Mathematics of Digitalization: Case Study

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Seventh lecture

GABOR FRAMES IN FINITE DIMENSIONS

Let $d = 1$ and we consider $a, b, L \in \mathbb{N}$ such that $a|L$ and $b|L$ and $a \cdot b \leq L$. Let us set $N = L/a$ e $M = L/b$. Then we define the discrete Gabor system

$$\mathbf{g}_{m,n} = M_{\frac{mb}{L}} T_{an} \mathbf{g}, \quad m = 0, ..., M-1, \quad n = 0, ..., N-1, \quad (1)$$

where $\mathbf{g} \in \mathbb{Z}_L$. Note that $N \cdot M \geq L$. We seek for frames for $\ell^2(\mathbb{Z}_L)$ of the type $\mathcal{G}(\mathbf{g}, a, b) = \{\mathbf{g}_{m,n}\}_{m=0,\ldots,M-1,n=0,\ldots,N-1}$. Let us consider $\{e_i\}_{i=0}^{L-1}$ the canonical basis of $\mathcal{H} = \ell^2(\mathbb{Z}_L)$; we can define the matrix

$$\mathcal{G}_{L\times N\cdot M} = \left(\mathbf{g}_{mod(k,M),\frac{k-mod(k,M)}{M}}\right)_{k=0,\ldots,N\cdot M-1}, \quad (2)$$

whose columns represent the valued of $\mathcal{G}(\mathbf{g}, a, b) = \{\mathbf{g}_{m,n}\}_{m=0,\ldots,M-1,n=0,\ldots,N-1}$.

As we already showed, the matrix of the frame operator is given by

$$\mathcal{S}_g = \mathcal{G}_{L\times N\cdot M} * \mathcal{G}^T_{L\times N\cdot M}. \tag{3}$$

In fact we have $\mathcal{S}_g : \ell^2(\mathbb{Z}_L) \to \ell^2(\mathbb{Z}_L)$. The the system is a Gabor frame is the matrix $\mathcal{S}_g$ is invertible, i.e., when $\det(\mathcal{S}_g) \neq 0$.

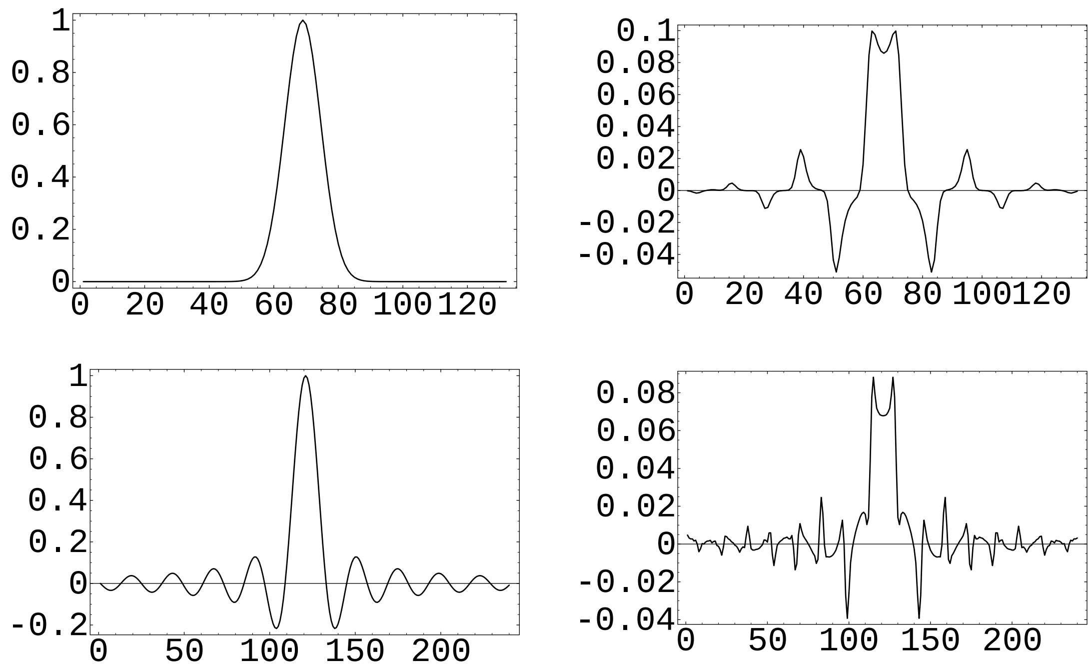

Figure 1: Gabor atoms and corresponding duals. On the top it is represented the Gaussian and on its right its corresponding dual for $L = 132$, $a = b = 11$, with redundancy $L/(ab) = 1.09$. On the bottom, we represented the cardinal sin and its relative dual for $L = 240$, $a = b = 15$, with redundancy $L/(ab) = 1.06$.

Signal compression

Once we constructed a Gabor frame $\mathcal{G}(\mathbf{g}, a, b)$ and its corresponding canonical dual $\mathcal{G}(\tilde{\mathbf{g}}, a, b)$ every signal $\mathbf{f}$ of length L can be (de)composed as follows:

$$\mathbf{f} = \sum_{m,n} \langle \mathbf{f}, \tilde{\mathbf{g}}_{m,n} \rangle \mathbf{g}_{m,n}. \tag{4}$$

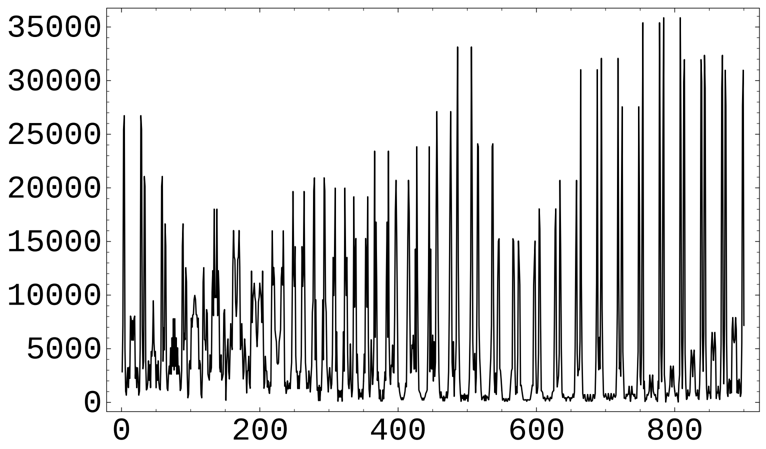

Figure 2: Coefficients $|\langle \mathbf{f}, \tilde{\mathbf{g}}_{m,n} \rangle|^2$.

The number of coefficients is $N \cdot M \geq L$ and it is a redundant digitalization. The information is spread on a larger number of digital samples.

The redundancy enlarges the possibility of identification. *The larger is the dictionary, the shorter are the sentences I can write!*

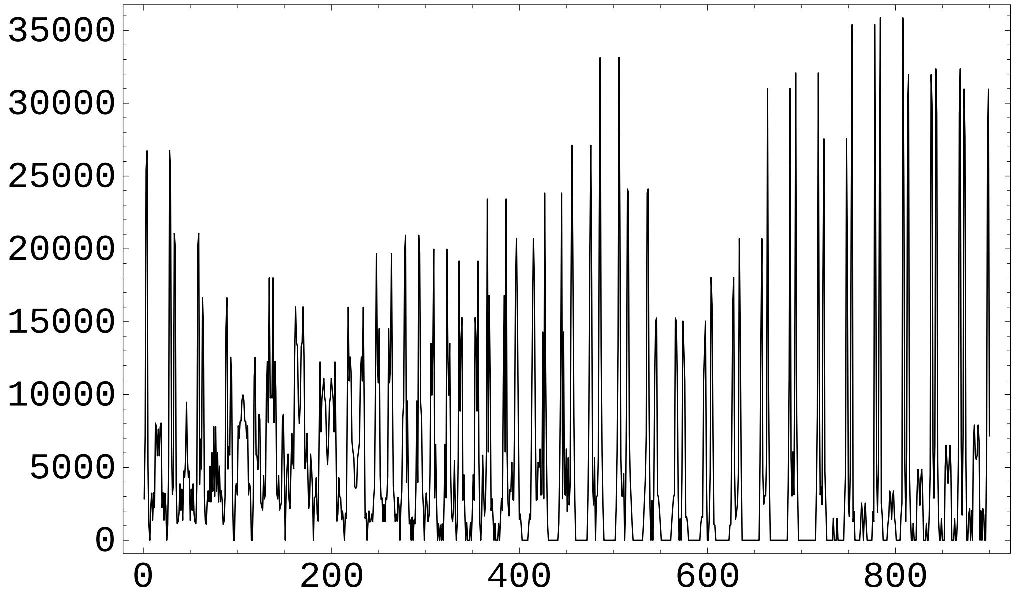

Figure 3: Coefficients $|\langle \mathbf{f}, \tilde{\mathbf{g}}_{m,n} \rangle|^2$ over a give threshold.

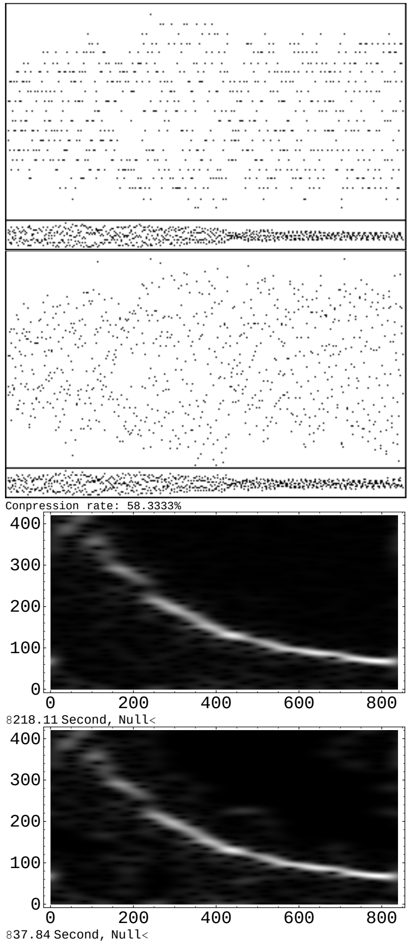

Figure 4: Coefficients $|\langle \mathbf{f}, \tilde{\mathbf{g}}_{m,n} \rangle|^2$ over a give threshold.

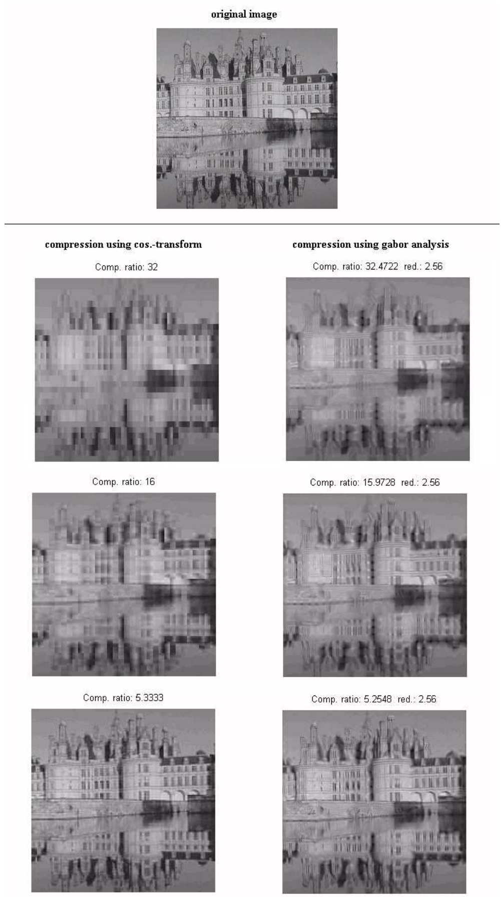

Figure 5: Compression of an image by Gabor frames and JPEG