Mathematics of Digitalization: Case Study

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Sixth lecture

Frames in finite dimensions

One can prove that any finite dimensional system $\mathcal{G}_n = \{g_0, ..., g_{n-1}\}$ is always a frame for its span $\mathcal{H} = \text{span}\{\mathcal{G}_n\} = \{\sum_{i=0}^{n} c_i g_i : c_i \in \mathbb{C}\}$ (exercise!).

How do we compute the canonical dual frame in finite dimension? Let us consider an orthonormal basis $\{e_i\}_{i=0}^{m-1}$, $m \leq n$ of $\mathcal{H}$ and we consider the matrix $\mathcal{G}_{m\times n} = (\langle g_k, e_i \rangle)_{i=0,...,m-1,k=0,...n-1}$. In particular we have that for all $f \in \mathcal{H}$ and considered $\overrightarrow{f} = (\langle f, e_i \rangle)_{i=0,...,m-1}$

$$C_g f = \mathcal{G}^T_{m\times n} * \overrightarrow{f} \ . \tag{1}$$

Singular Value Decomposition

The so-called Singular Value Decomposition (SVD) is a procedure applied on a matrix $M \in \mathbb{C}^{n \times m}$ and it is a very important one in numerical linear algebra. The basic idea is to decompose the matrix M in the form $M = U^* * D * V$ where D is a diagonal matrix with nonnegative diagonal entries and U, V are two matrices whose rows are orthonormal, $U^* = \overline{U^T}$ is the Hermitian transpose of U and coincides with its inverse. The nonzero elements of D are known as the *singular values* of M. The SVD of a matrix is unique.

http://en.wikipedia.org/wiki/Singular_value_decomposition

But then a left inverse of M can be computed as follows

$$M^{\dagger} = V^{*} * D^{-1} * U.$$

In fact $M^{\dagger} * M = (U^{*} * D * V) * \left(V^{*} * D^{-1} * U\right) = I$. One can also show that the sum of the squared entries of the matrix $(M * M^{\dagger} - I)$ is minimal and hence $M^{\dagger}$ is the so-called *pseudo-inverse* matrix of M. Moreover, we have

$$M^{\dagger} * M = M^{T} * (M^{\dagger})^{T} = M^{T} * (M^{T})^{\dagger}.$$

Hence, for a matrix $N \in \mathbb{C}^{m \times n}$ with $n \geq m$ we define $N^{\dagger} = (N^{T})^{\dagger}$ and we have $N * N^{\dagger} = I$.

The columns of the matrix $\left(\mathcal{G}^{\dagger}_{m\times n}\right)^{T}$ represents the coefficients $\{\langle\tilde{g}_k, e_i\rangle\}_{i=0,\dots,m-1}$ of the canonical dual of $\mathcal{G}_n$. If I consider a set of coefficients $\overrightarrow{c} = (c_k)_{k=0,\dots,n-1}$ we have that $\left(\mathcal{G}^{\dagger}_{m\times n}\right)^{T} * \overrightarrow{c}$ returns a vector of length m. In particular we have

$$R_{\tilde{g}} \overrightarrow{c} = \left(\left(\mathcal{G}^{\dagger}_{m\times n}\right)^{T} * \overrightarrow{c}\right)^{T} \cdot \{e_i\}_{i=0}^{m-1}. \tag{2}$$

Hence, we have the identities

$$f = R_{\tilde{g}} \circ C_g f = \left(\left(\mathcal{G}^{\dagger}_{m\times n} \right)^T * \mathcal{G}^T_{m\times n} * \overrightarrow{f} \right)^T \cdot \{e_i\}_{i=0}^{m-1} =$$

$$= \left(\overrightarrow{f}^T * \mathcal{G}_{m\times n} * \mathcal{G}^{\dagger}_{m\times n} \right) \cdot \{e_i\}_{i=0}^{m-1} \qquad (3)$$

Moreover we have,

$$R_g \overrightarrow{c} = \left(\mathcal{G}_{m\times n} * \overrightarrow{c}\right)^T \cdot \{e_i\}_{i=0}^{m-1}.$$

Hence, defined the frame operator $\mathcal{S}_g = \mathcal{G}_{m\times n} * \mathcal{G}_{m\times n}^T$ we obtain

$$S_g f = R_g \circ C_g f = \left(\mathcal{G}_{m\times n} * \mathcal{G}_{m\times n}^T * \overrightarrow{f}\right)^T \cdot \{e_i\}_{i=0}^{m-1} = \left(\mathcal{S}_g * \overrightarrow{f}\right)^T \cdot \{e_i\}_{i=0}^{m-1}.$$

But then we can compute the canonical dual vectors by using

$$\tilde{g}_k = \left(\mathcal{S}_g^{-1} * \overrightarrow{g}_k\right)^T \cdot \{e_i\}_{i=0}^{m-1}.$$

Definizione 0.1 *Given* $g \in L^2(\mathbb{R})$ *and* $a, b > 0$*, we say that (g,a,b)* generates a Gabor *for* $L^2(\mathbb{R})$ *if* $\{M_{mb}T_{an}g\}_{m,n\in\mathbb{Z}}$ *is a frame for* $L^2(\mathbb{R})$*.*
The function g *is called the* window or analyzing function*.*
The number a, b *are called* frame lattice parameters *and* a *is the* translation parameter*, while* b *is the* modulation parameter*.*

It is clear that $\{M_{mb}T_{an}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ if and only if $\{T_{an}M_{bn}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ (exercise!). Later we will use one or the other definition depending on the context.

Necessary and sufficient conditions for Gabor frames

Theorem 0.2 (Necessary condition) *For $g \in L^2(\mathbb{R})$, $a, b > 0$, if (g, a, b) generates and Gabor frame for $L^2(\mathbb{R})$, then $ab \leq 1$.*

Theorem 0.3 (Sufficient conditions) *Let $g \in L^2(\mathbb{R})$ and let $a > 0$ be such that*

1. *there exist constants A, B such that $0 < A \leq \sum_{n\in\mathbb{Z}} \mid g(x - na) \mid^2 \leq B < \infty$ a.e.*
2. *$\lim_{b\to 0} \sum_{k\neq 0, k\in\mathbb{Z}} \beta(k/b) = 0$ where*

$$\beta(s) = \operatorname*{ess\,sup}_{x\in\mathbb{R}} \mid \sum_{n\in\mathbb{Z}} g(x-na)\overline{g(x - s - na)} \mid = \Big\| \sum_{n\in\mathbb{Z}} T_{na}g \cdot T_{na+s}\bar{g} \Big\|_\infty$$

then there exists $b_0 > 0$ such that (g, a, b) generates a Gabor frame for $L^2(\mathbb{R})$ for all $0 < b < b_0$ (of course $ab < 1$ must hold!).

Proof of the Sufficient Conditions

First assume that f is continuous with compact support. This allows us to exchange the order of sums and integrals without troubles. For a fixed n we consider the 1/b-periodic function given by

$$F_n(t) = \sum_{k\in\mathbb{Z}} f(t-k/b)\overline{g(t-na-k/b)}$$

Now $F_n \in L^1[0,1/b]$ since f e g are bounded and holds

$$\int_{\mathbb{R}} f(t)\overline{g(t-na)}e^{-2\pi imbt}dt = \int_0^{1/b} F_n(t)e^{-2\pi imbt}dt.$$

Since $\{b^{1/2}M_{mb}\chi_{[0,1/b]}\}_{m\in\mathbb{Z}}$ is an orthonormal basis for

$L^2[0,1/b]$, by the Plancharel formula we obtain

$$\sum_{m\in\mathbb{Z}} \mid \int_0^{1/b} F_n(t)e^{-2\pi imbt}dt \mid^2 = b^{-1} \int_0^{1/b} \mid F_n(t) \mid^2 dt$$

Hence

$$\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\langle f, M_{mb}T_{na}g\rangle|^2 = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \left| \int_{\mathbb{R}} f(t)\overline{g(t-na)}e^{-2\pi imbt}dt \right|^2$$

$$= b^{-1}\sum_{n\in\mathbb{Z}}\int_0^{1/b} \left| \sum_{k\in\mathbb{Z}} f(t-k/b)\overline{g(t-na-k/b)} \right|^2 dt$$

$$= b^{-1}\sum_{n\in\mathbb{Z}}\int_0^{1/b} \sum_{l\in\mathbb{Z}} \overline{f(t-l/b)}g(t-na-l/b)\cdot\sum_{k\in\mathbb{Z}} f(t-k/b)\overline{g(t-na-k/b)}dt$$

$$= b^{-1}\sum_{n\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\int_0^{1/b} \overline{f(t-l/b)}g(t-na-l/b)\cdot\sum_{k\in\mathbb{Z}} f(t-k/b)\overline{g(t-na-k/b)}dt$$

$$= b^{-1}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} \overline{f(t)}g(t-na)\cdot\sum_{k\in\mathbb{Z}} f(t-k/b)\overline{g(t-na-k/b)}dt$$

$$= b^{-1}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} \overline{f(t)}f(t-k/b)\cdot\sum_{n\in\mathbb{Z}} g(t-na)\overline{g(t-na-k/b)}dt$$

$$= \quad b^{-1} \int_{\mathbb{R}} \mid f(t) \mid^2 \cdot \sum_{n \in \mathbb{R}} \mid g(t-na) \mid^2 dt +$$

$$b^{-1} \sum_{k \in \mathbb{Z}; k \neq 0} \int_{\mathbb{R}} \overline{f(t)} f(t-k/b) \cdot \sum_{n \in \mathbb{Z}} g(t-na) \overline{g(t-na-k/b)} dt = (*)$$

By using the Cauchy-Schwarz inequality we get

$$(*) \leq b^{-1} B \|f\|_2^2 + b^{-1} \sum_{k \in \mathbb{Z}; k \neq 0} \beta(k/b) \int_{\mathbb{R}} \overline{f(t)} f(t-k/b) dt \leq B_0(b) \|f\|_2^2$$

e

$$(*) \geq b^{-1} A \|f\|_2^2 - b^{-1} \sum_{k \in \mathbb{Z}; k \neq 0} \beta(k/b) \int_{\mathbb{R}} \overline{f(t)} f(t-k/b) dt \geq A_0(b) \|f\|_2^2$$

where

$$A_0(b) = b^{-1} A - b^{-1} \sum_{k \in \mathbb{Z}; k \neq 0} \beta(k/b) \quad \text{e} \quad B_0(b) = b^{-1} B + b^{-1} \sum_{k \in \mathbb{Z}; k \neq 0} \beta(k/b)$$

From condition 2. there exists, by definition of limit and by $A > 0$, $B < \infty$, a constant $b_0 > 0$ such that $A_0(b) > 0$ e $B_0(b) < \infty$ for all $0 < b < b_0$.

Assume now that $f \in L^2(\mathbb{R})$ is arbitrary. Then we can find a sequence of continuous functions of compact support f_j, such that $f_j \to f$ in $L^2(\mathbb{R})$ for $j \to \infty$. From the result just obtained, it holds

$A_o(b)\|f_j\|_2^2 \leq \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} | \langle f_j, M_{mb}T_{na}g\rangle |^2 \leq B_o(b)\|f_j\|_2^2$. It is not difficult to show that these inequalities hold also for the limit $j \to \infty$, from which we deduce that (g, a, b) generates a Gabor frame with frame bounds $A_0(b), B_0(b)$ for all $0 < b < b_0$.

If $a, b > 0$ and if $\{M_{aj}T_{bk}g\}_{j,k\in\mathbb{Z}^d}$ is a Gabor frame for $L^2(\mathbb{R}^d)$ then

$$A\|f\|_2^2 \leq \sum_{j,k} |\langle f, M_{aj}T_{bk}g\rangle|^2 \leq B\|f\|_2^2, \tag{4}$$

and $a \cdot b \leq 1$. In particular the frame operator

$$S_g f = \sum_{j,k} \langle f, M_{aj}T_{bk}g\rangle M_{aj}T_{bk}g,$$

commutes with the modulation and translation operators, in the sense that

$$S_g M_{aj}T_{bk} = M_{aj}T_{bk}S_g.$$

(exercise!)

This implies that the canonical dual $S_g^{-1} M_{aj} T_{bk} g = M_{aj} T_{bk} S_g^{-1} g$. We define $\tilde{g} = S_g^{-1} g$ and we have

$$f = \sum_{j,k} \langle f, M_{aj} T_{bk} \tilde{g} \rangle M_{aj} T_{bk} g. \quad (5)$$

We observe now that $\langle f, M_{aj} T_{bk} \tilde{g} \rangle = V_{\tilde{g}}(f)(bk, aj)$.

Theorem 0.4 (Existence of Gabor frames) *If $g \in L^2(\mathbb{R}^d)$ is a well-localized function both in time and frequency (for instance a very smooth function with fast decay), the there exist $a, b > 0$ sufficiently small such that $\mathcal{G}(g, a, b) = \{M_{aj} T_{bk} g\}_{j,k}$ and $\mathcal{G}(\mathcal{F}g, b, a) = \{M_{bj} T_{ak} \mathcal{F}g\}_{j,k}$ are Gabor frames. In particular if $g(x) = e^{-\pi|x|^2}$ is the Gaussian $\mathcal{G}(g, a, b)$ is a Gabor frame for all $a, b > 0$ such that $a \cdot b < 1$.*