Mathematics of Digitalization: Case Study

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Fifth lecture

Introduction

Main topics:

1. After a short introduction on the Hilbert spaces, we describe the concepts of Fourier series, Fourier transforms and their discrete implementation, in particular the algorithm of the Fast Fourier Transform (FFT).

2. To connect the continuous Fourier transforms and the discrete one, we need to address the Shannon sampling theory (analog-to-digital conversion). For that it will be fundamental to understand the Poisson summation formula which will allow us to estimate also the errors in the analog-to-digital conversion, the so-called *aliasing* errors.

3. We shall introduce systems for robust analog-to-digital

conversion called *frames* in Hilbert spaces. We address special cases of frames, in particular Gabor frames in their continuous and discrete versions and the implementation of their so-called canonical duals. Applications to audio signals and in image compression will be showed.

4. We address in more detail the study case of the fresco restoration and how to use the tools we considered so far to solve it.

The importance of harmonic analysis

- We can synthesize a variety of complicated functions by means of pure sinusoids in the same way we can produce a chord of C major pushing the keyboard (C, E, G) of a piano.

- It was Joseph Fourier (http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html) who first developed the modern methods of trigonometric series and integrals in the study of the heat conduction in the solids (*Analytical Theory of Heat*, 1815).

SOME OF THE MILESTONES

- The most cited paper in mathematics
  J. W. Cooley and J. W. Tukey, An algorithm for the machine computation of complex Fourier series, Math. Comp., 19 (1965), 297-301.

- Almost $3/4$ of the Nobel prizes in Physics are given on work using methods and tools from Fourier analysis.
  F. N. Magill, ed., *The Nobel Prize Winners — Physics*, Vol 1-3, Salem Press. Englewood Cliffs, NJ. 1989.

- Francis Crick, James Watson and Maurice Wilkins got the Nobel prize for the medicine in 1962 for the discovery of the molecular structure of the DNA. This was the first example of the use of Fourier methods on data coming from X ray diffraction.

- Herbert Hauptmann (a mathematician) and Jerome Karle shared in 1985 the Nobel prize for chemistry after having shown how to use systematically Fourier analysis in order to determine the structure of large molecules from X ray diffraction data.
  W. A. Hendrickson, *the 1985 Nobel Prize in Chemistery*, Science 231 (1986), 362-364.
- The starting of the time-frequency analysis and the wavelets (1960-1990).
- A large number of technological products (CD-DVD, high definition TV, digital phones, medical imaging instruments ...) are based on discrete Fourier analysis.

Hilbert spaces

Let $\mathcal{H}$ be a vector space. A scalar product $\langle u, v\rangle$ is a map from $\mathcal{H} \times \mathcal{H}$ to $\mathbb{C}$ such that:

(i) $\langle au + bv, z\rangle = a\langle u, z\rangle + b\langle v, z\rangle$ for all u,v,z $\in \mathcal{H}$ and $a, b \in \mathbb{C}$.

(ii) $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all u,v $\in \mathcal{H}$.

(iii) $\langle u, u\rangle \in \mathbb{R}$, $\langle u, u\rangle \geq 0$ for all $u \in \mathcal{H}$ and $\langle u, u\rangle \neq 0$ if $u \neq 0$.

We recall that a scalar product verify the Cauchy-Schwarz inequality (which will be proved in the exercises):

$$|\langle u, v\rangle| \leq \langle u, u\rangle^{1/2}\langle v, v\rangle^{1/2} \qquad \forall u, v \in \mathcal{H}. \tag{1}$$

and that $\|u\| = \langle u, u\rangle^{1/2}$ define a norm for $\mathcal{H}$.

A Hilbert spaces is a vector space $\mathcal{H}$ endowed with a scalar product $\langle u, v\rangle$ and is complete with respect to the norm $\langle u, u\rangle^{1/2}$.

Example 0.1 *Some relevant examples:*

- *Let* $\Omega \subset \mathbb{R}^n$*. The vector space of functions* $L^2(\Omega) = \{f : \Omega \to \mathbb{C} | \int_\Omega |f|^2 dx < \infty\}$ *is a Hilbert spaces with the scalar product:*

$$\langle u, v \rangle = \int_\Omega u(x)\overline{v(x)}dx.$$

- *Let* $\Omega_d \subset \mathbb{Z}^n$*. The vector space* $\ell^2(\Omega_d) = \{f : \Omega_d \to \mathbb{C} | \sum_{k \in \Omega_d} |f(k)|^2 < \infty\}$ *is a Hilbert spaces with the scalar product:*

$$\langle u, v \rangle = \sum_{k \in \Omega_d} u(k)\overline{v(k)}.$$

In particular if $|\Omega_d| = d < \infty$ *then* $\ell^2(\Omega_d) = \mathbb{C}^d$*.*

Definition 0.2 *Let E be a topological vector space. We say that E' is the topological dual space of E, i.e., E'={φ from E to $\mathbb{C} \mid \varphi$ is continuous and linear}. Given two Hilbert spaces $\mathcal{H}, \mathcal{K}$, the set of linear operators $T : \mathcal{H} \to \mathcal{K}$, i.e., $T(au + bv) = aT(v) + bT(v)$, which are also continuous is a vector space which, endowed with the norm $|||T|||_{\mathcal{H}\to\mathcal{K}} \equiv \sup_{u\in\mathcal{H}, u\neq 0} \frac{\|Tu\|_{\mathcal{K}}}{\|u\|_{\mathcal{H}}}$ is a complete normed space, i.e., a Banach space, and we indicate it with $\mathcal{L}(\mathcal{H}, \mathcal{K})$. In particular, if $\mathcal{H} = \mathcal{K}$ the we set $\mathcal{L}(\mathcal{H}) \equiv \mathcal{L}(\mathcal{H}, \mathcal{H})$ and if $\mathcal{K} = \mathbb{C}$ then we have $\mathcal{L}(\mathcal{H}, \mathcal{K}) = \mathcal{H}'$.*

Theorem 0.3 (Representation theorem of Riesz-Frechet) *Let* $\mathcal{H}$ *be a Hilbert space. For every* $\varphi \in \mathcal{H}'$ *there exists a unique* $f \in \mathcal{H}$ *such that*

$$\langle \varphi, v \rangle \equiv \varphi(v) = \langle v, f \rangle \qquad \forall v \in \mathcal{H}.$$

Moreover we have

$$\|f\| = \|\varphi\|_{\mathcal{H}'} \equiv \sup_{\|v\|=1} |\langle \varphi, v \rangle|.$$

Proposition 0.4 (Bessel inequality) *Let* $\{u_\alpha\}_{\alpha\in A}$ *be an orthonormal system in* $\mathcal{H}$*, i.e.,* $\langle u_\alpha, u_\beta\rangle = \delta_{\alpha,\beta}$ *where* $\delta_{\cdot,\cdot}$ *is the Kronecker symbol. Then for* $x \in \mathcal{H}$*:*

$$\sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2 \leq \|x\|^2.$$

In particular $\{\alpha | \langle x, u_\alpha\rangle \neq 0\}$ *is countable.*

Theorem 0.5 (Fourier) *Let $\{u_\alpha\}_{\alpha \in A}$ be a countable orthonormal system. Then the following are equivalent:*

(i) $x = \sum_{\alpha \in A} \langle x, u_\alpha \rangle u_\alpha$ *for all* $x \in \mathcal{H}$.

(ii) *(Parseval idendity)* $\|x\|^2 = \sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2$ *for all* $x \in \mathcal{H}$.

(iii) *(Completeness) If* $x \in \mathcal{H}$ *and if* $\langle x, u_\alpha \rangle = 0$ *for all* α*, then* $x = 0$.

The sequence $\langle x, u_\alpha \rangle$ *is such that* $\sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2 < \infty$*, then the series* $\sum_{\alpha \in A} \langle x, u_\alpha \rangle u_\alpha$ *is convergent in* $\mathcal{H}$ *and converges to* x*. A orthonormal set which verifies one of the previous conditions is called an orthonormal basis for the Hilbert space* $\mathcal{H}$*.*

Theorem 0.6 *Every Hilbert space has an orthonormal basis.*

Definition 0.7 *A Hilbert space* $\mathcal{H}$ *is separable if an only if it has a countable orthonormal basis.*

Fourier series and transform

A trigonometric series (or Fourier series) in complex form of period $\tau > 0$ is a function series of the type:

$$T(x) = \frac{1}{\sqrt{\tau}} \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x/\tau} \tag{2}$$

Set

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}) \tag{3}$$

we have, by the Euler formulas ($e^{iw} = \cos(w) + i\sin(w)$), that

$$T(x) = \frac{1}{\sqrt{\tau}} \left(\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(2\pi k x/\tau) + b_k \sin(2\pi k x/\tau)) \right) \tag{4}$$

We observe now that $\{\frac{1}{\sqrt{\tau}}e^{2\pi inx/\tau}\}_{n\in\mathbb{Z}}$ e $\{\sqrt{\frac{\tau}{2}}\cos(2\pi kx/\tau)\}_{k\in\mathbb{N}}\bigcup\{\sqrt{\frac{\tau}{2}}\sin(2\pi kx/\tau)\}_{k\in\mathbb{N},k\neq 0}$ are an orthonormal system in the Hilbert space $\mathcal{H} = L^2(0,\tau)$. For example

$$\begin{aligned}
\langle\frac{1}{\sqrt{\tau}}e^{2\pi inx/\tau}, \frac{1}{\sqrt{\tau}}e^{2\pi imx/\tau}\rangle &= \frac{1}{\tau}\int_0^\tau e^{2\pi inx/\tau}\overline{e^{2\pi imx/\tau}}dx \\
&= \frac{1}{\tau}\int_0^\tau e^{2\pi i(n-m)x/\tau}dx \\
&= \int_0^1 e^{2\pi i(n-m)x}dx = \delta_{m,n}.
\end{aligned}$$

We wonder whether they consitute systems of orthonormal bases. We consider the function

$$f(x) = \frac{\pi x}{2} - \frac{x^2}{4}.$$

Note that it is symmetric and therefore the coefficients $b_k(f) = \langle f, \sqrt{\frac{\tau}{2}} \sin(2\pi kx/\tau)\rangle$ vanish for all k. By computing (Exercise! Integrate twice by parts) instead the coefficients $a_k(f) = \langle f, \sqrt{\frac{\tau}{2}} \cos(2\pi kx/\tau)\rangle$ we obtain the trigonometric series associated to f given by

$$T_f(x) = \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}. \tag{5}$$

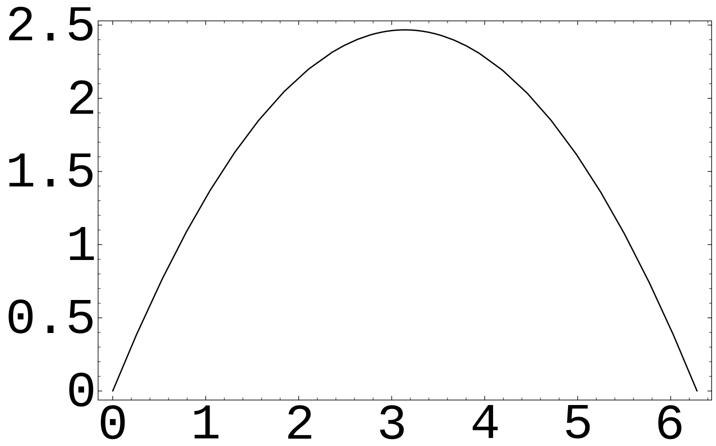
Theorem 0.8 (Pointwise convergence) *Let $f : \mathbb{R} \to \mathbb{C}$ periodic with period $\tau > 0$ and locally integrable. Assume that in $x_0 \in \mathbb{R}$ the limits $f(x_0)^+$ e $f(x_0^-)$ exist and are finite, and that the left and right finite difference*

$$\frac{f(x_0 + h) - f(x_0^+)}{h}, \qquad \frac{f(x_0 - h) + f(x_0^-)}{h},$$

are bounded for any $h > 0$ small. Then the series T_f of f converges in x_0 to the mean of the values of the right and left limits of f in x_0:

$$T_f(x_0) = \frac{1}{\tau} \sum_{n \in \mathbb{Z}} \langle f, e^{2\pi i n \xi/\tau} \rangle e^{2\pi i n x/\tau} = \frac{f(x_0^+) - f(x_0^-)}{2}.$$

In particular if $f'(x_0)$ exists then $T_f(x_0) = f(x_0)$.



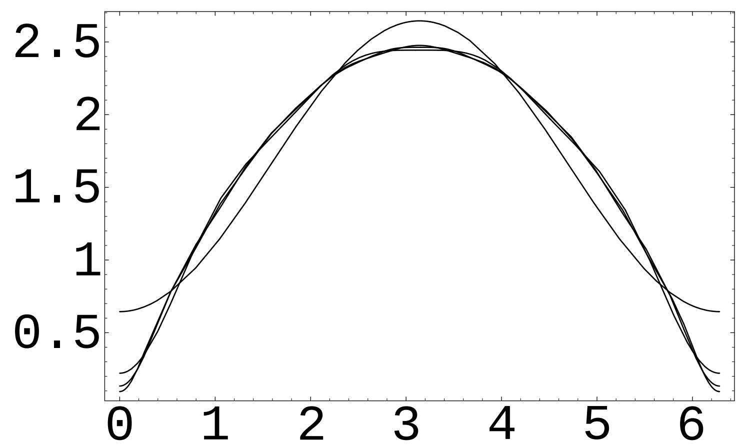


Figure 1: $f(x) = \frac{\pi x}{2} - \frac{x^2}{4} = \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$.

- The set of step functions S are dense in $L^2(0,\tau)$;
- We have

$$\sum_{n\in\mathbb{Z}} |\langle \chi_{[0,a]}, \frac{1}{\sqrt{\tau}} e^{2\pi i n x/\tau}\rangle|^2 = \frac{a^2}{\tau} + \frac{\tau}{\pi^2}\left(\frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2\pi n a}{\tau}\right)}{n^2}\right)$$

- From the previous theorem applied on f

$$a = \|\chi_{[0,a]}\|_2^2 = \sum_{n\in\mathbb{Z}} |\langle \chi_{[0,a]}, \frac{1}{\sqrt{\tau}} e^{2\pi i n x/\tau}\rangle|^2.$$

- By density of S we conclude that $\{\frac{1}{\sqrt{\tau}} e^{2\pi i n x/\tau}\}_{n\in\mathbb{Z}}$ is an orthonormal basis.

For functions $f \in L^2(0,\tau)$ we define the Fourier transform $\mathcal{F} : L^2(0,\tau) \to \ell^2(\mathbb{Z})$ as

$$\mathcal{F}f(n) = \langle f, \frac{1}{\sqrt{\tau}} e^{2\pi i n x/\tau} \rangle = \frac{1}{\sqrt{\tau}} \int_{(0,\tau)} f(x) e^{-2\pi i n x/\tau} dx. \qquad (6)$$

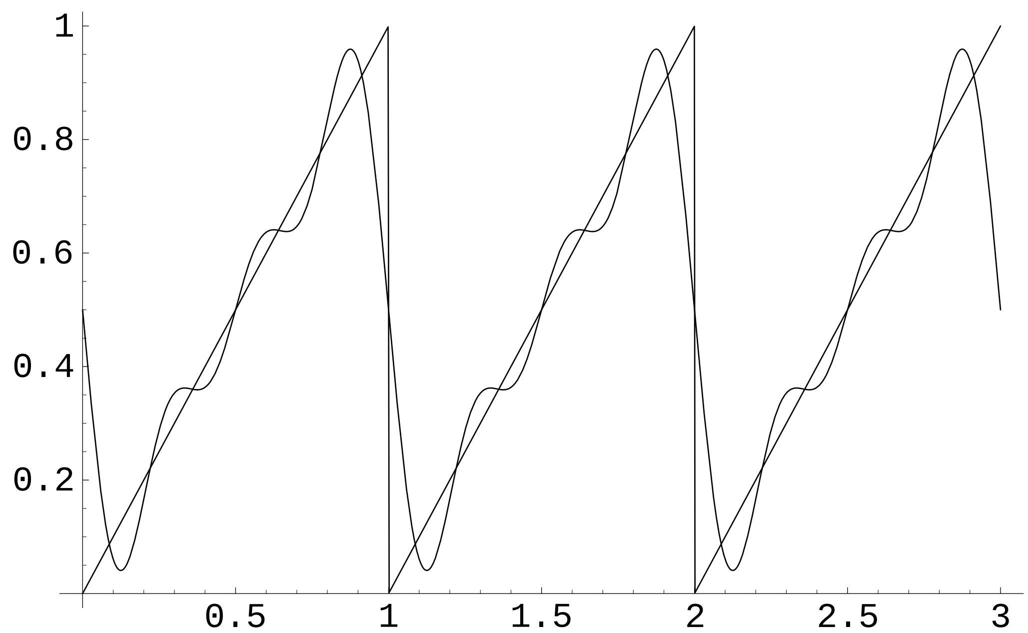


Figure 2: Fourier series associtated to $f(x) = x - [x]_-$.

What happens if we make $\tau \to \infty$?

$$\sum_{n\in\mathbb{Z}} \langle f, \frac{1}{\sqrt{\tau}} e^{2\pi i n x/\tau} \rangle \frac{1}{\sqrt{\tau}} e^{2\pi i n x/\tau} = \frac{1}{\tau} \sum_{n\in\mathbb{Z}} \left(\int_{\tau/2}^{-\tau/2} f(\xi) e^{-2\pi i n \xi/\tau} d\xi \right) e^{2\pi i n x/\tau}.$$

The last sums is in fact a Riemann sum. This makes us supposing that if f is an integrable function on $\mathbb{R}$ then we could write something like

$$f(x) = \lim_{\tau\to\infty} f(x)\chi_{[\tau/2,-\tau/2]} =$$

$$= \lim_{\tau\to\infty} \sum_{n\in\mathbb{Z}} \langle f, \frac{1}{\sqrt{\tau}} e^{2\pi i n x/\tau} \rangle \frac{1}{\sqrt{\tau}} e^{2\pi i n x/\tau} = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(\xi) e^{-2\pi i w \xi} d\xi \right) e^{2\pi i \xi w} dw.$$

When and in which sense this writing makes sense is though still to be proved.

Theorem 0.9 (Fourier transform and Plancherel identity) *We define for $f \in L^1(\mathbb{R})$ the Fourier transform, given by*

$$\mathcal{F}f(w) = \int_{\mathbb{R}} f(x)e^{-2\pi i w x}dx. \tag{7}$$

Moreover, if also $\mathcal{F}f \in L^1(\mathbb{R})$ then the Fourier transform is invertible and

$$f(x) = \int_{\mathbb{R}} \mathcal{F}f(w)e^{2\pi i w x}dw. \tag{8}$$

If $f \in L^1 \bigcap L^2$ then $\mathcal{F} : L^1 \bigcap L^2 \to L^2$ is a continuous linear operator which can be extended uniquely on the entire L^2. In particular, $\mathcal{F}$ is an isometric isomorphism, i.e., the Plancherel identity holds:

$$\|f\|_2 = \|\mathcal{F}f\|_2. \tag{9}$$

Moreover, if $f, g \in L^2$ then $\langle f, g\rangle = \langle \mathcal{F}f, \mathcal{F}g\rangle$.

Remark 0.10 *This definition is used in the literature of time-frequency analysis. The other definition*

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{-i\omega x} dx$$

is also used, for instance in the literature related to wavelets. In these lectures we use both in a consistent way with the corresponding literature. Anyway observe that $\mathcal{F}f(\omega) = 2\pi \hat{f}(2\pi\omega)$ *and, up to rescaling, one can transform all the result with one definition to results with another definition.*

What is the meaning of the Fourier transform? The Fourier transform represents the frequency content of a function/signal. It gives us which are the important oscillating building blocks of a signal and their dinstinctive frequency of oscillation.

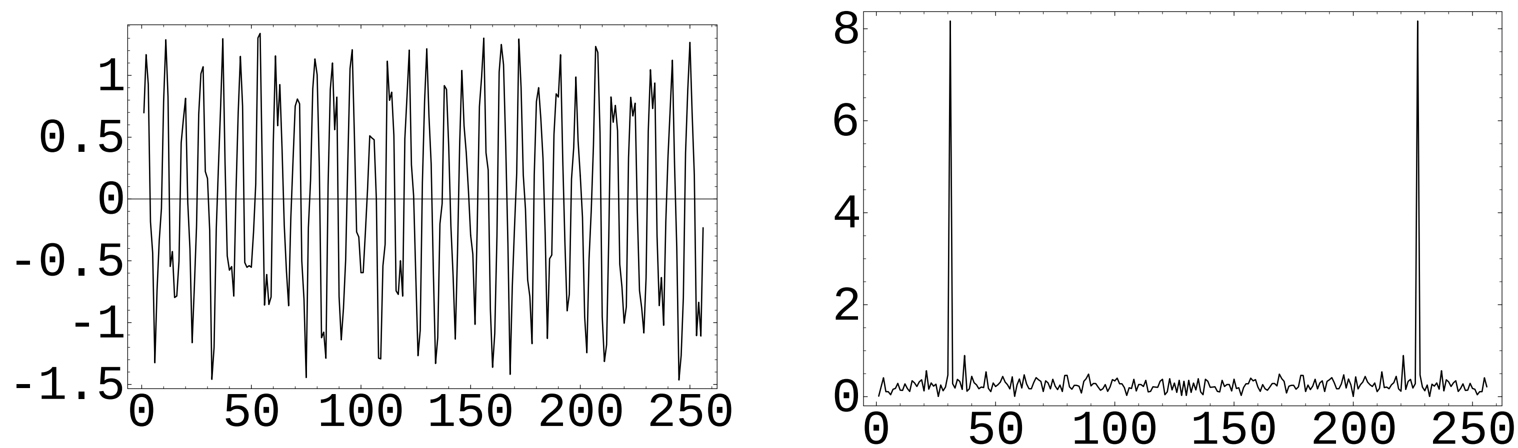


Figure 3: Sinosoidal signal affected by noise: there are two fundamental frequency components

The fortune of Fourier analysis stays essentially in the fact that it is able to describe one of the most frequent behavior in the nature: the wave phanomena, most of which are governed by ovelapping laws of the type

$$y_{\alpha,s_0}(t) = \begin{cases} e^{-\alpha t} e^{2\pi i s_0 t}, & t > 0 \\ 0, & t < 0. \end{cases}$$

$y_{\alpha,s_0}(t)$ is called *Lorenzian function*.

When, for example, some molecules are excited by electomagnetic radiation, damped oscillations are induced and described by the law $y_{\alpha,s_0}(t)$.

Every building block of the molecule has its own unique special oscillations. The Lorenzians are called in this case the *molecular spectrum*.

From here comes the idea which “costed” the Nobel prize to Richard Ernst for chemistry (1991) for the development of a powerful instrument to determine the structure of large complex organic molecules.

We want now to discuss how in practice we can use the tools of the Fourier analysis.

The execises we showed on simple functions show us that it is not so trivial to compute Fourier series or transforms.

For relatively simple functions such as $f(x) = \frac{1}{a^2+x^2}$ the computation of the Fourier transform is far from being trivial. In this case we have $\mathcal{F}f(w) = \frac{\pi}{a}e^{-|w|a}$.

A holomorphic complex function $f(z)$ (i.e., differentiable) on a domain Ω except for a point a, called singularity, can be expanded in a power series, called the Laurent series, $f(z) = \sum_{n\in\mathbb{Z}} c_n (z-a)^n$. The specific coefficient c_{-1} is called *residual* and we write $Res(f, a) = c_{-1}$. One can show that if f is holomorphic on a domain Ω which contains the halfplane $\{\mathrm{Im}(z) \geq 0\}$ except for a finite number of non real singularities and if the limit $\lim_{|z|\to\infty, \mathrm{Im}(z)\geq 0} f(z) = 0$ then we have for every $\alpha > 0$,

$$\lim_{r\to\infty} \int_{-r}^{r} f(x) e^{i\alpha x} dx = 2\pi i \sum \{Res(f(\cdot)e^{i\alpha\cdot}, a), \mathrm{Im}(a) > 0\}.$$

Analogous result holds for $\alpha < 0$ and for $\{\mathrm{Im}(z) < 0\}$, but changing the sign of the integral.

It is certainly more reasonable to search for an approximation on a discrete domain and a proper definition of discrete Fourier transform, which, in some sense, can approximate well the behavior of the continuous Fourier transform. Moreover, we need to make sure that the computation of the discrete Fourier transform (DFT) can be executed in reasonable time: otherwise it does not pay off.

Let's proceed as usual in the language of the Hilbert spaces. We consider now the cyclic group $\mathbb{Z}_n = \frac{\mathbb{Z}}{n\mathbb{Z}}$, $n \in \mathbb{N}\backslash\{0\}$ of the rest classes modulo n. In practice we can write

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\},$$

reminding us that such elements represent the rest of the integer number with respect to the division by n. For example, in $\mathbb{Z}_3$ the element $h = 4$ corresponds to 1 as $3 = 3 \cdot 1 + \mathbf{1}$. Hence, in particular, $2 + 2 = 1$ in $\mathbb{Z}_3$. Moreover, $\ell^2(\mathbb{Z}_n) = \{(c_0, \ldots, c_{n-1}) | \sum_{k=0}^{n-1} |c_i|^2 < \infty\} = \mathbb{C}^n$.

One can show by induction (exercise!)

$$1 + z + z^2 + \cdots + z^{n-1} = \begin{cases} n, & z = 1 \\ (z^n - 1)/(z - 1), & \text{otherwise.} \end{cases} \tag{10}$$

But then it is not difficult to show (exercise!) that for $z = e^{2\pi i(k-l)/n}$ we have

$$\langle \frac{1}{\sqrt{n}}(e^{2\pi ikl/n})_{l\in\mathbb{Z}_n}, \frac{1}{\sqrt{n}}(e^{2\pi ilm/n})_{m\in\mathbb{Z}_n}\rangle = \sum_{m=0}^{n-1} e^{2\pi im(k-l)/n} = \delta_{k,l}.$$

Hence $\{\frac{1}{\sqrt{n}}(e^{2\pi ikl/n})_{l\in\mathbb{Z}_n}\}_{k\in\mathbb{Z}_n}$ is an orthonormal basis for the finite dimensional (Hilbert) space $\ell^2(\mathbb{Z}_n)$.

Every discrete signal $\mathbf{f}$ of length n can be written as:

$$\mathbf{f} = \frac{1}{n}\sum_{k=0}^{n-1}\langle \mathbf{f}, (e^{2\pi ikl/n})_{l\in\mathbb{Z}_n}\rangle (e^{2\pi ikl/n})_{l\in\mathbb{Z}_n}. \tag{11}$$

Moreover we have that

$$\mathcal{F}\mathbf{f}(k) = \frac{1}{\sqrt{n}}\langle \mathbf{f}, (e^{2\pi ikl/n})_{l\in\mathbb{Z}_n}\rangle = \frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}\mathbf{f}(l)e^{-2\pi ikl/n}, \tag{12}$$

is again a signal of length n. The operator $\mathcal{F}: \ell^2(\mathbb{Z}_n) \to \ell^2(\mathbb{Z}_n)$ is called discrete Fourier transform or DFT and it realizes all the properties of the Fourier transforms already defined before for periodic functions and functions on the real line. For example, prove by exercize that

$$\|\mathbf{f}\|_{\ell^2} = \|\mathcal{F}\mathbf{f}\|_{\ell^2}.$$

If we assume that one operation is either a sum or a multiplication, then the computational cost of a DFG is of $2n$ sums and multiplications times n, i.e., $2n^2$. As one complex operation costs twice a single real, the total costs is $C(DFT)(n) = 4n^2$. If we had a PC able to compute 10^7 operations per second and we are able to compute a DFT of a signal of length $n = 1000$ in

$$4 \cdot 1000^2 \cdot \frac{1}{10^7} = 0.4\,\mathrm{sec}\,.$$

But already for a signal of lenght $n = 16384 = 2^{14}$ the cost is of $107\,\mathrm{sec}$. A simple audio signal of few seconds, sufficiently densely sampled, can easily reach length $n = 2^{14}$!.

Given a signal $\mathbf{f}$ of length n, we define the translation operator the one given by

$$T_m\mathbf{f}(k) = \mathbf{f}(k-m), \quad m \in \mathbb{Z}_n. \tag{13}$$

and the modulation operator by

$$M_m\mathbf{f}(k) = e^{2\pi imk/n}\mathbf{f}(k), \quad m \in \mathbb{Z}_n. \tag{14}$$

Moreover, we define the operator of *upsampling* and *doubling* as

$$U\mathbf{f}(h) = \begin{cases} \mathbf{f}(h/2), & \mathrm{mod}(h,2) = 0 \\ 0, & \text{altrimenti}, \end{cases} \tag{15}$$

$$D\mathbf{f} = \frac{1}{2}(\mathbf{f},\mathbf{f}), \tag{16}$$

for $h \in \mathbb{Z}_{2n}$.

The action of the DFT with respect to these operators is given by (exercise!)

$$\mathcal{F} \cdot T_m \mathbf{f}(k) = M_{-m} \cdot \mathcal{F}\mathbf{f}(k), \quad \mathcal{F} \cdot M_m \mathbf{f}(k) = T_m \cdot \mathcal{F}\mathbf{f}(k). \tag{17}$$

$$\mathcal{F} \cdot U\mathbf{f}(h) = D \cdot \mathcal{F}\mathbf{f}(h). \tag{18}$$

Let's now consider a vector of length $n = 2^2 = 4$ given by $\mathbf{f} = (f_0, f_1, f_2, f_3)$. Let us show how is it possible to compute $\mathbf{f}$ from the single f_i by simple operations:

$$
\begin{array}{ccccc}
f_0 & f_2 & & f_1 & f_3 \\
\downarrow U & \downarrow U & & \downarrow U & \downarrow U \\
(f_0, 0) & (f_2, 0) & & (f_1, 0) & (f_3, 0) \\
\downarrow I & \downarrow T_{-1} & & \downarrow I & \downarrow T_{-1} \\
(f_0, 0) & (0, f_2) & & (f_1, 0) & (0, f_3) \\
\searrow & \downarrow & & \downarrow & \swarrow
\end{array}
$$

$$
\begin{array}{ccc}
(f_0, f_2) & & (f_1, f_3) \\
\downarrow U & & \downarrow U \\
(f_0, 0, f_2, 0) & & (f_1, 0, f_3, 0) \\
\downarrow I & & \downarrow T_{-1} \\
(f_0, 0, f_2, 0) & & (0, f_1, 0, f_3) \\
\searrow & & \swarrow \\
& \mathbf{f} &
\end{array}
$$

Observing now that $\mathcal{F} f_i = f_i$ for all $i = 0, ...f_{n-1}$, by applying the DFT on the previous diagram and substituting to U, T_{-1} resp. D, M_1 as given in the commutation rules, we generate another diagram, actually a recursive algorithm to compute the DFT $\mathcal{F}\mathbf{f}$.

$$
\begin{array}{cccc}
f_0 & f_2 & f_1 & f_3 \\
\downarrow D & \downarrow D & \downarrow D & \downarrow D \\
(*,*) & (*,*) & (*,*) & (*,*) \\
\downarrow I & \downarrow M_1 & \downarrow I & \downarrow M_1 \\
(*,*) & (*,*) & (*,*) & (*,*) \\
\searrow & \downarrow & \downarrow & \swarrow \\
 & (*,*) & (*,*) & \\
 & \downarrow D & \downarrow D & \\
 & (*,*,*,*) & (*,*,*,*) & \\
 & \downarrow I & \downarrow M_1 & \\
 & (*,*,*,*) & (*,*,*,*) & \\
 & \searrow & \swarrow & \\
\end{array}
$$

$$\mathcal{F}\mathbf{f}$$

Let us assume that the computation of I and D are negligible (as it is in practice!). We need to compute only the cost of a single M_1 which, applied to a vector of length l costs $l-1$. If we assume now that $n = 2^m$, starting from the bottom of the diagram we do only one M_1 on the length of the original vector and then a cost given by $2^0(\frac{n}{2^0} - 1)$. This cost has to summed to the one of the level above, where one has to execute $2(\frac{n}{2} - 1)$ operations corresponding to twice M_1 computed on vector of half the original length. And so on, obtaining that the total cost is given by

$$C(FFT)(n) = \sum_{k=0}^{m-1} 2^k(\frac{n}{2^k} - 1) =$$

$$= \sum_{k=0}^{m-1} \left(2^m - 2^k\right) = m2^m - 2^m + 1 = n\log_2(n) - n + 1. \quad (19)$$

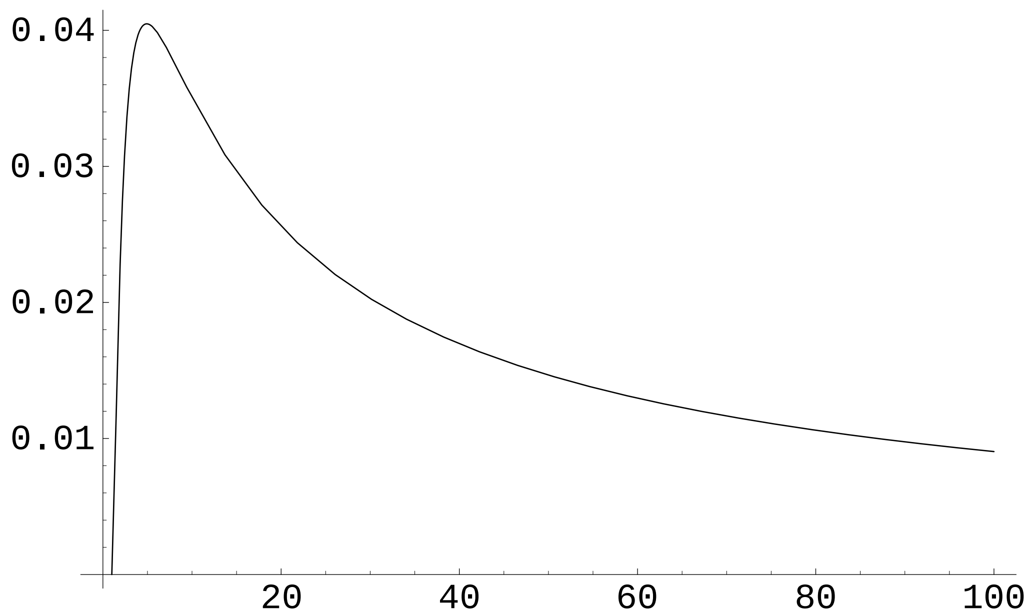


Figure 4: Ratio $\frac{C(FFT)(n)}{C(DFT)(n)}$ for growing n

Hence a PC is able to produce the FFT of a signal of length $n = 16384$ in

$$(2^{14}14 - 2^{14} + 1) \cdot \frac{1}{10^7} = 0.021\,\mathrm{sec},$$

against the $107\,\mathrm{sec}$ which one may need by means of the direct DFT!

Sampling and DFT

Definition 0.11 *Let* $f \in C(\mathbb{R}^d)$ *and* $\tau = (\tau_1, ..., \tau_d) \in (0, \infty)^d$. *We call the sampling of step* τ *the operator which associates to* f *the function* $f^c \equiv f_{|\tau\mathbb{Z}^d}$. *If* $f \in C_c(\mathbb{R}^d)$, *clearly* f^c *is vanishing out of a compact set of* $\mathbb{Z}^d$. *Moreover, up to a translation, we can always assume that* f^c *is defined on a set* $\tau\mathbb{Z}^d/\tau m\mathbb{Z}^d$ *isomorphic to* $\mathbb{Z}_m^d$. *Hence, the sampling induces an operator acting from* $C_c(\mathbb{R}^d)$ *to* $\ell^2(\mathbb{Z}_m^d)$, *for some* $m = (m_1, ..., m_d) \in \mathbb{N}^d \backslash \{0\}$.

We want to study some of the properties of the operator $\cdot^c$. In particular we wonder which is relationship between $\mathcal{F}\mathbf{f^c}$ (meant here as DFT!) and $\mathcal{F}f$ (meant as Fourier transform for functions on $\mathbb{R}^d$).

Lemma 0.12 *Let* $\omega = (\omega_1, ..., \omega_d) \in \mathbb{R}^d_+$*. If* $f \in L^1(\mathbb{R}^d)$ *then the series* $\sum_{k\in\mathbb{Z}^d} f(x - \omega k)$ *converges in* $L^1(\omega T^d)$ *to the periodic function* Pf *such that* $\|Pf\|_1 \leq \|f\|_1$*. Moreover for* $k \in \omega\mathbb{Z}^d$ *the Fourier transform for periodic functions on '*$(\omega T^d$ *the function* $\mathcal{F}(Pf)(k)$ *is equal to the Fourier transform for functions on* $\mathbb{R}^d$ *given by* $(\det(\omega))^{-1/2}\mathcal{F}f(k/\omega)$*.*

Proof of the Lemma

Let $Q = \prod_{i=1}^{d} \omega_i[-\frac{1}{2}, \frac{1}{2})$. We have $\int_Q \sum_{k\in\mathbb{Z}^d} |f(x-\omega k)|dx = \sum_{k\in\mathbb{Z}^d} \int_{Q+\omega k} |f(x)|dx = \int_{\mathbb{R}^d} |f(x)|dx < \infty$. By the Dominated Convergence Theorem we obtain the convergence of the series $\sum_{k\in\mathbb{Z}^d} f(x-\omega k)$ in L^1 to a function Pf such that $\|Pf\|_1 \le \|f\|_1$. By definition of Fourier transform of periodic functions defined on ωT^d we have

$$\mathcal{F}(Pf)(h) = \frac{1}{(\det(\omega))^{1/2}} \int_Q \left(\sum_{k\in\mathbb{Z}^d} f(x-\omega k) \right) e^{-2\pi i (h,x)/\omega} dx =$$

$$= \frac{1}{(\det(\omega))^{1/2}} \sum_{k\in\mathbb{Z}^d} \int_{Q+\omega k} f(x) e^{-2\pi i (h,x+\omega k)/\omega} dx =$$

$$= \frac{1}{(\det(\omega))^{1/2}} \int_{\mathbb{R}^d} f(x) e^{-2\pi i (h,x)/\omega} dx = (\det(\omega))^{-1/2} \mathcal{F}f(h/\omega).$$

Theorem 0.13 (Poisson summation formula) *Let $f \in C(\mathbb{R}^d)$, $|f(x)| \leq C(1+|x|)^{-d-\varepsilon}$ and $|\mathcal{F}f(\hat{x})| \leq C(1+|\hat{x}|)^{-d-\varepsilon}$ for some $C, \varepsilon > 0$. Let $\omega = (\omega_1, ..., \omega_d) \in (0,\infty)^d$. Then the following formula holds*

$$(\det(\omega))^{1/2} \sum_{k \in \mathbb{Z}^d} \mathcal{F}f(\hat{x} - \omega k) = (\det(\frac{1}{\omega}))^{1/2} \sum_{n \in \mathbb{Z}^d} f(n/\omega) e^{-2\pi i (\hat{x}, n/\omega)}.$$

where both sides converge uniformly in ωT^d. In particular for $\hat{x} = 0$ we obtain

$$(\det(\omega))^{1/2} \sum_{k \in \mathbb{Z}^d} \mathcal{F}f(\omega k) = (\det(\frac{1}{\omega}))^{1/2} \sum_{n \in \mathbb{Z}^d} f(n/\omega).$$

Proof of the Theorem

The absolute and uniform convergence of the series follows by the application of the criterion of the integral estimate for the series $\sum_{k\in\mathbb{Z}^d}(1+|k|)^{-n-\varepsilon}$. Then $Pf(x)=\sum_{k\in\mathbb{Z}^d} f(x-\omega k)$ is in $C(\omega T^d)$ and therefore in $L^2(\omega T^d)$. By the Theorem of Fourier and by the previous Lemma we have

$$Pf(x)=\frac{1}{(\det(\omega))^{1/2}}\sum_{h\in\mathbb{Z}^d}\mathcal{F}(Pf)(h)e^{2\pi i(h,x)/\omega}=$$

$$=\frac{1}{\det(\omega)}\sum_{h\in\mathbb{Z}^d}\mathcal{F}f(h/\omega)e^{2\pi i(h,x)/\omega}.$$

Exchanging the roles of f and $\mathcal{F}f$ and observing that $\mathcal{F}\mathcal{F}f=f(-x)$ we obtain the final result.

Corollary 0.14 (Shannon sampling theorem) *Let* $f \in C_c(\mathbb{R}^d)$, $|f(x)| \leq C(1+|x|)^{-d-\varepsilon}$ *and* $|\mathcal{F}f(\hat{x})| \leq C(1+|\hat{x}|)^{-d-\varepsilon}$ *for some* $C, \varepsilon > 0$. *If, for example,* $\mathrm{supp}(f) \subset \prod_{i=1}^{d}[0, a_i]$, *then it holds*

$$\mathcal{F}\mathbf{f^c}(l) = \frac{(\det(m))^{1/2}}{(\det(a))} \left[\sum_{k \in \mathbb{Z}^d} \mathcal{F}f(\hat{x} - \omega k) \right]^c (l/a), \tag{20}$$

where the sampling on the left is made with step $\tau = 1/\omega$ *and the one on the right with step* $1/a$, *so that* $\omega a = m \in \mathbb{N}^d$.

Proof of the Corollary

Let us assument $\operatorname{supp}(f) \subset \prod_{i=1}^{d}[0, a_i]$ and denote $a = (a_1, \ldots, a_d)$. We also assume that $a > 0$ is such that $\omega a = m \in \mathbb{N}^d$. But then

$$(\det(\frac{1}{w}))^{1/2} \sum_{n \in \mathbb{Z}^d} f(n/\omega) e^{-2\pi i(\hat{x}, n/\omega)} = \frac{(\det(a))^{1/2}}{(\det(m))^{1/2}} \sum_{n \in \mathbb{Z}_m^d} f(n/\omega) e^{-2\pi i(\hat{x}a, n)/m}.$$

If we assume now that $\hat{x}a = l \in \mathbb{Z}^d$ then by the Poisson summation formula, we get

$$\frac{(\det(m))^{1/2}}{(\det(a))^{1/2}} \sum_{k \in \mathbb{Z}^d} \mathcal{F}f(l/a - \omega k) = \frac{(\det(a))^{1/2}}{(\det(m))^{1/2}} \sum_{n \in \mathbb{Z}_m^d} f(n/\omega) e^{-2\pi i(l, n)/m}.$$

Set $\tau = 1/\omega$, we obtain by definition of the DFT

$$\mathcal{F}\mathbf{f^c}(l) = \frac{(\det(m))^{1/2}}{(\det(a))} \sum_{k \in \mathbb{Z}^d} \mathcal{F}f(l/a - \omega k) = \frac{(\det(m))^{1/2}}{(\det(a))} \left[\sum_{k \in \mathbb{Z}^d} \mathcal{F}f(\hat{x} - \omega k) \right]^c (l$$

where the last sampling is made with step $1/a$.

SAMPLING THEORY

Lemma 0.15 *Define* $\mathrm{sinc}(x) \equiv \frac{sin(\pi x)}{\pi x}$*,* $x \in \mathbb{R}$ *and* $Q = \prod_{i=1}^{d} \omega_i[-1/2, 1/2)$*. Then it holds*

$$\frac{1}{\det(\omega)^{1/2}} \int_Q e^{2\pi i(k,\xi)/\omega} e^{2\pi i(\xi,x)} dx = \det(\omega)^{1/2} \prod_{i=1}^{d} \mathrm{sinc}(w_i x_i - k_i), \quad x \in \mathbb{R}^d.$$

Proof. *(Exercise!) Just use Fubini-Tonelli Theorem.*

Theorem 0.16 (of the perturbed sampling in L^2) *Let $Q = \prod_{i=1}^{d} \omega_i[-1/2, 1/2)$ and $f \in C(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that $f_{|\tau\mathbb{Z}^d} \in \ell^2$. We write $f = \eta + \epsilon$, where $\mathcal{F}\eta = \mathcal{F}f$ on Q. Then it holds*

$$f(x) = \sum_{k\in\mathbb{Z}^n} (f^c(\tau k) - \epsilon^c(\tau k)) \prod_{i=1}^{d} \operatorname{sinc}(\tau_i^{-1} x_i - k_i) + \epsilon(x), \quad in \quad L^2(\mathbb{R}^d). \tag{21}$$

Proof of the Theorem

Let us consider $P[\mathcal{F}\eta](\xi) = \sum_{k\in\mathbb{Z}^d} \mathcal{F}\eta(\xi - \omega k)$. We know already that $P[\mathcal{F}\eta] \in L^2(\omega T^d)$ and thus

$$P[\mathcal{F}\eta](\xi) = \frac{1}{\det(\omega)^{1/2}} \sum_{k\in\mathbb{Z}^d} \mathcal{F}(P[\mathcal{F}\eta])(k) e^{2\pi i(k,\xi)/\omega},$$

where

$$\mathcal{F}(P[\mathcal{F}\eta])(k) = \frac{1}{\det(\omega)^{1/2}} \int_{\omega T^d} P[\mathcal{F}\eta](x) e^{-2\pi i(k,x)/\omega} dx.$$

Clearly the restriction of this function to Q gives $P[\mathcal{F}\eta]_{|Q} = \mathcal{F}\eta$. But then we see immediately that $\mathcal{F}(P[\mathcal{F}\eta])(k) = \frac{1}{\det(\omega)^{1/2}} (f^c(-k/\omega) - \epsilon^c(-k/\omega))$, since $\mathcal{F}\eta$ is not vanishing only on a compact set and thus $\mathcal{F}\eta \in L^1(\mathbb{R}^d)$. But then, substituting $P[\mathcal{F}\eta]$ instead of $\mathcal{F}\eta$ on Q and extending it to 0 out of Q we obtain

$$f(x) = \frac{1}{\det(\omega)^{1/2}} \mathcal{F}^{-1} \left(\sum_{k \in \mathbb{Z}^d} \mathcal{F}(P[\mathcal{F}\eta])(k) e^{2\pi i (k,\xi)/\omega} \chi_Q(\xi) \right) (x) + \epsilon(x) =$$

$$= \sum_{k \in \mathbb{Z}^n} \mathcal{F}(P[\mathcal{F}\eta])(k) \mathcal{F}^{-1} \left(\frac{1}{\det(\omega)^{1/2}} e^{2\pi i (k,\xi)/\omega} \chi_Q(\xi) \right) (x) + \epsilon(x) =$$

$$= \sum_{k \in \mathbb{Z}^n} (f^c(\tau k) - \epsilon^c(\tau k)) \prod_{i=1}^{d} \operatorname{sinc}(\tau_i^{-1} x_i - k_i) + \epsilon(x),$$

where we used the previous Lemma in the last equality and all the equalities hold in L^2.

Definition 0.17 *For* $Q = \prod_{i=1}^{d} \omega_i[-1/2, 1/2)$*, we define*

$$L_Q^2(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : \mathrm{supp}(\mathcal{F}f) \subset Q\}.$$

We say that $f \in L_Q^2(\mathbb{R}^d)$ *is a* ω*-bandlimited function.*

Corollary 0.18 (Whittacker-Shannon) *If $f \in L^2(\mathbb{R}^d)$ is a ω-bandlimited function, there exists a $\tau_0 > 0$ such that for all $0 < \tau \leq \tau_0$*

$$f(x) = \sum_{k \in \mathbb{Z}^d} f^c(\tau k) \prod_{i=1}^{d} \operatorname{sinc}(\tau_i^{-1} x_i - k_i). \tag{22}$$

Proposition 0.19 *If $f \in C_c(\mathbb{R}^d)$ then its Fourier transformation $\mathcal{F}f$ on $\mathbb{R}^d$ cannot be compactly supported.*

Proof. If $f \in C_c(\mathbb{R}^d)$ then $\mathcal{F}f$ is an analitic function (because it has infinitely many bounded derivatives). But a nonzero analitic function has only isolated zeros, hence it cannot be compactly supported (see http://planetmath.org/zeroesofanalyticfunctionsareisolated for a proof).

Summary of the results obtained so far

So far we showed that the sampling followed by a DFT

$$\begin{array}{ccccc} & .^c & & \mathcal{F} & \\ C_c & \to & \ell^2(\mathbb{Z}^d_m) & \to & \ell^2(\mathbb{Z}^d_m) \\ f & \to & \mathbf{f^c} & \to & \mathcal{F}\mathbf{f^c} \end{array}$$

is equivalent to apply a continuous Fourier transform, one periodization and a sampling again

$$\begin{array}{ccccccc} & \mathcal{F} & & P & & .^c & \\ C_c & \to & L^2(\mathbb{R}^d) & \to & L^2(\omega T^d) & \to & \ell^2(\mathbb{Z}^d) \\ f & \to & \mathcal{F}f & \to & P[\mathcal{F}f] & \to & (P[\mathcal{F}f])^c \end{array}$$

Moreover **IF** $\mathcal{F}f$ is a function decaying very rapidly we have

$$\mathcal{F}f(w) \approx P[\mathcal{F}f](w), \quad \forall w \in Q$$

and we can therefore assume the following good approximations

$$\begin{aligned} \mathcal{F}f(w) &\approx \sum_{l\in\mathbb{Z}^d, l/a\in Q} (P[\mathcal{F}f])^c(l/a) \prod_{i=1}^{d} \mathrm{sinc}(a_i w_i - l_i) \\ &= \frac{\det(a))}{(\det(m))^{1/2}} \sum_{l\in\mathbb{Z}^d, l/a\in Q} \mathcal{F}\mathbf{f}^{\mathbf{c}}(l) \prod_{i=1}^{d} \mathrm{sinc}(a_i w_i - l_i). \end{aligned}$$

THEREFORE if f is a function of compact support, whose Fourier transform decay fast (and this is ensured as soon as a function is sufficiently smooth!) then the Fourier transform can be well-approximated by applying a DFT on the samples of f and the formula given by the Wittacker-Shannon sampling theorem. Hence, for functions, where it is difficult to apply the techniques of the residuals, we can attempt the approximation by means of discrete Fourier transform.

In the following we assume that $\ell^2(\tau\mathbb{Z}^d)$ is endowed with the scalar product $\langle f, g\rangle_{\ell^2} := \det(\tau)\sum_{k\in\mathcal{Z}^d} f(\tau k)\overline{g(\tau k)}$.

Theorem 0.20 (Perturbed scalar product) *Let us assume that $f, g \in C(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ tali che $f_{|\tau\mathbb{Z}^d}, g_{|\tau\mathbb{Z}^d} \in \ell^2$. Then the following approximation on the scalar products of f and g holds:*

$$|\langle f, g\rangle_{L^2} - \langle f_{|\tau Z^d}, g_{|\tau Z^d}\rangle_{\ell^2}| \leq$$

$$\leq \|\varepsilon_f\|_{\ell^2}\|g\|_{\ell^2} + \|\varepsilon_g\|_{\ell^2}\|f\|_{\ell^2} + \|\varepsilon_f\|_{\ell^2}\|\varepsilon_g\|_{\ell^2} + \|\varepsilon_f\|_{L^2}\|\varepsilon_f\|_{L^2}. \quad (23)$$

Proof of the Theorem

By the Theorem of the Perturbed Sampling in L^2 we obtain by considering the scalar products:

$$\begin{aligned}\langle f, g\rangle_{L^2} &= \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx \\ &= \int_{\mathbb{R}^d} \left(\sum_{k\in\mathbb{Z}^d} (f(\tau k) - \varepsilon_f(\tau k)) \prod_{i=1}^{d} \mathrm{sinc}(\tau_i^{-1}x_i - k_i) + \varepsilon_f(x) \right) \times \\ &\qquad \times \overline{\left(\sum_{h\in\mathbb{Z}^d} (g(\tau h) - \varepsilon_g(\tau h)) \prod_{i=1}^{d} \mathrm{sinc}(\tau_i^{-1}x_i - h_i) + \varepsilon_g(x) \right)}\end{aligned}$$

As the functions of the type η_* are orthogonal in L^2 to functions of the type ε_*, we have:

$$\langle f, g\rangle_{L^2} = \sum_{h,k\in\mathbb{Z}^d} (f(\tau k) - \varepsilon_f(\tau k))\overline{(g(\tau h) - \varepsilon_g(\tau h))}\times$$

$$\times \left(\int_{\mathbb{R}^d} \prod_{i=1}^{d} \mathrm{sinc}(\tau_i^{-1} x_i - k_i)\, \mathrm{sinc}(\tau_i^{-1} x_i - h_i) dx \right) + \langle \varepsilon_f, \varepsilon_g \rangle_{L^2} =$$

and by the orthogonality of the "sinc" these identities continue as follows

$$\begin{aligned} = \quad & \det(\tau) \cdot \sum_{k \in \mathbb{Z}^d} (f(\tau k) - \varepsilon_f(\tau k))\overline{(g(\tau k) - \varepsilon_g(\tau k))} + \langle \varepsilon_f, \varepsilon_g \rangle_{L^2} \\ = \quad & \langle f_{|\tau\mathbb{Z}^d}, g_{|\tau\mathbb{Z}^d} \rangle_{\ell^2} + \langle f_{|\tau\mathbb{Z}^d}, (\varepsilon_g)_{|\tau\mathbb{Z}^d} \rangle_{\ell^2} + \langle (\varepsilon_f)_{|\tau\mathbb{Z}^d}, g_{|\tau\mathbb{Z}^d} \rangle_{\ell^2} \\ & + \langle (\varepsilon_f)_{|\tau\mathbb{Z}^d}, (\varepsilon_g)_{|\tau\mathbb{Z}^d} \rangle_{\ell^2} + \langle \varepsilon_f, \varepsilon_g \rangle_{L^2} \end{aligned}$$

Hence, by using the Cauchy-Schwarz inequalty, we obtain the following estimate:

$$|\langle f, g \rangle_{L^2} - \langle f_{|\tau\mathbb{Z}^d}, g_{|\tau\mathbb{Z}^d} \rangle_{\ell^2}| \leq$$

$$\leq \|\varepsilon_f\|_{\ell^2} \|g\|_{\ell^2} + \|\varepsilon_g\|_{\ell^2} \|f\|_{\ell^2} + \|\varepsilon_f\|_{\ell^2} \|\varepsilon_g\|_{\ell^2} + \|\varepsilon_f\|_{L^2} \|\varepsilon_f\|_{L^2}.$$

Remark 0.21 *If f, g are two ω-bandlimited functions we have the identity*

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}dx \;\; dx = \det(\tau) \sum_{k\in\mathbb{Z}^n} f^c(\tau k)\overline{g^c(\tau k)}.$$

Digitalization and recovery: redundancy

We considered so far a system of digitalization by sampling: for every $f \in L^2_Q$ the coefficient map $C : L^2_Q \to \ell^2$ is given by $C(f) = \{f(\tau k)\}_{k\in\mathbb{Z}^d}$ and it can be inverted by the recovery map $R : \ell^2 \to L^2_Q$ defined by $R(\overrightarrow{c}) = \sum_{k\in\mathbb{Z}^d} c_k \prod_{i=1}^d \mathrm{sinc}(\tau_i^{-1} x_i - k_i)$. Indeed we have $I = R \circ C$. Moreover, by the previous remark, we know that this system of digitalization is stable, since we have an equivalence of norms. Unfortunately the system is NOT redundant as C is the unique coefficient map which is the right-inverse R. In fact the $\mathrm{sinc}(\tau_i^{-1} x_i - k_i)$ constitute an orthonormal basis! Can we construct redundant systems?

Translation and Modulation

As for discrete signals $\mathbb{Z}_n^d$, also for functions on $\mathbb{R}^d$ we can define the operators of translation and modulation:

$$T_x f(t) = f(t-x), \quad M_w f(t) = e^{2\pi i w t} f(t). \tag{24}$$

and it holds (exercise!)

$$\mathcal{F} T_x = M_{-x} \mathcal{F}, \quad \mathcal{F} M_w = T_w \mathcal{F}. \tag{25}$$

Moreover if $f, g \in L^2(\mathbb{R}^d)$ we define the $L^2(\mathbb{R}^d)$ function given by the convolution of f and g by

$$f \star g(x) = \int_{\mathbb{R}^d} f(x-y) g(y) dy. \tag{26}$$

and we have that (exercise!)

$$\mathcal{F}(f \star g) = \mathcal{F}(f) \cdot \mathcal{F}(g). \tag{27}$$

Proposition 0.22 *The translation operator is given by $T_x g(t) = g(t - x)$ for any function g on $\mathbb{R}^d$. Given band-limited function $g \in L^1(\mathbb{R}^d)$ such that $\mathcal{F}g \neq 0$ on a compact set $\Omega \subset \mathbb{R}^d$, then there exists $\tau > 0$ such that for all $f \in L^2_\Omega(\mathbb{R}^d)$*

$$f = \sum_{k \in \mathbb{Z}^d} c_k(f) T_{\tau k} g, \tag{28}$$

for suitable coefficients $(c_k(f))_{k \in \mathbb{Z}^d} \in \ell^2$ depending continuously on f. Such coefficients are NOT in general unique.

PROOF OF THE PROPOSITION

We known that $\{\tau^{d/2}e^{2\pi i\tau kx}\}_{k\in\mathbb{Z}^d}$ is and orthonormal basis for $L^2([-1/(2\tau),1/(2\tau)]^d)$. For $\tau>0$ sufficiently small $\Omega\subset[-1/(2\tau),1/(2\tau)]^d$. There exists $g_1\in L^1$ a bandlimited function, such that $\mathcal{F}g_1\cdot\mathcal{F}g\equiv 1$ on Ω. Hence for $f\in L^2_\Omega(\mathbb{R}^d)$ and for $\tau>0$ sufficiently small

$$\mathcal{F}f(w)=[(\mathcal{F}f\mathcal{F}g_1)\mathcal{F}g](w)=\sum_{k\in\mathbb{Z}^d}\tau^d\langle\mathcal{F}f\mathcal{F}g_1,e^{2\pi i\tau kx}\rangle e^{2\pi i\tau kw}\mathcal{F}g(w). \tag{29}$$

By applying the inverse Fourier transform we obtain

$$f=\sum_{k\in\mathbb{Z}^d}\tau^d(f\star g_1)(\tau k)T_{\tau k}g, \tag{30}$$

where $\star$ is the convolution symbol

$$f \star g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy.$$

In particular we can prove

$$\sum_{k \in \mathbb{Z}^d} \tau^d |(f \star g_1)(\tau k)|^2 = \|f \star g_1\|_2^2 \leq \|f\|_2^2 \|g_1\|_1^2. \tag{31}$$

Hence, if we set $c_k(f) = \tau^d (f \star g_1)(\tau k)$ we have

$$\sum_{k \in \mathbb{Z}} |c_k(f)|^2 \leq \left(\tau^d \|g_1\|_1^2\right) \|f\|_2^2. \tag{32}$$

In particular,

$\sum_{k \in \mathbb{Z}} |\langle f, T_{\tau k} g \rangle|^2 = \sum_{k \in \mathbb{Z}} |\langle \mathcal{F} f, e^{2\pi i \tau k w} \mathcal{F} g \rangle|^2 \leq \tau^{-d} \|f\|_2^2 \|\mathcal{F} g\|_\infty^2.$