Mathematics of Digitalization: Case Study

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Second lecture

INTRODUCTION

Main topics:

1. After a short introduction on the Hilbert spaces, we describe the concepts of Fourier series, Fourier transforms and their discrete implementation, in particular the algorithm of the Fast Fourier Transform (FFT).

2. To connect the continuous Fourier transforms and the discrete one, we need to address the Shannon sampling theory (analog-to-digital conversion). For that it will be fundamental to understand the Poisson summation formula which will allow us to estimate also the errors in the analog-to-digital conversion, the so-called *aliasing* errors.

3. We shall introduce systems for robust analog-to-digital

conversion called *frames* in Hilbert spaces. We address special cases of frames, in particular Gabor frames in their continuous and discrete versions and the implementation of their so-called canonical duals. Applications to audio signals and in image compression will be showed.

4. We address in more detail the study case of the fresco restoration and how to use the tools we considered so far to solve it.

THE IMPORTANCE OF HARMONIC ANALYSIS

- We can synthesize a variety of complicated functions by means of pure sinusoids in the same way we can produce a chord of C major pushing the keyboard (C, E, G) of a piano.

- It was Joseph Fourier (http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html) who first developed the modern methods of trigonometric series and integrals in the study of the heat conduction in the solids (*Analytical Theory of Heat*, 1815).

SOME OF THE MILESTONES

- The most cited paper in mathematics
  J. W. Cooley and J. W. Tukey, An algorithm for the machine computation of complex Fourier series, Math. Comp., 19 (1965), 297-301.

- Almost $3/4$ of the Nobel prizes in Physics are given on work using methods and tools from Fourier analysis.
  F. N. Magill, ed., *The Nobel Prize Winners — Physics*, Vol 1-3, Salem Press. Englewood Cliffs, NJ. 1989.

- Francis Crick, James Watson and Maurice Wilkins got the Nobel prize for the medicine in 1962 for the discovery of the molecular structure of the DNA. This was the first example of the use of Fourier methods on data coming from X ray diffraction.

- Herbert Hauptmann (a mathematician) and Jerome Karle shared in 1985 the Nobel prize for chemistry after having shown how to use systematically Fourier analysis in order to determine the structure of large molecules from X ray diffraction data.
  W. A. Hendrickson, *the 1985 Nobel Prize in Chemistery*, Science 231 (1986), 362-364.
- The starting of the time-frequency analysis and the wavelets (1960-1990).
- A large number of technological products (CD-DVD, high definition TV, digital phones, medical imaging instruments ...) are based on discrete Fourier analysis.

HILBERT SPACES

Let $\mathcal{H}$ be a vector space. A scalar product $\langle u, v\rangle$ is a map from $\mathcal{H} \times \mathcal{H}$ to $\mathbb{C}$ such that:

(i) $\langle au + bv, z\rangle = a\langle u, z\rangle + b\langle v, z\rangle$ for all u,v,z $\in \mathcal{H}$ and $a, b \in \mathbb{C}$.

(ii) $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all u,v $\in \mathcal{H}$.

(iii) $\langle u, u\rangle \in \mathbb{R}$, $\langle u, u\rangle \geq 0$ for all $u \in \mathcal{H}$ and $\langle u, u\rangle \neq 0$ if $u \neq 0$.

We recall that a scalar product verify the Cauchy-Schwarz inequality (which will be proved in the exercises):

$$|\langle u, v\rangle| \leq \langle u, u\rangle^{1/2}\langle v, v\rangle^{1/2} \qquad \forall u, v \in \mathcal{H}. \tag{1}$$

and that $\|u\| = \langle u, u\rangle^{1/2}$ define a norm for $\mathcal{H}$.

A Hilbert spaces is a vector space $\mathcal{H}$ endowed with a scalar product $\langle u, v\rangle$ and is complete with respect to the norm $\langle u, u\rangle^{1/2}$.

Example 0.1 *Some relevant examples:*

- *Let* $\Omega \subset \mathbb{R}^n$*. The vector space of functions* $L^2(\Omega) = \{f : \Omega \to \mathbb{C} | \int_\Omega |f|^2 dx < \infty\}$ *is a Hilbert spaces with the scalar product:*

$$\langle u, v \rangle = \int_\Omega u(x)\overline{v(x)}dx.$$

- *Let* $\Omega_d \subset \mathbb{Z}^n$*. The vector space* $\ell^2(\Omega_d) = \{f : \Omega_d \to \mathbb{C} | \sum_{k \in \Omega_d} |f(k)|^2 < \infty\}$ *is a Hilbert spaces with the scalar product:*

$$\langle u, v \rangle = \sum_{k \in \Omega_d} u(k)\overline{v(k)}.$$

In particular if $|\Omega_d| = d < \infty$ *then* $\ell^2(\Omega_d) = \mathbb{C}^d$*.*

Definition 0.2 *Let E be a topological vector space. We say that E' is the topological dual space of E, i.e., E'={φ from E to $\mathbb{C}$ $\mid$ φ is continuous and linear}. Given two Hilbert spaces $\mathcal{H}, \mathcal{K}$, the set of linear operators $T : \mathcal{H} \to \mathcal{K}$, i.e., $T(au + bv) = aT(v) + bT(v)$, which are also continuous is a vector space which, endowed with the norm $|||T|||_{\mathcal{H}\to\mathcal{K}} \equiv \sup_{u\in\mathcal{H}, u\neq 0} \frac{\|Tu\|_{\mathcal{K}}}{\|u\|_{\mathcal{H}}}$ is a complete normed space, i.e., a Banach space, and we indicate it with $\mathcal{L}(\mathcal{H}, \mathcal{K})$. In particular, if $\mathcal{H} = \mathcal{K}$ the we set $\mathcal{L}(\mathcal{H}) \equiv \mathcal{L}(\mathcal{H}, \mathcal{H})$ and if $\mathcal{K} = \mathbb{C}$ then we have $\mathcal{L}(\mathcal{H}, \mathcal{K}) = \mathcal{H}'$.*

Theorem 0.3 (Representation theorem of Riesz-Frechet) *Let $\mathcal{H}$ be a Hilbert space. For every $\varphi \in \mathcal{H}'$ there exists a unique $f \in \mathcal{H}$ such that*

$$\langle \varphi, v \rangle \equiv \varphi(v) = \langle v, f \rangle \qquad \forall v \in \mathcal{H}.$$

Moreover we have

$$\|f\| = \|\varphi\|_{\mathcal{H}'} \equiv \sup_{\|v\|=1} |\langle \varphi, v \rangle|.$$

Proposition 0.4 (Bessel inequality) *Let $\{u_\alpha\}_{\alpha \in A}$ be an orthonormal system in $\mathcal{H}$, i.e., $\langle u_\alpha, u_\beta \rangle = \delta_{\alpha,\beta}$ where $\delta_{\cdot,\cdot}$ is the Kronecker symbol. Then for $x \in \mathcal{H}$:*

$$\sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2 \leq \|x\|^2.$$

In particular $\{\alpha | \langle x, u_\alpha \rangle \neq 0\}$ is countable.

Theorem 0.5 (Fourier) *Let $\{u_\alpha\}_{\alpha \in A}$ be a countable orthonormal system. Then the following are equivalent:*

(i) $x = \sum_{\alpha \in A} \langle x, u_\alpha \rangle u_\alpha$ *for all* $x \in \mathcal{H}$.

(ii) (Parseval idendity) $\|x\|^2 = \sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2$ *for all* $x \in \mathcal{H}$.

(iii) (Completeness) If $x \in \mathcal{H}$ *and if* $\langle x, u_\alpha \rangle = 0$ *for all* α*, then* $x = 0$.

The sequence $\langle x, u_\alpha \rangle$ *is such that* $\sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2 < \infty$*, then the series* $\sum_{\alpha \in A} \langle x, u_\alpha \rangle u_\alpha$ *is convergent in* $\mathcal{H}$ *and converges to* x*. A orthonormal set which verifies one of the previous conditions is called an orthonormal basis for the Hilbert space* $\mathcal{H}$*.*

Theorem 0.6 *Every Hilbert space has an orthonormal basis.*

Definition 0.7 *A Hilbert space $\mathcal{H}$ is separable if an only if it has a countable orthonormal basis.*

Fourier series and transform

A trigonometric series (or Fourier series) in complex form of period $\tau > 0$ is a function series of the type:

$$T(x) = \frac{1}{\sqrt{\tau}} \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x/\tau} \tag{2}$$

Set

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}) \tag{3}$$

we have, by the Euler formulas ($e^{iw} = \cos(w) + i\sin(w)$), that

$$T(x) = \frac{1}{\sqrt{\tau}} \left(\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(2\pi k x/\tau) + b_k \sin(2\pi k x/\tau)) \right) \tag{4}$$

We observe now that $\{\frac{1}{\sqrt{\tau}} e^{2\pi i n x/\tau}\}_{n\in\mathbb{Z}}$ e $\{\sqrt{\frac{\tau}{2}}\cos(2\pi k x/\tau)\}_{k\in\mathbb{N}} \bigcup \{\sqrt{\frac{\tau}{2}}\sin(2\pi k x/\tau)\}_{k\in\mathbb{N},k\neq 0}$ are an orthonormal system in the Hilbert space $\mathcal{H} = L^2(0,\tau)$. For example

$$\begin{aligned}
\langle \frac{1}{\sqrt{\tau}} e^{2\pi i n x/\tau}, \frac{1}{\sqrt{\tau}} e^{2\pi i m x/\tau} \rangle &= \frac{1}{\tau}\int_0^\tau e^{2\pi i n x/\tau}\overline{e^{2\pi i m x/\tau}}dx \\
&= \frac{1}{\tau}\int_0^\tau e^{2\pi i (n-m) x/\tau} dx \\
&= \int_0^1 e^{2\pi i (n-m) x} dx = \delta_{m,n}.
\end{aligned}$$

We wonder whether they consitute systems of orthonormal bases. We consider the function

$$f(x) = \frac{\pi x}{2} - \frac{x^2}{4}.$$

Note that it is symmetric and therefore the coefficients $b_k(f) = \langle f, \sqrt{\frac{\tau}{2}} \sin(2\pi kx/\tau) \rangle$ vanish for all k. By computing (Exercise! Integrate twice by parts) instead the coefficients $a_k(f) = \langle f, \sqrt{\frac{\tau}{2}} \cos(2\pi kx/\tau) \rangle$ we obtain the trigonometric series associated to f given by

$$T_f(x) = \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}. \tag{5}$$

Theorem 0.8 (Pointwise convergence) *Let $f : \mathbb{R} \to \mathbb{C}$ periodic with period $\tau > 0$ and locally integrable. Assume that in $x_0 \in \mathbb{R}$ the limits $f(x_0)^+$ e $f(x_0^-)$ exist and are finite, and that the left and right finite difference*

$$\frac{f(x_0 + h) - f(x_0^+)}{h}, \quad \frac{f(x_0 - h) + f(x_0^-)}{h},$$

are bounded for any $h > 0$ small. Then the series T_f of f converges in x_0 to the mean of the values of the right and left limits of f in x_0:

$$T_f(x_0) = \frac{1}{\tau} \sum_{n \in \mathbb{Z}} \langle f, e^{2\pi i n \xi/\tau} \rangle e^{2\pi i n x/\tau} = \frac{f(x_0^+) - f(x_0^-)}{2}.$$

In particular if $f'(x_0)$ exists then $T_f(x_0) = f(x_0)$.

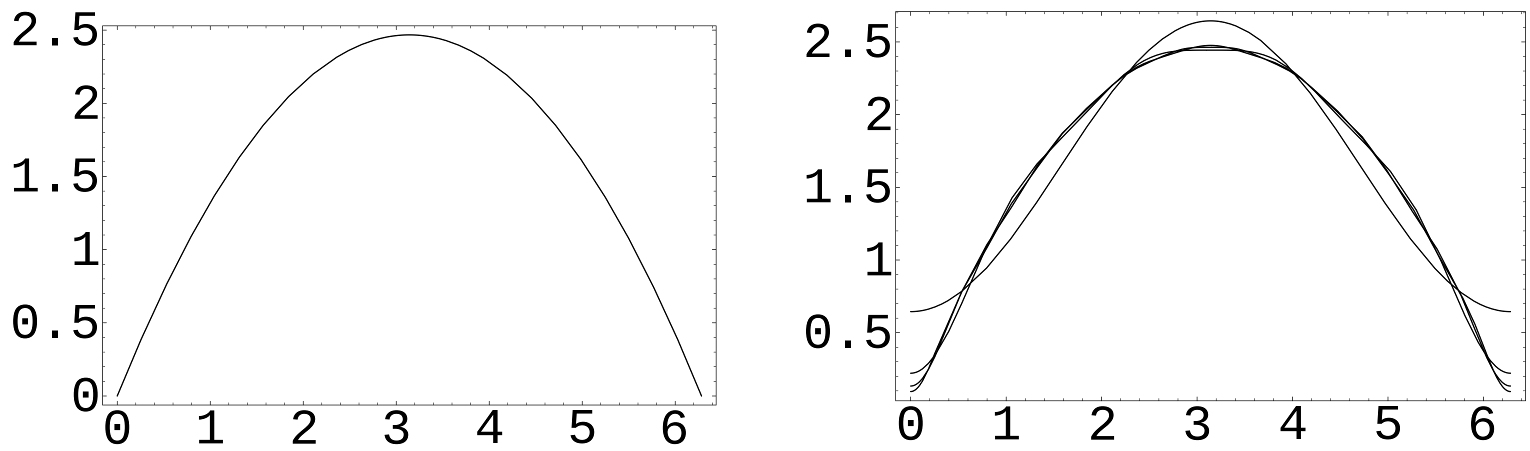

Figure 1: $f(x) = \frac{\pi x}{2} - \frac{x^2}{4} = \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$.

- The set of step functions S are dense in $L^2(0,\tau)$;
- We have

$$\sum_{n\in\mathbb{Z}} |\langle \chi_{[0,a]}, \frac{1}{\sqrt{\tau}} e^{2\pi inx/\tau} \rangle|^2 = \frac{a^2}{\tau} + \frac{\tau}{\pi^2}\left(\frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2\pi na}{\tau}\right)}{n^2}\right)$$

- From the previous theorem applied on f

$$a = \|\chi_{[0,a]}\|_2^2 = \sum_{n\in\mathbb{Z}} |\langle \chi_{[0,a]}, \frac{1}{\sqrt{\tau}} e^{2\pi inx/\tau} \rangle|^2.$$

- By density of S we conclude that $\{\frac{1}{\sqrt{\tau}} e^{2\pi inx/\tau}\}_{n\in\mathbb{Z}}$ is an orthonormal basis.

For functions $f \in L^2(0,\tau)$ we define the Fourier transform $\mathcal{F} : L^2(0,\tau) \to \ell^2(\mathbb{Z})$ as

$$\mathcal{F}f(n) = \langle f, \frac{1}{\sqrt{\tau}} e^{2\pi i n x/\tau} \rangle = \frac{1}{\sqrt{\tau}} \int_{(0,\tau)} f(x) e^{-2\pi i n x/\tau} dx. \qquad (6)$$

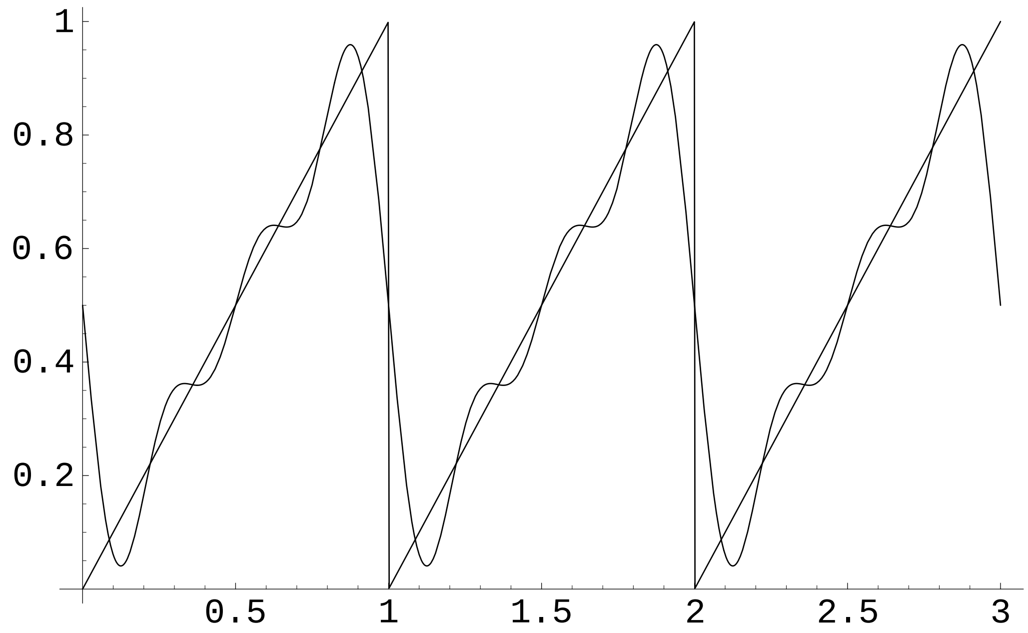

Figure 2: Fourier series associtated to $f(x) = x - [x]_-$.

What happens if we make $\tau \to \infty$?

$$\sum_{n\in\mathbb{Z}} \langle f, \frac{1}{\sqrt{\tau}} e^{2\pi i n x/\tau} \rangle \frac{1}{\sqrt{\tau}} e^{2\pi i n x/\tau} = \frac{1}{\tau} \sum_{n\in\mathbb{Z}} \left(\int_{\tau/2}^{-\tau/2} f(\xi) e^{-2\pi i n \xi/\tau} d\xi \right) e^{2\pi i n x/\tau}.$$

The last sums is in fact a Riemann sum. This makes us supposing that if f is an integrable function on $\mathbb{R}$ then we could write something like

$$f(x) = \lim_{\tau\to\infty} f(x)\chi_{[\tau/2,-\tau/2]} =$$

$$= \lim_{\tau\to\infty} \sum_{n\in\mathbb{Z}} \langle f, \frac{1}{\sqrt{\tau}} e^{2\pi i n x/\tau} \rangle \frac{1}{\sqrt{\tau}} e^{2\pi i n x/\tau} = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(\xi) e^{-2\pi i w \xi} d\xi \right) e^{2\pi i \xi w} dw.$$

When and in which sense this writing makes sense is though still to be proved.

Theorem 0.9 (Fourier transform and Plancherel identity) *We define for $f \in L^1(\mathbb{R})$ the Fourier transform, given by*

$$\mathcal{F}f(w) = \int_{\mathbb{R}} f(x)e^{-2\pi i w x}dx. \tag{7}$$

Moreover, if also $\mathcal{F}f \in L^1(\mathbb{R})$ then the Fourier transform is invertible and

$$f(x) = \int_{\mathbb{R}} \mathcal{F}f(w)e^{2\pi i w x}dw. \tag{8}$$

If $f \in L^1 \bigcap L^2$ then $\mathcal{F} : L^1 \bigcap L^2 \to L^2$ is a continuous linear operator which can be extended uniquely on the entire L^2. In particular, $\mathcal{F}$ is an isometric isomorphism, i.e., the Plancherel identity holds:

$$\|f\|_2 = \|\mathcal{F}f\|_2. \tag{9}$$

Moreover, if $f, g \in L^2$ then $\langle f, g\rangle = \langle \mathcal{F}f, \mathcal{F}g\rangle$.

Remark 0.10 *This definition is used in the literature of time-frequency analysis. The other definition*

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{-i\omega x} dx$$

is also used, for instance in the literature related to wavelets. In these lectures we use both in a consistent way with the corresponding literature. Anyway observe that $\mathcal{F}f(\omega) = 2\pi \hat{f}(2\pi\omega)$ *and, up to rescaling, one can transform all the result with one definition to results with another definition.*

What is the meaning of the Fourier transform? The Fourier transform represents the frequency content of a function/signal. It gives us which are the important oscillating building blocks of a signal and their dinstinctive frequency of oscillation.

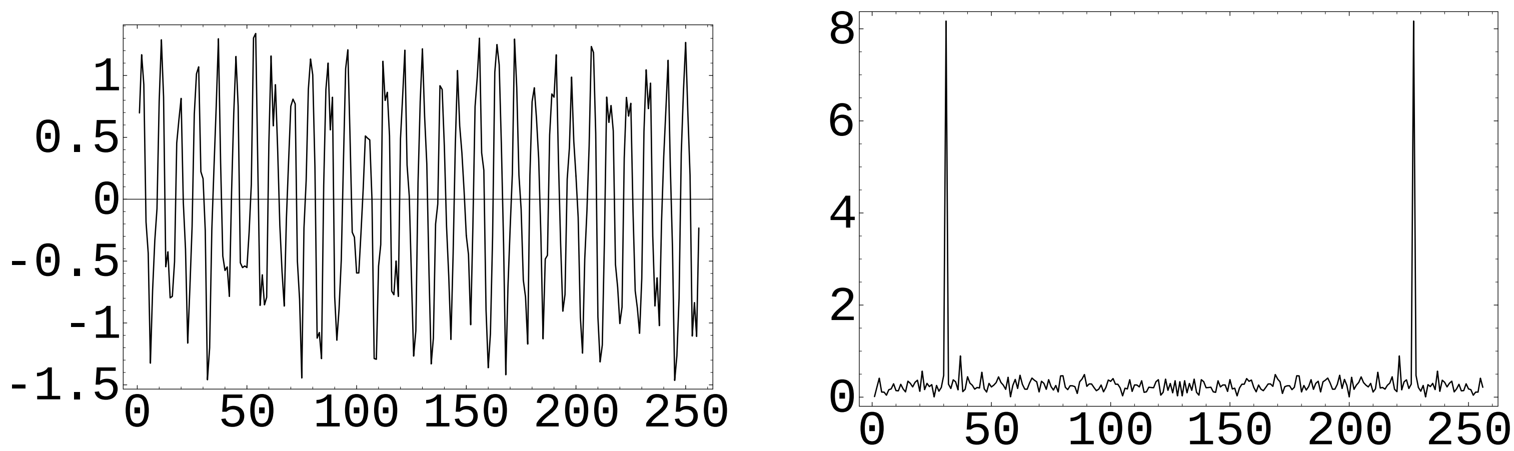

Figure 3: Sinosoidal signal affected by noise: there are two fundamental frequency components

The fortune of Fourier analysis stays essentially in the fact that it is able to describe one of the most frequent behavior in the nature: the wave phanomena, most of which are governed by ovelapping laws of the type

$$y_{\alpha,s_0}(t) = \begin{cases} e^{-\alpha t} e^{2\pi i s_0 t}, & t > 0 \\ 0, & t < 0. \end{cases}$$

$y_{\alpha,s_0}(t)$ is called *Lorenzian function*.

When, for example, some molecules are excited by electomagnetic radiation, damped oscillations are induced and described by the law $y_{\alpha,s_0}(t)$.

Every building block of the molecule has its own unique special oscillations. The Lorenzians are called in this case the *molecular spectrum*.

From here comes the idea which “costed” the Nobel prize to Richard Ernst for chemistry (1991) for the development of a powerful instrument to determine the structure of large complex organic molecules.

We want now to discuss how in practice we can use the tools of the Fourier analysis.

The execises we showed on simple functions show us that it is not so trivial to compute Fourier series or transforms.

For relatively simple functions such as $f(x) = \frac{1}{a^2+x^2}$ the computation of the Fourier transform is far from being trivial. In this case we have $\mathcal{F}f(w) = \frac{\pi}{a}e^{-|w|a}$.

A holomorphic complex function $f(z)$ (i.e., differentiable) on a domain Ω except for a point a, called singularity, can be expanded in a power series, called the Laurent series, $f(z) = \sum_{n \in \mathbb{Z}} c_n (z-a)^n$. The specific coefficient c_{-1} is called *residual* and we write $Res(f, a) = c_{-1}$. One can show that if f is holomorphic on a domain Ω which contains the halfplane $\{\mathrm{Im}(z) \geq 0\}$ except for a finite number of non real singularities and if the limit $\lim_{|z| \to \infty, \mathrm{Im}(z) \geq 0} f(z) = 0$ then we have for every $\alpha > 0$,

$$\lim_{r \to \infty} \int_{-r}^{r} f(x) e^{i\alpha x} dx = 2\pi i \sum \{Res(f(\cdot)e^{i\alpha \cdot}, a), \mathrm{Im}(a) > 0\}.$$

Analogous result holds for $\alpha < 0$ and for $\{\mathrm{Im}(z) < 0\}$, but changing the sign of the integral.

It is certainly more reasonable to search for an approximation on a discrete domain and a proper definition of discrete Fourier transform, which, in some sense, can approximate well the behavior of the continuous Fourier transform. Moreover, we need to make sure that the computation of the discrete Fourier transform (DFT) can be executed in reasonable time: otherwise it does not pay off.