small. Then the series  $T_f$  of  $f$  converges in  $x_0$  to the mean of the values of the right and left limits of  $f$  in  $x_0$ :*

$$T_f(x_0) = \frac{1}{\tau} \sum_{n \in \mathbb{Z}} \langle f, e^{2\pi i n \xi / \tau} \rangle e^{2\pi i n x / \tau} = \frac{f(x_0^+) + f(x_0^-)}{2}.$$

*In particular if  $f'(x_0)$  exists then  $T_f(x_0) = f(x_0)$ .*

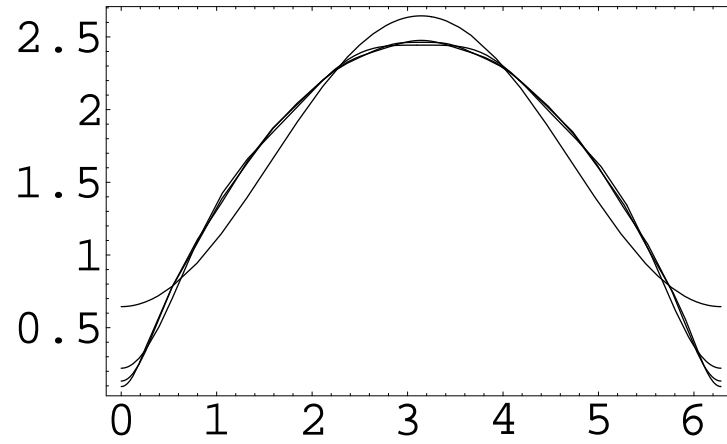
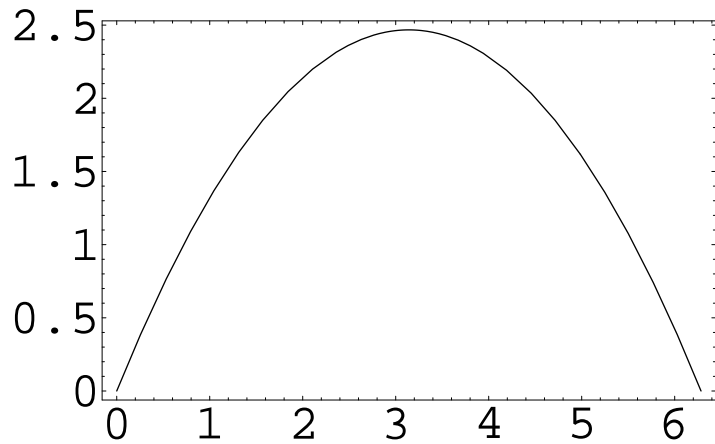


Figure 1:  $f(x) = \frac{\pi x}{2} - \frac{x^2}{4} = \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$ .

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- The set of step functions  $S$  are dense in  $L^2(0, \tau)$ ;
  - We have

$$\sum_{n \in \mathbb{Z}} \left| \left\langle \chi_{[0,a]}, \frac{1}{\sqrt{\tau}} e^{2\pi i n x / \tau} \right\rangle \right|^2 = \frac{a^2}{\tau} + \frac{\tau}{\pi^2} \left( \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2\pi n a}{\tau}\right)}{n^2} \right)$$

- From the previous theorem applied on  $f$

$$a = \|\chi_{[0,a]}\|_2^2 = \sum_{n \in \mathbb{Z}} \left| \left\langle \chi_{[0,a]}, \frac{1}{\sqrt{\tau}} e^{2\pi i n x / \tau} \right\rangle \right|^2.$$

- By density of  $S$  we conclude that  $\left\{ \frac{1}{\sqrt{\tau}} e^{2\pi i n x / \tau} \right\}_{n \in \mathbb{Z}}$  is an orthonormal basis.

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For functions  $f \in L^2(0, \tau)$  we define the Fourier transform  $\mathcal{F} : L^2(0, \tau) \rightarrow \ell^2(\mathbb{Z})$  as

$$\mathcal{F}f(n) = \left\langle f, \frac{1}{\sqrt{\tau}} e^{2\pi i n x / \tau} \right\rangle = \frac{1}{\sqrt{\tau}} \int_{(0, \tau)} f(x) e^{-2\pi i n x / \tau} dx. \quad (6)$$

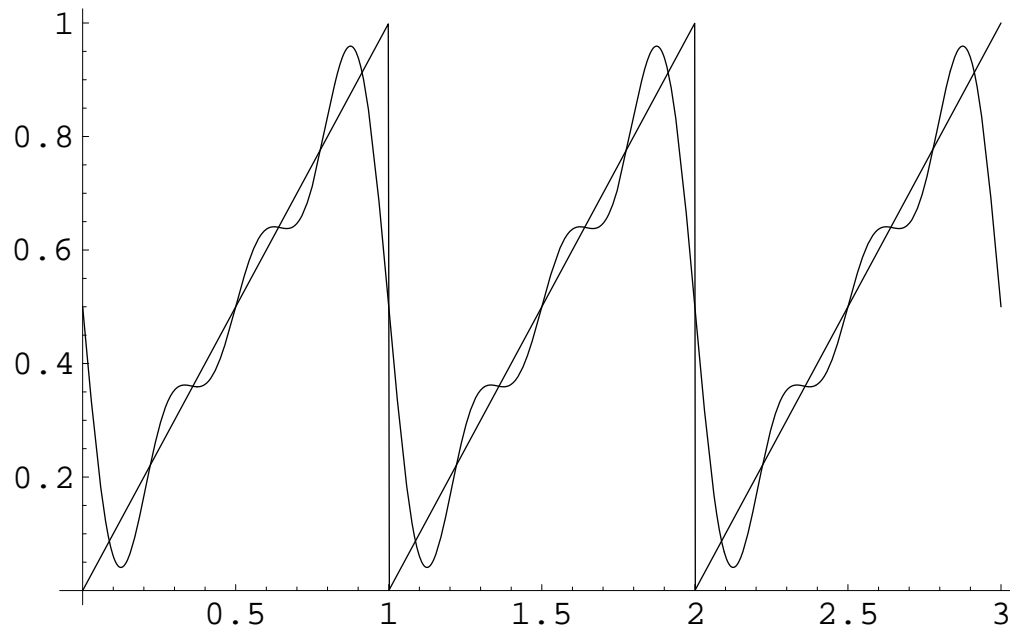


Figure 2: Fourier series associated to  $f(x) = x - [x]_-$ .

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What happens if we make  $\tau \rightarrow \infty$ ?

$$\sum_{n \in \mathbb{Z}} \left\langle f, \frac{1}{\sqrt{\tau}} e^{2\pi i n x / \tau} \right\rangle \frac{1}{\sqrt{\tau}} e^{2\pi i n x / \tau} = \frac{1}{\tau} \sum_{n \in \mathbb{Z}} \left( \int_{\tau/2}^{-\tau/2} f(\xi) e^{-2\pi i n \xi / \tau} d\xi \right) e^{2\pi i n x / \tau}.$$

The last sum is in fact a Riemann sum. This makes us supposing that if  $f$  is an integrable function on  $\mathbb{R}$  then we could write something like

$$\begin{aligned} f(x) &= \lim_{\tau \rightarrow \infty} f(x) \chi_{[\tau/2, -\tau/2]} = \\ &= \lim_{\tau \rightarrow \infty} \sum_{n \in \mathbb{Z}} \left\langle f, \frac{1}{\sqrt{\tau}} e^{2\pi i n x / \tau} \right\rangle \frac{1}{\sqrt{\tau}} e^{2\pi i n x / \tau} = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(\xi) e^{-2\pi i w \xi} d\xi \right) e^{2\pi i \xi w} dw. \end{aligned}$$

When and in which sense this writing makes sense is though still to be proved.

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**Theorem 0.9 (Fourier transform and Plancherel identity)** *We define for  $f \in L^1(\mathbb{R})$  the Fourier transform, given by*

$$\mathcal{F}f(w) = \int_{\mathbb{R}} f(x)e^{-2\pi iw x} dx. \quad (7)$$

*Moreover, if also  $\mathcal{F}f \in L^1(\mathbb{R})$  then the Fourier transform is invertible and*

$$f(x) = \int_{\mathbb{R}} \mathcal{F}f(w)e^{2\pi iw x} dw. \quad (8)$$

*If  $f \in L^1 \cap L^2$  then  $\mathcal{F} : L^1 \cap L^2 \rightarrow L^2$  is a continuous linear operator which can be extended uniquely on the entire  $L^2$ . In particular,  $\mathcal{F}$  is an isometric isomorphism, i.e., the Plancherel identity holds:*

$$\|f\|_2 = \|\mathcal{F}f\|_2. \quad (9)$$

*Moreover, if  $f, g \in L^2$  then  $\langle f, g \rangle = \langle \mathcal{F}f, \mathcal{F}g \rangle$ .*

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**Remark 0.10** *This definition is used in the literature of time-frequency analysis. The other definition*

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{-i\omega x} dx$$

*is also used, for instance in the literature related to wavelets. In these lectures we use both in a consistent way with the corresponding literature. Anyway observe that  $\mathcal{F}f(\omega) = 2\pi\hat{f}(2\pi\omega)$  and, up to rescaling, one can transform all the result with one definition to results with another definition.*



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What is the meaning of the Fourier transform? The Fourier transform represents the frequency content of a function/signal. It gives us which are the important oscillating building blocks of a signal and their distinctive frequency of oscillation.

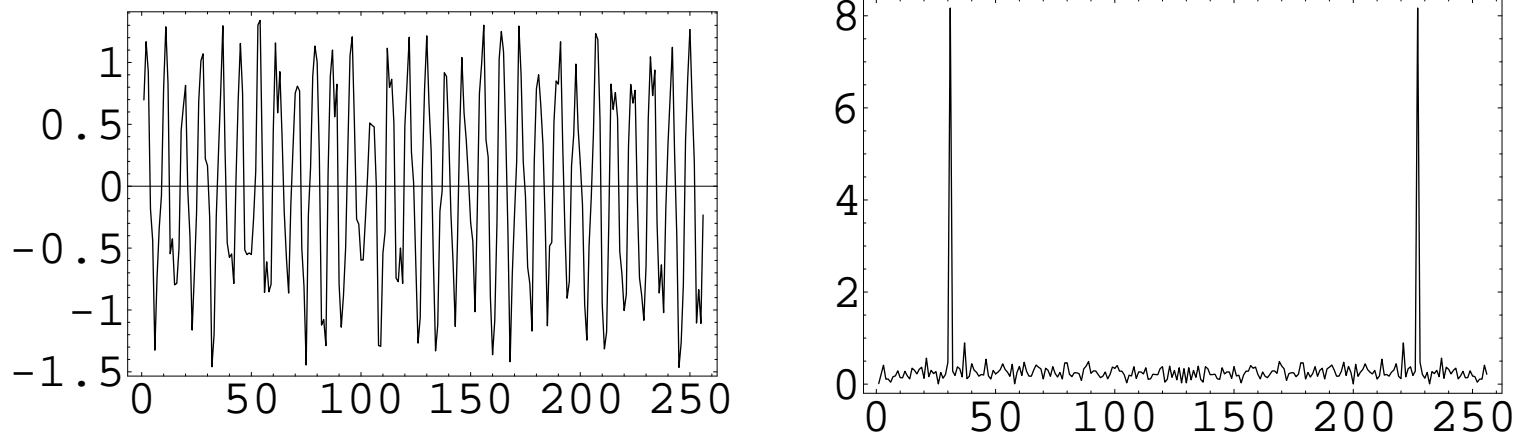


Figure 3: Sinusoidal signal affected by noise: there are two fundamental frequency components

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The fortune of Fourier analysis stays essentially in the fact that it is able to describe one of the most frequent behavior in the nature: the wave phenomena, most of which are governed by overlapping laws of the type

$$y_{\alpha, s_0}(t) = \begin{cases} e^{-\alpha t} e^{2\pi i s_0 t}, & t > 0 \\ 0, & t < 0. \end{cases}$$

$y_{\alpha, s_0}(t)$  is called *Lorenzian function*.

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When, for example, some molecules are excited by electromagnetic radiation, damped oscillations are induced and described by the law  $y_{\alpha, s_0}(t)$ .

Every building block of the molecule has its own unique special oscillations. The Lorenzians are called in this case the *molecular spectrum*.

From here comes the idea which “costed” the Nobel prize to Richard Ernst for chemistry (1991) for the development of a powerful instrument to determine the structure of large complex organic molecules.

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We want now to discuss how in practice we can use the tools of the Fourier analysis.

The exercises we showed on simple functions show us that it is not so trivial to compute Fourier series or transforms.

For relatively simple functions such as  $f(x) = \frac{1}{a^2+x^2}$  the computation of the Fourier transform is far from being trivial. In this case we have  $\mathcal{F}f(w) = \frac{\pi}{a}e^{-|w|a}$ .

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A holomorphic complex function  $f(z)$  (i.e., differentiable) on a domain  $\Omega$  except for a point  $a$ , called singularity, can be expanded in a power series, called the Laurent series,  $f(z) = \sum_{n \in \mathbb{Z}} c_n (z - a)^n$ . The specific coefficient  $c_{-1}$  is called *residual* and we write  $Res(f, a) = c_{-1}$ . One can show that if  $f$  is holomorphic on a domain  $\Omega$  which contains the halfplane  $\{\text{Im}(z) \geq 0\}$  except for a finite number of non real singularities and if the limit  $\lim_{|z| \rightarrow \infty, \text{Im}(z) \geq 0} f(z) = 0$  then we have for every  $\alpha > 0$ ,

$$\lim_{r \rightarrow \infty} \int_{-r}^r f(x) e^{i\alpha x} dx = 2\pi i \sum \{Res(f(\cdot) e^{i\alpha \cdot}, a), \text{Im}(a) > 0\}.$$

Analogous result holds for  $\alpha < 0$  and for  $\{\text{Im}(z) < 0\}$ , but changing the sign of the integral.

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It is certainly more reasonable to search for an approximation on a discrete domain and a proper definition of discrete Fourier transform, which, in some sense, can approximate well the behavior of the continuous Fourier transform. Moreover, we need to make sure that the computation of the discrete Fourier transform (DFT) can be executed in reasonable time: otherwise it does not pay off.

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Let's proceed as usual in the language of the Hilbert spaces. We consider now the cyclic group  $\mathbb{Z}_n = \frac{\mathbb{Z}}{n\mathbb{Z}}$ ,  $n \in \mathbb{N} \setminus \{0\}$  of the rest classes modulo  $n$ . In practice we can write

$$\mathbb{Z}_n = \{0, 1, \dots, n-1\},$$

reminding us that such elements represent the rest of the integer number with respect to the division by  $n$ . For example, in  $\mathbb{Z}_3$  the element  $h = 4$  corresponds to 1 as  $3 = 3 \cdot 1 + 0$ . Hence, in particular,  $2 + 2 = 1$  in  $\mathbb{Z}_3$ . Moreover,  $\ell^2(\mathbb{Z}_n) = \{(c_0, \dots, c_{n-1}) \mid \sum_{k=0}^{n-1} |c_k|^2 < \infty\} = \mathbb{C}^n$ .

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One can show by induction (exercise!)

$$1 + z + z^2 + \cdots + z^{n-1} = \begin{cases} n, & z = 1 \\ (z^n - 1)/(z - 1), & \text{otherwise.} \end{cases} \quad (10)$$

But then it is not difficult to show (exercise!) that for

$z = e^{2\pi i(k-l)/n}$  we have

$$\left\langle \frac{1}{\sqrt{n}} (e^{2\pi ikl/n})_{l \in \mathbb{Z}_n}, \frac{1}{\sqrt{n}} (e^{2\pi ilm/n})_{m \in \mathbb{Z}_n} \right\rangle = \sum_{m=0}^{n-1} e^{2\pi im(k-l)/n} = \delta_{k,l}.$$

Hence  $\left\{ \frac{1}{\sqrt{n}} (e^{2\pi ikl/n})_{l \in \mathbb{Z}_n} \right\}_{k \in \mathbb{Z}_n}$  is an orthonormal basis for the finite dimensional (Hilbert) space  $\ell^2(\mathbb{Z}_n)$ .



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Every discrete signal  $\mathbf{f}$  of length  $n$  can be written as:

$$\mathbf{f} = \frac{1}{n} \sum_{k=0}^{n-1} \langle \mathbf{f}, (e^{2\pi ikl/n})_{l \in \mathbb{Z}_n} \rangle (e^{2\pi ikl/n})_{l \in \mathbb{Z}_n}. \quad (11)$$

Moreover we have that

$$\mathcal{F}\mathbf{f}(k) = \frac{1}{\sqrt{n}} \langle \mathbf{f}, (e^{2\pi ikl/n})_{l \in \mathbb{Z}_n} \rangle = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \mathbf{f}(l) e^{-2\pi ikl/n}, \quad (12)$$

is again a signal of length  $n$ . The operator  $\mathcal{F} : \ell^2(\mathbb{Z}_n) \rightarrow \ell^2(\mathbb{Z}_n)$  is called discrete Fourier transform or DFT and it realizes all the properties of the Fourier transforms already defined before for periodic functions and functions on the real line. For example, prove by exercise that

$$\|\mathbf{f}\|_{\ell^2} = \|\mathcal{F}\mathbf{f}\|_{\ell^2}.$$

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If we assume that one operation is either a sum or a multiplication, then the computational cost of a DFG is of  $2n$  sums and multiplications times  $n$ , i.e.,  $2n^2$ . As one complex operation costs twice a single real, the total costs is  $C(DFT)(n) = 4n^2$ . If we had a PC able to compute  $10^7$  operations per second and we are able to compute a DFT of a signal of length  $n = 1000$  in

$$4 \cdot 1000^2 \cdot \frac{1}{10^7} = 0.4 \text{ sec.}$$

But already for a signal of length  $n = 16384 = 2^{14}$  the cost is of 107 sec. A simple audio signal of few seconds, sufficiently densely sampled, can easily reach length  $n = 2^{14}$ !

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Given a signal  $\mathbf{f}$  of length  $n$ , we define the translation operator the one given by

$$T_m \mathbf{f}(k) = \mathbf{f}(k - m), \quad m \in \mathbb{Z}_n. \quad (13)$$

and the modulation operator by

$$M_m \mathbf{f}(k) = e^{2\pi i m k / n} \mathbf{f}(k), \quad m \in \mathbb{Z}_n. \quad (14)$$

Moreover, we define the operator of *upsampling* and *doubling* as

$$U\mathbf{f}(h) = \begin{cases} \mathbf{f}(h/2), & \text{mod}(h, 2) = 0 \\ 0, & \text{altrimenti,} \end{cases} \quad (15)$$

$$D\mathbf{f} = \frac{1}{2}(\mathbf{f}, \mathbf{f}), \quad (16)$$

for  $h \in \mathbb{Z}_{2n}$ .

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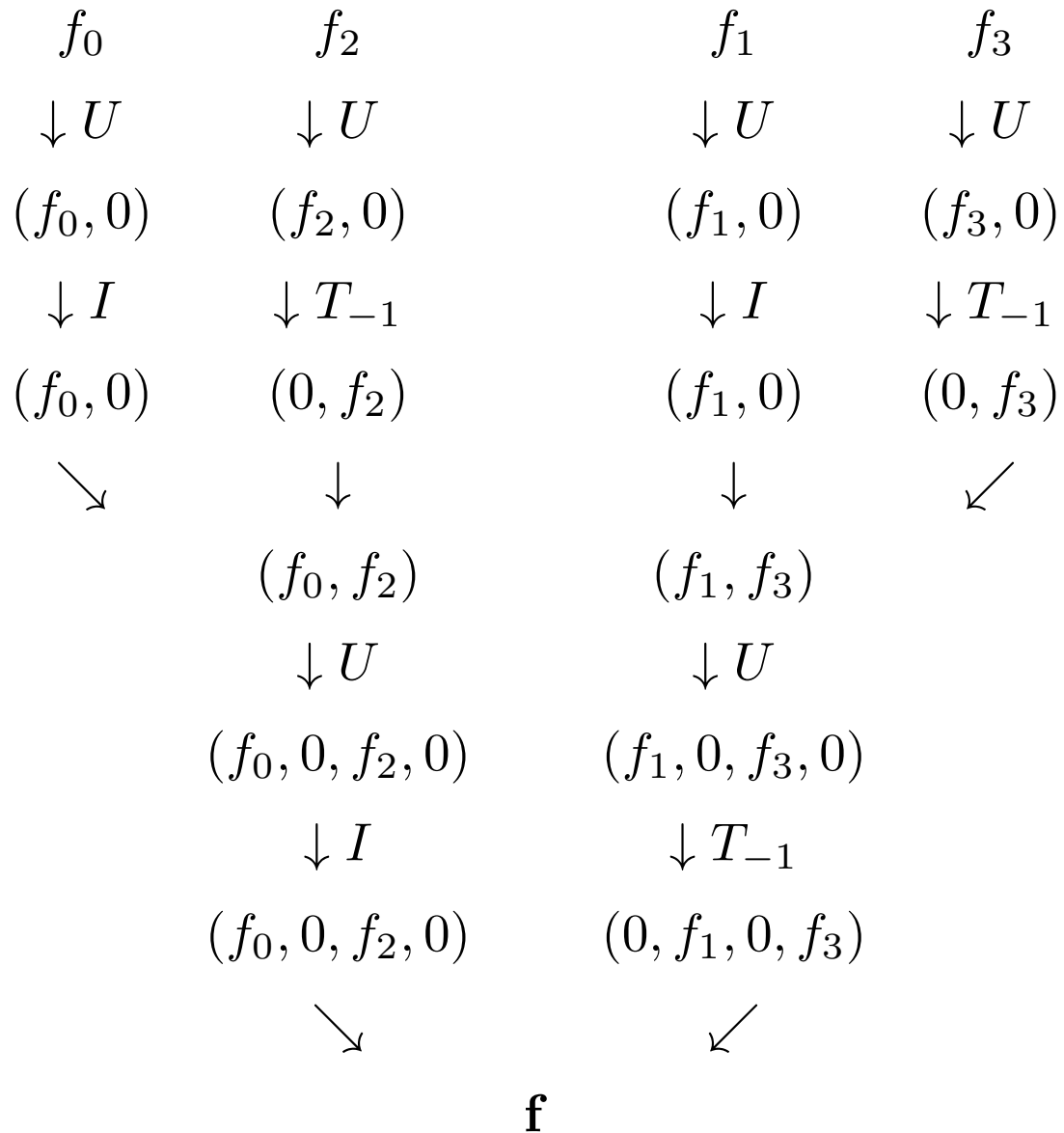
The action of the DFT with respect to these operators is given by (exercise!)

$$\mathcal{F} \cdot T_m \mathbf{f}(k) = M_{-m} \cdot \mathcal{F} \mathbf{f}(k), \quad \mathcal{F} \cdot M_m \mathbf{f}(k) = T_m \cdot \mathcal{F} \mathbf{f}(k). \quad (17)$$

$$\mathcal{F} \cdot U \mathbf{f}(h) = D \cdot \mathcal{F} \mathbf{f}(h). \quad (18)$$

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Let's now consider a vector of length  $n = 2^2 = 4$  given by  $\mathbf{f} = (f_0, f_1, f_2, f_3)$ . Let us show how is it possible to compute  $\mathbf{f}$  from the single  $f_i$  by simple operations:

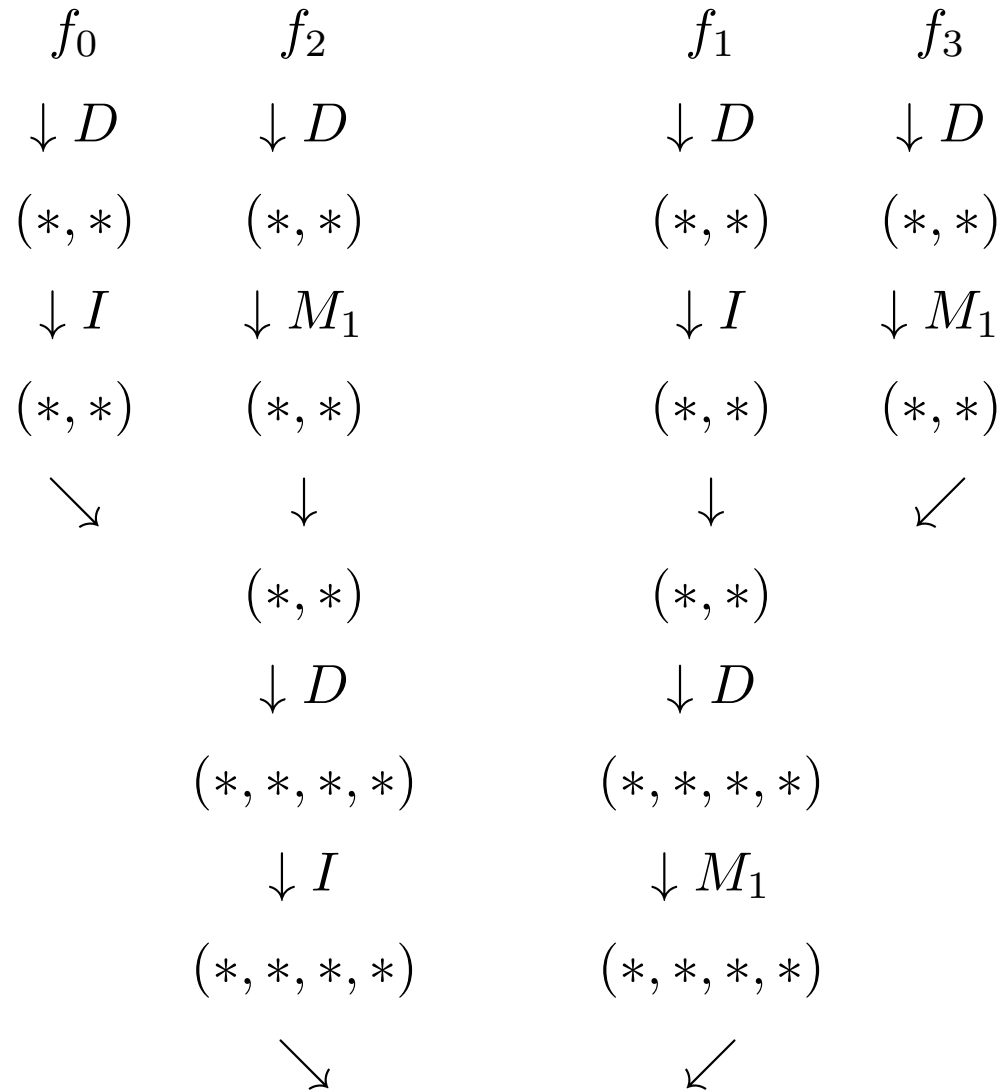


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Observing now that  $\mathcal{F}f_i = f_i$  for all  $i = 0, \dots, f_{n-1}$ , by applying the DFT on the previous diagram and substituting to  $U, T_{-1}$  resp.  $D, M_1$  as given in the commutation rules, we generate another diagram, actually a recursive algorithm to compute

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the DFT  $\mathcal{F}f$ .



$\mathcal{F}f$

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Let us assume that the computation of  $I$  and  $D$  are negligible (as it is in practice!). We need to compute only the cost of a single  $M_1$  which, applied to a vector of length  $l$  costs  $l - 1$ . If we assume now that  $n = 2^m$ , starting from the bottom of the diagram we do only one  $M_1$  on the length of the original vector and then a cost given by  $2^0(\frac{n}{2^0} - 1)$ . This cost has to be summed to the one of the level above, where one has to execute  $2(\frac{n}{2} - 1)$  operations corresponding to twice  $M_1$  computed on vector of half the original length. And so on, obtaining that the total cost is given by

$$C(FFT)(n) = \sum_{k=0}^{m-1} 2^k \left( \frac{n}{2^k} - 1 \right) =$$

$$= \sum_{k=0}^{m-1} (2^m - 2^k) = m2^m - 2^m + 1 = n \log_2(n) - n + 1. \quad (19)$$

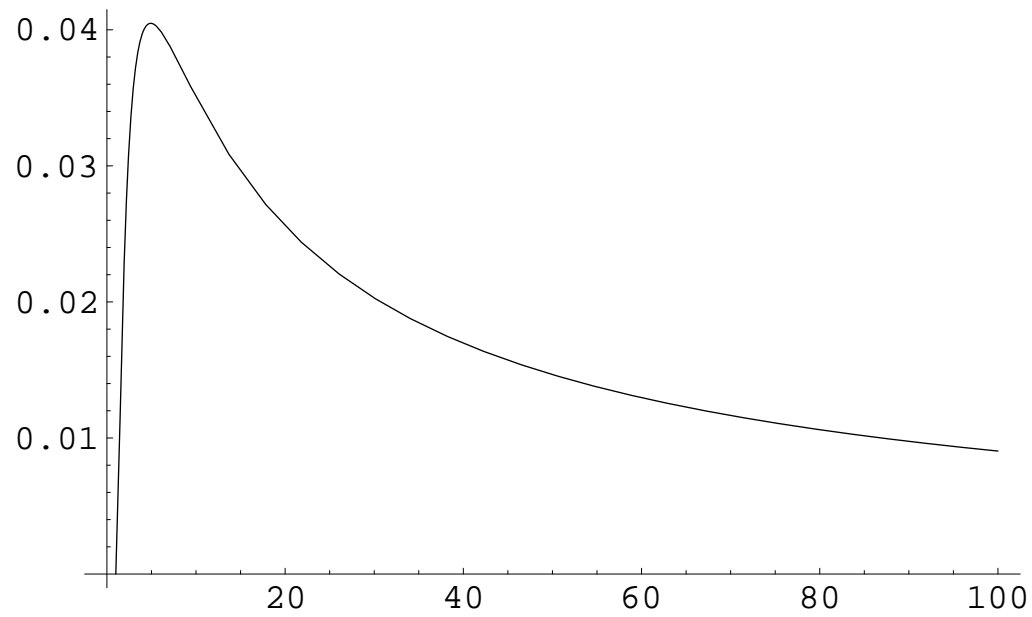


Figure 4: Ratio  $\frac{C(FFT)(n)}{C(DFT)(n)}$  for growing  $n$

---

Hence a PC is able to produce the FFT of a signal of length  $n = 16384$  in

$$(2^{14}14 - 2^{14} + 1) \cdot \frac{1}{10^7} = 0.021 \text{ sec},$$

against the  $10^7$  sec which one may need by means of the direct DFT!

---

## SAMPLING AND DFT

**Definition 0.11** *Let  $f \in C(\mathbb{R}^d)$  and  $\tau = (\tau_1, \dots, \tau_d) \in (0, \infty)^d$ . We call the sampling of step  $\tau$  the operator which associates to  $f$  the function  $f^c \equiv f|_{\tau\mathbb{Z}^d}$ . If  $f \in C_c(\mathbb{R}^d)$ , clearly  $f^c$  is vanishing out of a compact set of  $\mathbb{Z}^d$ . Moreover, up to a translation, we can always assume that  $f^c$  is defined on a set  $\tau\mathbb{Z}^d / \tau m\mathbb{Z}^d$  isomorphic to  $\mathbb{Z}_m^d$ . Hence, the sampling induces an operator acting from  $C_c(\mathbb{R}^d)$  to  $\ell^2(\mathbb{Z}_m^d)$ , for some  $m = (m_1, \dots, m_d) \in \mathbb{N}^d \setminus \{0\}$ .*

We want to study some of the properties of the operator  $\cdot^c$ . In particular we wonder which is relationship between  $\mathcal{F}f^c$  (meant here as DFT!) and  $\mathcal{F}f$  (meant as Fourier transform for functions on  $\mathbb{R}^d$ ).

---

**Lemma 0.12** *Let  $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{R}_+^d$ . If  $f \in L^1(\mathbb{R}^d)$  then the series  $\sum_{k \in \mathbb{Z}^d} f(x - \omega k)$  converges in  $L^1(\omega T^d)$  to the periodic function  $Pf$  such that  $\|Pf\|_1 \leq \|f\|_1$ . Moreover for  $k \in \omega \mathbb{Z}^d$  the Fourier transform for periodic functions on  $\omega T^d$  the function  $\mathcal{F}(Pf)(k)$  is equal to the Fourier transform for functions on  $\mathbb{R}^d$  given by  $(\det(\omega))^{-1/2} \mathcal{F}f(k/\omega)$ .*

---

## PROOF OF THE LEMMA

Let  $Q = \prod_{i=1}^d \omega_i [-\frac{1}{2}, \frac{1}{2})$ . We have  $\int_Q \sum_{k \in \mathbb{Z}^d} |f(x - \omega k)| dx = \sum_{k \in \mathbb{Z}^d} \int_{Q + \omega k} |f(x)| dx = \int_{\mathbb{R}^d} |f(x)| dx < \infty$ . By the Dominated Convergence Theorem we obtain the convergence of the series  $\sum_{k \in \mathbb{Z}^d} f(x - \omega k)$  in  $L^1$  to a function  $Pf$  such that  $\|Pf\|_1 \leq \|f\|_1$ . By definition of Fourier transform of periodic functions defined on  $\omega T^d$  we have

$$\begin{aligned} \mathcal{F}(Pf)(h) &= \frac{1}{(\det(\omega))^{1/2}} \int_Q \left( \sum_{k \in \mathbb{Z}^d} f(x - \omega k) \right) e^{-2\pi i(h,x)/\omega} dx = \\ &= \frac{1}{(\det(\omega))^{1/2}} \sum_{k \in \mathbb{Z}^d} \int_{Q + \omega k} f(x) e^{-2\pi i(h,x + \omega k)/\omega} dx = \\ &= \frac{1}{(\det(\omega))^{1/2}} \int_{\mathbb{R}^d} f(x) e^{-2\pi i(h,x)/\omega} dx = (\det(\omega))^{-1/2} \mathcal{F}f(h/\omega). \end{aligned}$$

---

**Theorem 0.13 (Poisson summation formula)** *Let  $f \in C(\mathbb{R}^d)$ ,  $|f(x)| \leq C(1 + |x|)^{-d-\varepsilon}$  and  $|\mathcal{F}f(\hat{x})| \leq C(1 + |\hat{x}|)^{-d-\varepsilon}$  for some  $C, \varepsilon > 0$ . Let  $\omega = (\omega_1, \dots, \omega_d) \in (0, \infty)^d$ . Then the following formula holds*

$$(\det(\omega))^{1/2} \sum_{k \in \mathbb{Z}^d} \mathcal{F}f(\hat{x} - \omega k) = (\det(\frac{1}{\omega}))^{1/2} \sum_{n \in \mathbb{Z}^d} f(n/\omega) e^{-2\pi i(\hat{x}, n/\omega)}.$$

*where both sides converge uniformly in  $\omega T^d$ . In particular for  $\hat{x} = 0$  we obtain*

$$(\det(\omega))^{1/2} \sum_{k \in \mathbb{Z}^d} \mathcal{F}f(\omega k) = (\det(\frac{1}{\omega}))^{1/2} \sum_{n \in \mathbb{Z}^d} f(n/\omega).$$

---

## PROOF OF THE THEOREM

The absolute and uniform convergence of the series follows by the application of the criterion of the integral estimate for the series  $\sum_{k \in \mathbb{Z}^d} (1 + |k|)^{-n-\varepsilon}$ . Then  $Pf(x) = \sum_{k \in \mathbb{Z}^d} f(x - \omega k)$  is in  $C(\omega T^d)$  and therefore in  $L^2(\omega T^d)$ . By the Theorem of Fourier and by the previous Lemma we have

$$\begin{aligned} Pf(x) &= \frac{1}{(\det(\omega))^{1/2}} \sum_{h \in \mathbb{Z}^d} \mathcal{F}(Pf)(h) e^{2\pi i(h,x)/\omega} = \\ &= \frac{1}{\det(\omega)} \sum_{h \in \mathbb{Z}^d} \mathcal{F}f(h/\omega) e^{2\pi i(h,x)/\omega}. \end{aligned}$$

Exchanging the roles of  $f$  and  $\mathcal{F}f$  and observing that  $\mathcal{F}\mathcal{F}f = f(-x)$  we obtain the final result.



---

**Corollary 0.14 (Shannon sampling theorem)** *Let  $f \in C_c(\mathbb{R}^d)$ ,  $|f(x)| \leq C(1 + |x|)^{-d-\varepsilon}$  and  $|\mathcal{F}f(\hat{x})| \leq C(1 + |\hat{x}|)^{-d-\varepsilon}$  for some  $C, \varepsilon > 0$ . If, for example,  $\text{supp}(f) \subset \prod_{i=1}^d [0, a_i]$ , then it holds*

$$\mathcal{F}f^c(l) = \frac{(\det(m))^{1/2}}{(\det(a))} \left[ \sum_{k \in \mathbb{Z}^d} \mathcal{F}f(\hat{x} - \omega k) \right]^c (l/a), \quad (20)$$

where the sampling on the left is made with step  $\tau = 1/\omega$  and the one on the right with step  $1/a$ , so that  $\omega a = m \in \mathbb{N}^d$ .

---

## PROOF OF THE COROLLARY

Let us assume  $\text{supp}(f) \subset \prod_{i=1}^d [0, a_i]$  and denote  $a = (a_1, \dots, a_d)$ . We also assume that  $a > 0$  is such that  $\omega a = m \in \mathbb{N}^d$ . But then

$$\left(\det\left(\frac{1}{\omega}\right)\right)^{1/2} \sum_{n \in \mathbb{Z}^d} f(n/\omega) e^{-2\pi i(\hat{x}, n/\omega)} = \frac{(\det(a))^{1/2}}{(\det(m))^{1/2}} \sum_{n \in \mathbb{Z}_m^d} f(n/\omega) e^{-2\pi i(\hat{x}a, n)/m}.$$

If we assume now that  $\hat{x}a = l \in \mathbb{Z}^d$  then by the Poisson summation formula, we get

$$\frac{(\det(m))^{1/2}}{(\det(a))^{1/2}} \sum_{k \in \mathbb{Z}^d} \mathcal{F}f(l/a - \omega k) = \frac{(\det(a))^{1/2}}{(\det(m))^{1/2}} \sum_{n \in \mathbb{Z}_m^d} f(n/\omega) e^{-2\pi i(l, n)/m}.$$

---

Set  $\tau = 1/\omega$ , we obtain by definition of the DFT

$$\mathcal{F}\mathbf{f}^c(l) = \frac{(\det(m))^{1/2}}{(\det(a))} \sum_{k \in \mathbb{Z}^d} \mathcal{F}f(l/a - \omega k) = \frac{(\det(m))^{1/2}}{(\det(a))} \left[ \sum_{k \in \mathbb{Z}^d} \mathcal{F}f(\hat{x} - \omega k) \right]^c (l)$$

where the last sampling is made with step  $1/a$ .

---

## SAMPLING THEORY

**Lemma 0.15** Define  $\text{sinc}(x) \equiv \frac{\sin(\pi x)}{\pi x}$ ,  $x \in \mathbb{R}$  and  $Q = \prod_{i=1}^d \omega_i[-1/2, 1/2)$ . Then it holds

$$\frac{1}{\det(\omega)^{1/2}} \int_Q e^{2\pi i(k,\xi)/\omega} e^{2\pi i(\xi,x)} dx = \det(\omega)^{1/2} \prod_{i=1}^d \text{sinc}(\omega_i x_i - k_i), \quad x \in \mathbb{R}^d.$$

*Proof. (Exercise!) Just use Fubini-Tonelli Theorem.*

---

**Theorem 0.16 (of the perturbed sampling in  $L^2$ )** *Let*

$Q = \prod_{i=1}^d \omega_i[-1/2, 1/2)$  *and*  $f \in C(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$  *such that*  
 $f|_{\tau\mathbb{Z}^d} \in \ell^2$ . *We write*  $f = \eta + \epsilon$ , *where*  $\mathcal{F}\eta = \mathcal{F}f$  *on*  $Q$ . *Then it holds*

$$f(x) = \sum_{k \in \mathbb{Z}^n} (f^c(\tau k) - \epsilon^c(\tau k)) \prod_{i=1}^d \text{sinc}(\tau_i^{-1} x_i - k_i) + \epsilon(x), \quad \text{in } L^2(\mathbb{R}^d). \quad (21)$$

---

## PROOF OF THE THEOREM

Let us consider  $P[\mathcal{F}\eta](\xi) = \sum_{k \in \mathbb{Z}^d} \mathcal{F}\eta(\xi - \omega k)$ . We know already that  $P[\mathcal{F}\eta] \in L^2(\omega T^d)$  and thus

$$P[\mathcal{F}\eta](\xi) = \frac{1}{\det(\omega)^{1/2}} \sum_{k \in \mathbb{Z}^d} \mathcal{F}(P[\mathcal{F}\eta])(k) e^{2\pi i(k, \xi)/\omega},$$

where

$$\mathcal{F}(P[\mathcal{F}\eta])(k) = \frac{1}{\det(\omega)^{1/2}} \int_{\omega T^d} P[\mathcal{F}\eta](x) e^{-2\pi i(k, x)/\omega} dx.$$

Clearly the restriction of this function to  $Q$  gives  $P[\mathcal{F}\eta]|_Q = \mathcal{F}\eta$ .

But then we see immediately that

$\mathcal{F}(P[\mathcal{F}\eta])(k) = \frac{1}{\det(\omega)^{1/2}} (f^c(-k/\omega) - \epsilon^c(-k/\omega))$ , since  $\mathcal{F}\eta$  is not vanishing only on a compact set and thus  $\mathcal{F}\eta \in L^1(\mathbb{R}^d)$ . But then, substituting  $P[\mathcal{F}\eta]$  instead of  $\mathcal{F}\eta$  on  $Q$  and extending it

---

to 0 out of  $Q$  we obtain

$$\begin{aligned}
f(x) &= \frac{1}{\det(\omega)^{1/2}} \mathcal{F}^{-1} \left( \sum_{k \in \mathbb{Z}^d} \mathcal{F}(P[\mathcal{F}\eta])(k) e^{2\pi i(k, \xi)/\omega} \chi_Q(\xi) \right) (x) + \epsilon(x) = \\
&= \sum_{k \in \mathbb{Z}^n} \mathcal{F}(P[\mathcal{F}\eta])(k) \mathcal{F}^{-1} \left( \frac{1}{\det(\omega)^{1/2}} e^{2\pi i(k, \xi)/\omega} \chi_Q(\xi) \right) (x) + \epsilon(x) = \\
&= \sum_{k \in \mathbb{Z}^n} (f^c(\tau k) - \epsilon^c(\tau k)) \prod_{i=1}^d \text{sinc}(\tau_i^{-1} x_i - k_i) + \epsilon(x),
\end{aligned}$$

where we used the previous Lemma in the last equality and all the equalities hold in  $L^2$ .

---

**Definition 0.17** For  $Q = \prod_{i=1}^d \omega_i[-1/2, 1/2)$ , we define

$$L_Q^2(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : \text{supp}(\mathcal{F}f) \subset Q\}.$$

We say that  $f \in L_Q^2(\mathbb{R}^d)$  is a  $\omega$ -bandlimited function.



---

**Corollary 0.18 (Whittaker-Shannon)** *If  $f \in L^2(\mathbb{R}^d)$  is a  $\omega$ -bandlimited function, there exists a  $\tau_0 > 0$  such that for all  $0 < \tau \leq \tau_0$*

$$f(x) = \sum_{k \in \mathbb{Z}^d} f^c(\tau k) \prod_{i=1}^d \text{sinc}(\tau_i^{-1} x_i - k_i). \quad (22)$$

---

**Proposition 0.19** *If  $f \in C_c(\mathbb{R}^d)$  then its Fourier transformation  $\mathcal{F}f$  on  $\mathbb{R}^d$  cannot be compactly supported.*

Proof. If  $f \in C_c(\mathbb{R}^d)$  then  $\mathcal{F}f$  is an analytic function (because it has infinitely many bounded derivatives). But a nonzero analytic function has only isolated zeros, hence it cannot be compactly supported (see <http://planetmath.org/zeroesofanalyticfunctionsareisolated> for a proof).

---

## SUMMARY OF THE RESULTS OBTAINED SO FAR

So far we showed that the sampling followed by a DFT

$$\begin{array}{ccccc}
 & & \cdot^c & & \mathcal{F} \\
 C_c & \rightarrow & \ell^2(\mathbb{Z}_m^d) & \rightarrow & \ell^2(\mathbb{Z}_m^d) \\
 f & \rightarrow & \mathbf{f}^c & \rightarrow & \mathcal{F}\mathbf{f}^c
 \end{array}$$

is equivalent to apply a continuous Fourier transform, one periodization and a sampling again

$$\begin{array}{ccccccc}
 & & \mathcal{F} & & P & & \cdot^c \\
 C_c & \rightarrow & L^2(\mathbb{R}^d) & \rightarrow & L^2(\omega T^d) & \rightarrow & \ell^2(\mathbb{Z}^d) \\
 f & \rightarrow & \mathcal{F}f & \rightarrow & P[\mathcal{F}f] & \rightarrow & (P[\mathcal{F}f])^c
 \end{array}$$

Moreover **IF**  $\mathcal{F}f$  is a function decaying very rapidly we have

$$\mathcal{F}f(w) \approx P[\mathcal{F}f](w), \quad \forall w \in Q$$

---

and we can therefore assume the following good approximations

$$\begin{aligned} \mathcal{F}f(w) &\approx \sum_{l \in \mathbb{Z}^d, l/a \in Q} (P[\mathcal{F}f])^c(l/a) \prod_{i=1}^d \text{sinc}(a_i w_i - l_i) \\ &= \frac{\det(a)}{(\det(m))^{1/2}} \sum_{l \in \mathbb{Z}^d, l/a \in Q} \mathcal{F}f^c(l) \prod_{i=1}^d \text{sinc}(a_i w_i - l_i). \end{aligned}$$

**THEREFORE** if  $f$  is a function of compact support, whose Fourier transform decay fast (and this is ensured as soon as a function is sufficiently smooth!) then the Fourier transform can be well-approximated by applying a DFT on the samples of  $f$  and the formula given by the Wittacker-Shannon sampling theorem. Hence, for functions, where it is difficult to apply the techniques of the residuals, we can attempt the approximation by means of discrete Fourier transform.

---

In the following we assume that  $\ell^2(\tau\mathbb{Z}^d)$  is endowed with the scalar product  $\langle f, g \rangle_{\ell^2} := \det(\tau) \sum_{k \in \mathbb{Z}^d} f(\tau k) \overline{g(\tau k)}$ .

**Theorem 0.20 (Perturbed scalar product)** *Let us assume that  $f, g \in C(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$  tali che  $f|_{\tau\mathbb{Z}^d}, g|_{\tau\mathbb{Z}^d} \in \ell^2$ . Then the following approximation on the scalar products of  $f$  and  $g$  holds:*

$$\begin{aligned}
 & |\langle f, g \rangle_{L^2} - \langle f|_{\tau\mathbb{Z}^d}, g|_{\tau\mathbb{Z}^d} \rangle_{\ell^2}| \leq \\
 & \leq \|\varepsilon_f\|_{\ell^2} \|g\|_{\ell^2} + \|\varepsilon_g\|_{\ell^2} \|f\|_{\ell^2} + \|\varepsilon_f\|_{\ell^2} \|\varepsilon_g\|_{\ell^2} + \|\varepsilon_f\|_{L^2} \|\varepsilon_g\|_{L^2}. \quad (23)
 \end{aligned}$$

---

## PROOF OF THE THEOREM

By the Theorem of the Perturbed Sampling in  $L^2$  we obtain by considering the scalar products:

$$\begin{aligned}
 \langle f, g \rangle_{L^2} &= \int_{\mathbb{R}^d} f(x) \overline{g(x)} dx \\
 &= \int_{\mathbb{R}^d} \left( \sum_{k \in \mathbb{Z}^d} (f(\tau k) - \varepsilon_f(\tau k)) \prod_{i=1}^d \text{sinc}(\tau_i^{-1} x_i - k_i) + \varepsilon_f(x) \right) \times \\
 &\quad \times \left( \sum_{h \in \mathbb{Z}^d} (g(\tau h) - \varepsilon_g(\tau h)) \prod_{i=1}^d \text{sinc}(\tau_i^{-1} x_i - h_i) + \varepsilon_g(x) \right)
 \end{aligned}$$

As the functions of the type  $\eta_*$  are orthogonal in  $L^2$  to functions of the type  $\varepsilon_*$ , we have:

$$\langle f, g \rangle_{L^2} = \sum_{h, k \in \mathbb{Z}^d} (f(\tau k) - \varepsilon_f(\tau k)) \overline{(g(\tau h) - \varepsilon_g(\tau h))} \times$$

---


$$\times \left( \int_{\mathbb{R}^d} \prod_{i=1}^d \operatorname{sinc}(\tau_i^{-1} x_i - k_i) \operatorname{sinc}(\tau_i^{-1} x_i - h_i) dx \right) + \langle \varepsilon_f, \varepsilon_g \rangle_{L^2} =$$

and by the orthogonality of the “sinc” these identities continue as follows

$$\begin{aligned} &= \det(\tau) \cdot \sum_{k \in \mathbb{Z}^d} (f(\tau k) - \varepsilon_f(\tau k)) \overline{(g(\tau k) - \varepsilon_g(\tau k))} + \langle \varepsilon_f, \varepsilon_g \rangle_{L^2} \\ &= \langle f|_{\tau\mathbb{Z}^d}, g|_{\tau\mathbb{Z}^d} \rangle_{\ell^2} + \langle f|_{\tau\mathbb{Z}^d}, (\varepsilon_g)|_{\tau\mathbb{Z}^d} \rangle_{\ell^2} + \langle (\varepsilon_f)|_{\tau\mathbb{Z}^d}, g|_{\tau\mathbb{Z}^d} \rangle_{\ell^2} \\ &\quad + \langle (\varepsilon_f)|_{\tau\mathbb{Z}^d}, (\varepsilon_g)|_{\tau\mathbb{Z}^d} \rangle_{\ell^2} + \langle \varepsilon_f, \varepsilon_g \rangle_{L^2} \end{aligned}$$

Hence, by using the Cauchy-Schwarz inequality, we obtain the following estimate:

$$\begin{aligned} &|\langle f, g \rangle_{L^2} - \langle f|_{\tau\mathbb{Z}^d}, g|_{\tau\mathbb{Z}^d} \rangle_{\ell^2}| \leq \\ &\leq \|\varepsilon_f\|_{\ell^2} \|g\|_{\ell^2} + \|\varepsilon_g\|_{\ell^2} \|f\|_{\ell^2} + \|\varepsilon_f\|_{\ell^2} \|\varepsilon_g\|_{\ell^2} + \|\varepsilon_f\|_{L^2} \|\varepsilon_g\|_{L^2}. \end{aligned}$$

---

**Remark 0.21** *If  $f, g$  are two  $\omega$ -bandlimited functions we have the identity*

$$\int_{\mathbb{R}^n} f(x) \overline{g(x)} dx = \det(\tau) \sum_{k \in \mathbb{Z}^n} f^c(\tau k) \overline{g^c(\tau k)}.$$



---

## DIGITALIZATION AND RECOVERY: REDUNDANCY

We considered so far a system of digitalization by sampling: for every  $f \in L^2_Q$  the coefficient map  $C : L^2_Q \rightarrow \ell^2$  is given by  $C(f) = \{f(\tau k)\}_{k \in \mathbb{Z}^d}$  and it can be inverted by the recovery map  $R : \ell^2 \rightarrow L^2_Q$  defined by

$R(\vec{c}) = \sum_{k \in \mathbb{Z}^d} c_k \prod_{i=1}^d \text{sinc}(\tau_i^{-1} x_i - k_i)$ . Indeed we have  $I = R \circ C$ . Moreover, by the previous remark, we know that this system of digitalization is stable, since we have an equivalence of norms. Unfortunately the system is NOT redundant as  $C$  is the unique coefficient map which is the right-inverse  $R$ . In fact the  $\text{sinc}(\tau_i^{-1} x_i - k_i)$  constitute an orthonormal basis! Can we construct redundant systems?

---

## TRANSLATION AND MODULATION

As for discrete signals  $\mathbb{Z}_n^d$ , also for functions on  $\mathbb{R}^d$  we can define the operators of translation and modulation:

$$T_x f(t) = f(t - x), \quad M_w f(t) = e^{2\pi i w t} f(t). \quad (24)$$

and it holds (exercise!)

$$\mathcal{F}T_x = M_{-x}\mathcal{F}, \quad \mathcal{F}M_w = T_w\mathcal{F}. \quad (25)$$

Moreover if  $f, g \in L^2(\mathbb{R}^d)$  we define the  $L^2(\mathbb{R}^d)$  function given by the convolution of  $f$  and  $g$  by

$$f \star g(x) = \int_{\mathbb{R}^d} f(x - y)g(y)dy. \quad (26)$$

and we have that (exercise!)

$$\mathcal{F}(f \star g) = \mathcal{F}(f) \cdot \mathcal{F}(g). \quad (27)$$

---

**Proposition 0.22** *The translation operator is given by  $T_x g(t) = g(t - x)$  for any function  $g$  on  $\mathbb{R}^d$ . Given band-limited function  $g \in L^1(\mathbb{R}^d)$  such that  $\mathcal{F}g \neq 0$  on a compact set  $\Omega \subset \mathbb{R}^d$ , then there exists  $\tau > 0$  such that for all  $f \in L^2_\Omega(\mathbb{R}^d)$*

$$f = \sum_{k \in \mathbb{Z}^d} c_k(f) T_{\tau k} g, \tag{28}$$

*for suitable coefficients  $(c_k(f))_{k \in \mathbb{Z}^d} \in \ell^2$  depending continuously on  $f$ . Such coefficients are NOT in general unique.*

---

## PROOF OF THE PROPOSITION

We know that  $\{\tau^{d/2} e^{2\pi i \tau k x}\}_{k \in \mathbb{Z}^d}$  is an orthonormal basis for  $L^2([-1/(2\tau), 1/(2\tau)]^d)$ . For  $\tau > 0$  sufficiently small  $\Omega \subset [-1/(2\tau), 1/(2\tau)]^d$ . There exists  $g_1 \in L^1$  a bandlimited function, such that  $\mathcal{F}g_1 \cdot \mathcal{F}g \equiv 1$  on  $\Omega$ . Hence for  $f \in L^2_\Omega(\mathbb{R}^d)$  and for  $\tau > 0$  sufficiently small

$$\mathcal{F}f(w) = [(\mathcal{F}f\mathcal{F}g_1)\mathcal{F}g](w) = \sum_{k \in \mathbb{Z}^d} \tau^d \langle \mathcal{F}f\mathcal{F}g_1, e^{2\pi i \tau k x} \rangle e^{2\pi i \tau k w} \mathcal{F}g(w). \quad (29)$$

By applying the inverse Fourier transform we obtain

$$f = \sum_{k \in \mathbb{Z}^d} \tau^d (f \star g_1)(\tau k) T_{\tau k} g, \quad (30)$$

---

where  $\star$  is the convolution symbol

$$f \star g(x) = \int_{\mathbb{R}^d} f(x - y)g(y)dy.$$

In particular we can prove

$$\sum_{k \in \mathbb{Z}^d} \tau^d |(f \star g_1)(\tau k)|^2 = \|f \star g_1\|_2^2 \leq \|f\|_2^2 \|g_1\|_1^2. \quad (31)$$

Hence, if we set  $c_k(f) = \tau^d (f \star g_1)(\tau k)$  we have

$$\sum_{k \in \mathbb{Z}} |c_k(f)|^2 \leq (\tau^d \|g_1\|_1^2) \|f\|_2^2. \quad (32)$$

In particular,

$$\sum_{k \in \mathbb{Z}} |\langle f, T_{\tau k} g \rangle|^2 = \sum_{k \in \mathbb{Z}} |\langle \mathcal{F}f, e^{2\pi i \tau k w} \mathcal{F}g \rangle|^2 \leq \tau^{-d} \|f\|_2^2 \|\mathcal{F}g\|_\infty^2.$$

---

Let us now consider  $Q = \prod_{i=1}^d \omega_i [-1/2, 1/2)$ . Surely we have that  $\{\omega j + Q\}_{j \in \mathbb{Z}^d}$  define a disjoint covering of  $\mathbb{R}^d$ , that means  $\mathbb{R}^d = \bigcup_{j \in \mathbb{Z}^d} (\omega j + Q)$ . Every function  $f \in L^2(\mathbb{R}^d)$  can be written as the following decomposition:

$$f = \mathcal{F}^{-1} \left( \sum_{j \in \mathbb{Z}^d} \mathcal{F} f \cdot \chi_{\omega j + Q} \right) = \sum_{j \in \mathbb{Z}^d} \mathcal{F}^{-1} (\mathcal{F} f \cdot \chi_{\omega j + Q}). \quad (33)$$

Each function  $\mathcal{F}^{-1} (\mathcal{F} f \cdot \chi_{\omega j + Q})$  is bandlimited with  $\text{supp}((\mathcal{F} f \cdot \chi_{\omega j + Q})) \subset \omega j + Q \Leftrightarrow \text{supp}(T_{\omega j} (\mathcal{F} f \cdot \chi_{\omega j + Q})) \subset Q$ .

---

Hence, we have

$$M_{-\omega_j} \mathcal{F}^{-1} (\mathcal{F} f \cdot \chi_{\omega_j+Q}) = M_{-\omega_j} f_j,$$

is a  $\omega$ -bandlimited function, where we set

$f_j = \mathcal{F}^{-1} (\mathcal{F} f \cdot \chi_{\omega_j+Q})$ . From the Wittacker-Shannon sampling Theorem we have immediately that

$$M_{-\omega_j} f_j = \sum_{k \in \mathbb{Z}^d} f_j(\tau k) \prod_{i=1}^d \text{sinc}(\tau_i^{-1} x_i - k_i),$$

and

$$f_j = \sum_{k \in \mathbb{Z}^d} f_j(\tau k) M_{\omega_j} \prod_{i=1}^d \text{sinc}(\tau_i^{-1} x_i - k_i). \quad (34)$$

---

Combining the previous expressions we get

$$f = \sum_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} f_j(\tau k) M_{\omega j} T_{\tau k} \left( \prod_{i=1}^d \text{sinc}(\tau_i^{-1} x_i) \right). \quad (35)$$



---

## THE GABOR TRANSFORM

Let us define

$$V_g(f)(b, \omega) = \langle f, M_\omega T_b g \rangle = \int_{\mathbb{R}^d} f(t) e^{-2\pi i \omega \cdot t} \overline{g(t - b)} dt \quad (36)$$

For  $\tau > 0$  fixed small and  $\omega = 1/\tau$  we have

$$\begin{aligned} f &= \sum_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} f_j(\tau k) M_{\omega j} T_{\tau k} \underbrace{\left( \prod_{i=1}^d \text{sinc}(\tau_i^{-1} x_i) \right)}_{:=g} \\ &= \sum_{j, k \in \mathbb{Z}^d} \underbrace{\frac{1}{\det(\tau)} V_g(f)(\tau k, \omega j)}_{:=f_j(\tau k)} M_{\omega j} T_{\tau k} g \end{aligned}$$

---

A signal  $f$  is mapped to the time-frequency plane by means of the transform  $V_g(f)(b, w)$ , realized by considering the scalar product in  $L^2$  of  $f$  with respect to an analyzing function  $g$  modified by an application of a translation and a modulation.

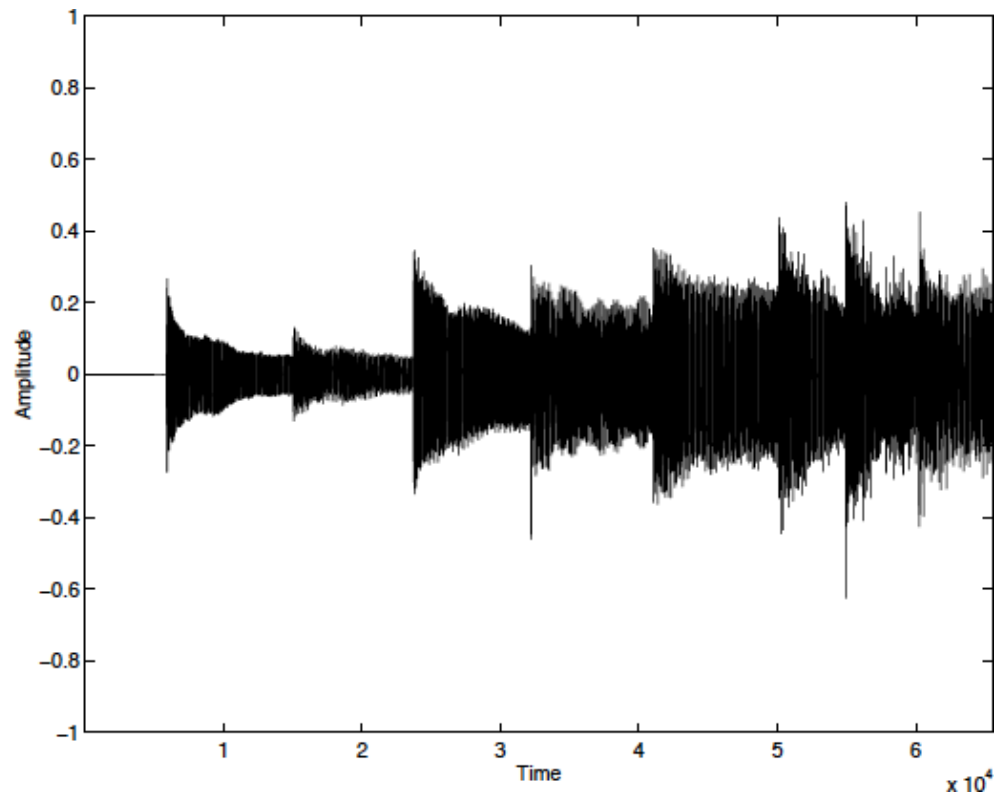


Figure 5: A test signal

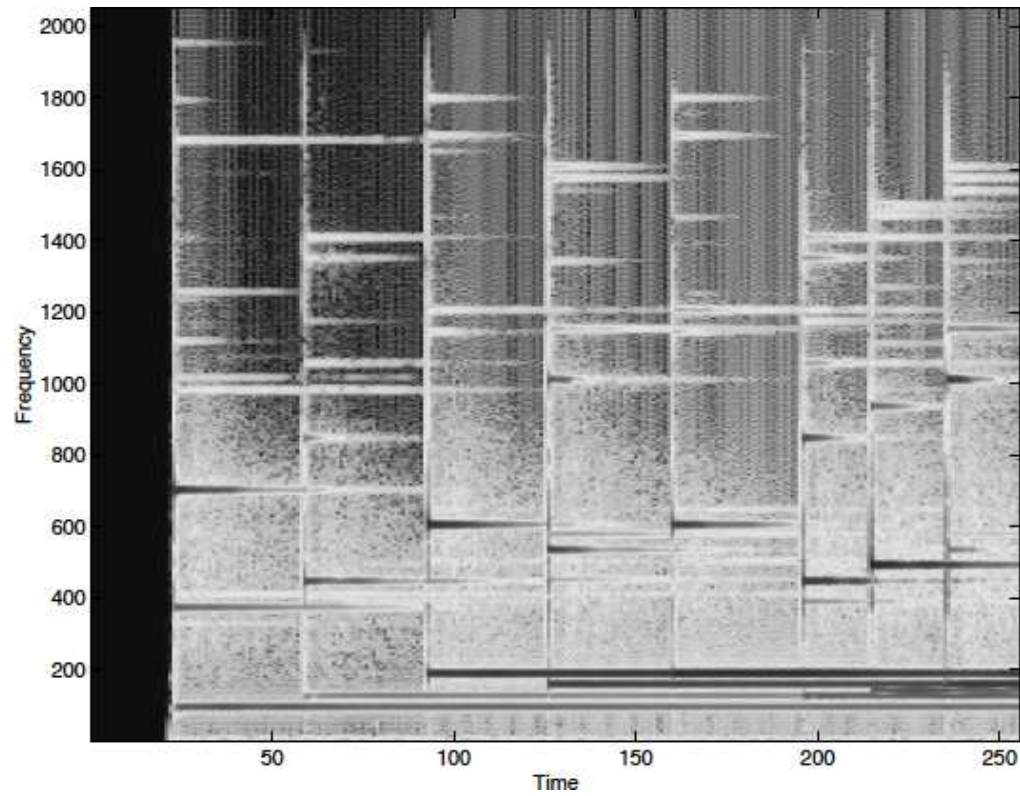


Figure 6: Spectrogram of  $|V_g f(b, w)|^2$  of the test signal

---

## UNCERTAINTY PRINCIPLE

**Theorem 0.23 (Heisenberg uncertainty principle)** *Let  $g \in L^2(\mathbb{R})$  and  $a, b \in \mathbb{R}$  two arbitrary scalars. Then*

$$\left( \int (x - a)^2 |g(x)|^2 dx \right)^{1/2} \left( \int (\omega - b)^2 |\mathcal{F}g(\omega)|^2 d\omega \right)^{1/2} \geq \frac{\|g\|_2^2}{4\pi}.$$

*If  $g(x) = e^{-\pi|x|^2}$  is a Gaussian, then it holds*

$$\left( \int (x - a)^2 |g(x)|^2 dx \right)^{1/2} \left( \int (\omega - b)^2 |\mathcal{F}g(\omega)|^2 d\omega \right)^{1/2} = \frac{\|g\|_2^2}{4\pi}.$$

A window (analyzing function) cannot be arbitrarily concentrated in time and frequency. The Gaussian is the function which is equally concentrated in time and frequency.

---

Hence for an arbitrary window function  $g$ , the Gabor transform has more resolution either in time or in frequency. In particular the function  $g(x) = \text{sinc}(x)$  is badly localized in time. We wonder how can we recover  $f$  by means of superpositions of functions of the type  $M_w T_x g$  with  $g$  different from  $\text{sinc}(x)$

$$f = \sum_{j,k \in \mathbb{Z}^d} c_{j,k} M_{\omega_j} T_{\tau k} g.$$

---

## FRAMES IN HILBERT SPACES

Let  $\mathcal{H}$  be a separable Hilbert space.

**Definition 0.24** *A set  $\{g_n\}_{n \in \mathcal{N}} \subset \mathcal{H}$  is a frame for  $\mathcal{H}$  if there exist  $A, B > 0$  such that*

$$A \cdot \|f\|^2 \leq \sum_{n \in \mathcal{N}} |\langle f, g_n \rangle|^2 \leq B \cdot \|f\|^2, \quad \forall f \in \mathcal{H}. \quad (37)$$

Note that this is a weaker version of the Parseval identity in the Fourier theorem. In particular orthonormal bases are frames!

---

Let us define the *frame operator* by  $S : \mathcal{H} \rightarrow \mathcal{H}$

$$Sf = \sum_{n \in \mathcal{N}} \langle f, g_n \rangle g_n. \quad (38)$$

In particular the frame definition implies that  $S$  is positive, self-adjoint, and invertible. Then we have

$$f = SS^{-1}f = \sum_{n \in \mathcal{N}} \langle f, S^{-1}g_n \rangle g_n = S^{-1}Sf = \sum_{n \in \mathcal{N}} \langle f, g_n \rangle S^{-1}g_n. \quad (39)$$

The system  $\{\tilde{g}_n = S^{-1}g_n\}_{n \in \mathcal{N}}$  is again a frame, called the *canonical dual* of  $\{g_n\}_{n \in \mathcal{N}}$  with corresponding frame operator  $S^{-1}$ .



---

Since a frame is typically redundant, in the sense that there is not only one coefficient map  $\{c_n\}_{n \in \mathcal{N}}$  such that

$$f = \sum_{n \in \mathcal{N}} c_n(f) g_n,$$

by the Riesz-Fischer duality theorem, there exist many possible duals  $\{\tilde{g}_n\}_{n \in \mathcal{N}} \subset \mathcal{H}$  such that

$$f = \sum_{n \in \mathcal{N}} \langle f, \tilde{g}_n \rangle g_n.$$

---

**Example 0.25** *Let*

$\mathcal{H} = \mathbb{R}^2$ ,  $f = (-1, 3)$  and  $g_1^T = (1, -1)$ ,  $g_2^T = (0, 1)$ ,  $g_3^T = (1, 1)$ .

*The coefficients  $(c_n)_{n \in \{1,2,3\}}$  of  $f$  are*

$$(c_n)_{n \in \{1,2,3\}} = (\langle f, g_1 \rangle, \langle f, g_2 \rangle, \langle f, g_3 \rangle) = (-4, 3, 2).$$

*The canonical dual frame of  $\{\tilde{g}_n\}_{n \in \{1,2,3\}}$  is given by*

$$\{\tilde{g}_n\}_{n \in \{1,2,3\}} = \{S^{-1}g_n\}_{n \in \{1,2,3\}} = \left\{ \begin{pmatrix} 1/2 \\ -1/3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix} \right\}.$$

*Hence I obtain the recovery of  $f$  by the identity:*

$$f = \sum_{n=1}^3 c_n \tilde{g}_n = \left( \begin{pmatrix} -2 \\ \frac{4}{3} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix} \right) = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

---

If I choose another dual, for instance

$$\{h_n\}_{n \in \{1,2,3\}} = \left\{ \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

we have again the recovery of  $f$

$$f = \sum_{n=1}^3 c_n h_n = \left( \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

---

## FRAMES IN FINITE DIMENSIONS

One can prove that any finite dimensional system

$\mathcal{G}_n = \{g_0, \dots, g_{n-1}\}$  is always a frame for its span

$\mathcal{H} = \text{span}\{\mathcal{G}_n\} = \{\sum_{i=0}^n c_i g_i : c_i \in \mathbb{C}\}$  (exercise!).

How do we compute the canonical dual frame in finite dimension? Let us consider an orthonormal basis  $\{e_i\}_{i=0}^{m-1}$ ,

$m \leq n$  of  $\mathcal{H}$  and we consider the matrix

$\mathcal{G}_{m \times n} = (\langle g_k, e_i \rangle)_{i=0, \dots, m-1, k=0, \dots, n-1}$ . In particular we have that

for all  $f \in \mathcal{H}$  and considered  $\vec{f} = (\langle f, e_i \rangle)_{i=0, \dots, m-1}$

$$C_g f = \mathcal{G}_{m \times n}^T * \vec{f}. \quad (40)$$

---

## SINGULAR VALUE DECOMPOSITION

The so-called Singular Value Decomposition (SVD) is a procedure applied on a matrix  $M \in \mathbb{C}^{n \times m}$  and it is a very important one in numerical linear algebra. The basic idea is to decompose the matrix  $M$  in the form  $M = U^* * D * V$  where  $D$  is a diagonal matrix with nonnegative diagonal entries and  $U, V$  are two matrices whose rows are orthonormal,  $U^* = \overline{U^T}$  is the Hermitian transpose of  $U$  and coincides with its inverse. The nonzero elements of  $D$  are known as the *singular values* of  $M$ . The SVD of a matrix is unique.

**[http://en.wikipedia.org/wiki/Singular\\_value\\_decomposition](http://en.wikipedia.org/wiki/Singular_value_decomposition)**

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But then a left inverse of  $M$  can be computed as follows

$$M^\dagger = V^* * D^{-1} * U.$$

In fact  $M^\dagger * M = (U^* * D * V) * (V^* * D^{-1} * U) = I$ . One can also show that the sum of the squared entries of the matrix  $(M * M^\dagger - I)$  is minimal and hence  $M^\dagger$  is the so-called *pseudo-inverse* matrix of  $M$ . Moreover, we have

$$M^\dagger * M = M^T * (M^\dagger)^T = M^T * (M^T)^\dagger.$$

Hence, for a matrix  $N \in \mathbb{C}^{m \times n}$  with  $n \geq m$  we define  $N^\dagger = (N^T)^\dagger$  and we have  $N * N^\dagger = I$ .

---

The columns of the matrix  $\left(\mathcal{G}_{m \times n}^\dagger\right)^T$  represents the coefficients  $\{\langle \tilde{g}_k, e_i \rangle\}_{i=0, \dots, m-1}$  of the canonical dual of  $\mathcal{G}_n$ . If I consider a set of coefficients  $\vec{c} = (c_k)_{k=0, \dots, n-1}$  we have that  $\left(\mathcal{G}_{m \times n}^\dagger\right)^T * \vec{c}$  returns a vector of length  $m$ . In particular we have

$$R_{\tilde{g}} \vec{c} = \left( \left( \mathcal{G}_{m \times n}^\dagger \right)^T * \vec{c} \right)^T \cdot \{e_i\}_{i=0}^{m-1}. \quad (41)$$

---

Hence, we have the identities

$$\begin{aligned} f &= R_{\tilde{g}} \circ C_g f = \left( \left( \mathcal{G}_{m \times n}^\dagger \right)^T * \mathcal{G}_{m \times n}^T * \vec{f} \right)^T \cdot \{e_i\}_{i=0}^{m-1} = \\ &= \left( \vec{f}^T * \mathcal{G}_{m \times n} * \mathcal{G}_{m \times n}^\dagger \right) \cdot \{e_i\}_{i=0}^{m-1} \end{aligned} \quad (42)$$



---

Moreover we have,

$$R_g \vec{c} = \left( \mathcal{G}_{m \times n} * \vec{c} \right)^T \cdot \{e_i\}_{i=0}^{m-1}.$$

Hence, defined the frame operator  $\mathcal{S}_g = \mathcal{G}_{m \times n} * \mathcal{G}_{m \times n}^T$  we obtain

$$S_g f = R_g \circ C_g f = \left( \mathcal{G}_{m \times n} * \mathcal{G}_{m \times n}^T * \vec{f} \right)^T \cdot \{e_i\}_{i=0}^{m-1} = \left( \mathcal{S}_g * \vec{f} \right)^T \cdot \{e_i\}_{i=0}^{m-1}.$$

But then we can compute the canonical dual vectors by using

$$\tilde{g}_k = \left( \mathcal{S}_g^{-1} * \vec{g}_k \right)^T \cdot \{e_i\}_{i=0}^{m-1}.$$

---

## GABOR FRAMES IN FINITE DIMENSIONS

Let  $d = 1$  and we consider  $a, b, L \in \mathbb{N}$  such that  $a|L$  and  $b|L$  and  $a \cdot b \leq L$ . Let us set  $N = L/a$  e  $M = L/b$ . Then we define the discrete Gabor system

$$\mathbf{g}_{m,n} = M \frac{mb}{L} T_{an} \mathbf{g}, \quad m = 0, \dots, M-1, \quad n = 0, \dots, N-1, \quad (43)$$

where  $\mathbf{g} \in \mathbb{Z}_L$ . Note that  $N \cdot M \geq L$ . We seek for frames for  $\ell^2(\mathbb{Z}_L)$  of the type  $\mathcal{G}(\mathbf{g}, a, b) = \{\mathbf{g}_{m,n}\}_{m=0,\dots,M-1,n=0,\dots,N-1}$ . Let us consider  $\{e_i\}_{i=0}^{L-1}$  the canonical basis of  $\mathcal{H} = \ell^2(\mathbb{Z}_L)$ ; we can define the matrix

$$\mathcal{G}_{L \times N \cdot M} = \left( \mathbf{g}_{\text{mod}(k,M)}, \frac{k - \text{mod}(k,M)}{M} \right)_{k=0,\dots,N \cdot M-1}, \quad (44)$$

whose columns represent the valued of  $\mathcal{G}(\mathbf{g}, a, b) = \{\mathbf{g}_{m,n}\}_{m=0,\dots,M-1,n=0,\dots,N-1}$ .

---

As we already showed, the matrix of the frame operator is given by

$$\mathcal{S}_g = \mathcal{G}_{L \times N \cdot M} * \mathcal{G}_{L \times N \cdot M}^T. \quad (45)$$

In fact we have  $\mathcal{S}_g : \ell^2(\mathbb{Z}_L) \rightarrow \ell^2(\mathbb{Z}_L)$ . The the system is a Gabor frame is the matrix  $\mathcal{S}_g$  is invertible, i.e., when  $\det(\mathcal{S}_g) \neq 0$ .

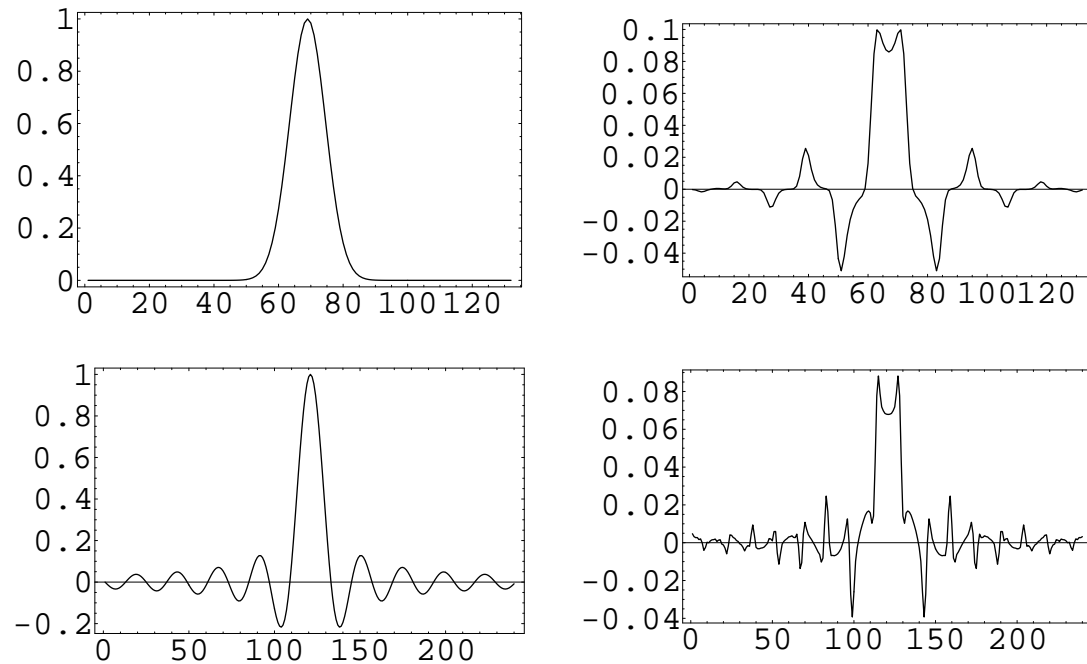


Figure 7: Gabor atoms and corresponding duals. On the top it is represented the Gaussian and on its right its corresponding dual for  $L = 132$ ,  $a = b = 11$ , with redundancy  $L/(ab) = 1.09$ . On the bottom, we represented the cardinal sin and its relative dual for  $L = 240$ ,  $a = b = 15$ , with redundancy  $L/(ab) = 1.06$ .

---

## SIGNAL COMPRESSION

Once we constructed a Gabor frame  $\mathcal{G}(\mathbf{g}, a, b)$  and its corresponding canonical dual  $\mathcal{G}(\tilde{\mathbf{g}}, a, b)$  every signal  $\mathbf{f}$  of length  $L$  can be (de)composed as follows:

$$\mathbf{f} = \sum_{m,n} \langle \mathbf{f}, \tilde{\mathbf{g}}_{m,n} \rangle \mathbf{g}_{m,n}. \quad (46)$$

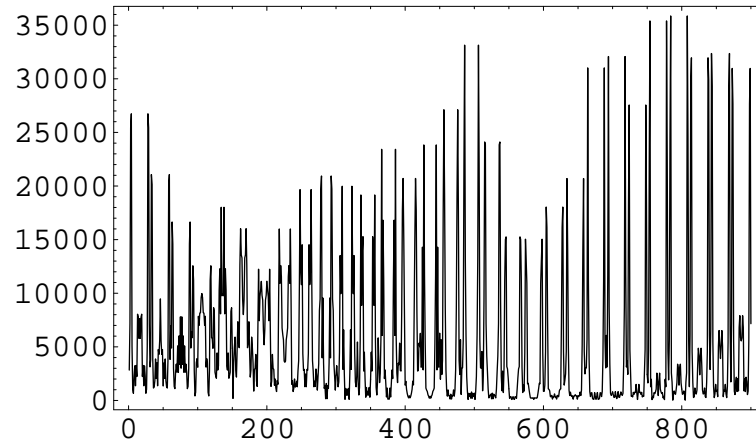


Figure 8: Coefficients  $|\langle \mathbf{f}, \tilde{\mathbf{g}}_{m,n} \rangle|^2$ .

---

The number of coefficients is  $N \cdot M \geq L$  and it is a redundant digitalization. The information is spread on a larger number of digital samples.

The redundancy enlarges the possibility of identification. *The larger is the dictionary, the shorter are the sentences I can write!*

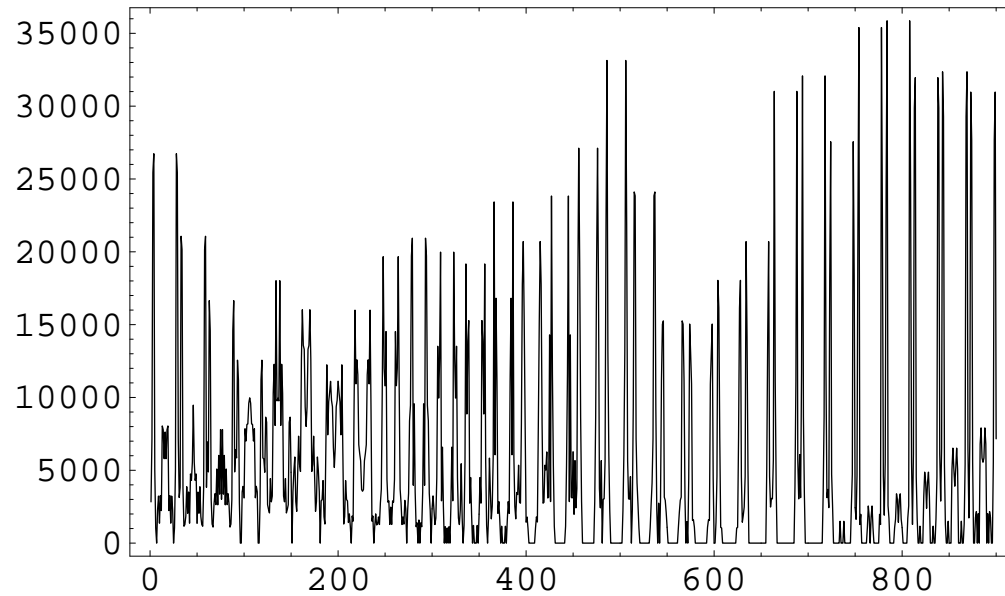


Figure 9: Coefficients  $|\langle \mathbf{f}, \tilde{\mathbf{g}}_{m,n} \rangle|^2$  over a give threshold.

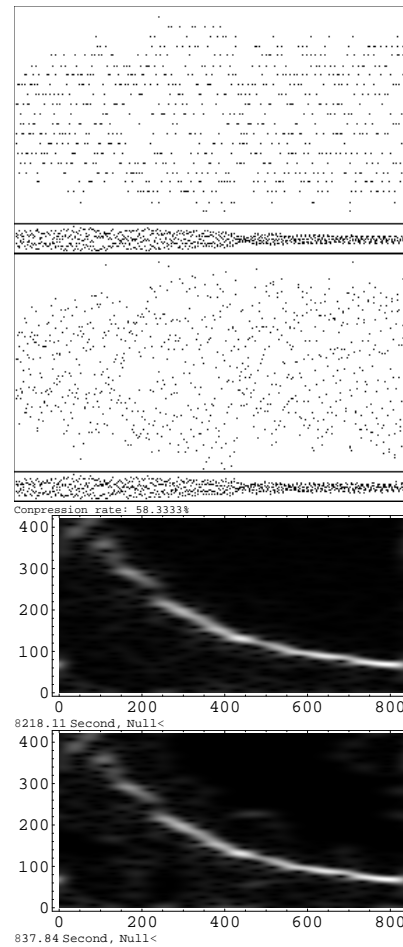


Figure 10: Coefficients  $|\langle \mathbf{f}, \tilde{\mathbf{g}}_{m,n} \rangle|^2$  over a give threshold.



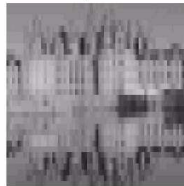
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original image



compression using cos.-transform

Comp. ratio: 32



Comp. ratio: 16



Comp. ratio: 5.3333



compression using gabor analysis

Comp. ratio: 32.4722 red.: 2.56



Comp. ratio: 15.9728 red.: 2.56



Comp. ratio: 5.2548 red.: 2.56



Figure 11: Compression of an image by Gabor frames and JPEG