Mathematics of Digitalization: Case Study

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Fifth lecture

Frames in Hilbert spaces

Let $\mathcal{H}$ be a separable Hilbert space.

Definizione 0.1 *A set* $\{g_n\}_{n\in\mathcal{N}} \subset \mathcal{H}$ *is a frame for* $\mathcal{H}$ *if there exist* $A, B > 0$ *such that*

$$A \cdot \|f\|^2 \leq \sum_{n\in\mathcal{N}} |\langle f, g_n\rangle|^2 \leq B \cdot \|f\|^2, \quad \forall f \in \mathcal{H}. \tag{1}$$

Note that this is a weaker version of the Parseval identity in the Fourier theorem. In particular orthonormal bases are frames!

Let us define the *frame operator* by $S : \mathcal{H} \to \mathcal{H}$

$$Sf = \sum_{n \in \mathcal{N}} \langle f, g_n \rangle g_n. \tag{2}$$

In particular the frame definition implies that S is positive, self-adjoint, and invertible. Then we have

$$f = SS^{-1}f = \sum_{n \in \mathcal{N}} \langle f, S^{-1}g_n \rangle g_n = S^{-1}Sf = \sum_{n \in \mathcal{N}} \langle f, g_n \rangle S^{-1}g_n. \tag{3}$$

The system $\{\tilde{g}_n = S^{-1}g_n\}_{n \in \mathcal{N}}$ is again a frame, called the *canonical dual* of $\{g_n\}_{n \in \mathcal{N}}$ with corresponding frame operator S^{-1}.

Since a frame is typically redundant, in the sense that there is not only one coefficient map $\{c_n\}_{n\in\mathcal{N}}$ such that

$$f = \sum_{n\in\mathcal{N}} c_n(f) g_n,$$

by the Riesz-Fischer duality theorem, there exist many possible duals $\{\tilde{g}_n\}_{n\in\mathcal{N}} \subset \mathcal{H}$ such that

$$f = \sum_{n\in\mathcal{N}} \langle f, \tilde{g}_n \rangle g_n.$$

Example 0.2 *Let*
$\mathcal{H} = \mathbb{R}^2$, $f = (-1, 3)$ *and* $g_1^T = (1, -1)$, $g_2^T = (0, 1)$, $g_3^T = (1, 1)$.
The coefficients $(c_n)_{n\in\{1,2,3\}}$ *of* f *are*

$$(c_n)_{n\in\{1,2,3\}} = \big(\langle f, g_1\rangle, \langle f, g_2\rangle, \langle f, g_3\rangle\big) = (-4, 3, 2).$$

The canonical dual frame of $\{\tilde{g}_n\}_{n\in\{1,2,3\}}$ *is given by*

$$\{\tilde{g}_n\}_{n\in\{1,2,3\}} = \{S^{-1}g_n\}_{n\in\{1,2,3\}} = \left\{ \begin{pmatrix} 1/2 \\ -1/3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix} \right\}.$$

Hence I obtain the recovery of f *by the identity:*

$$f = \sum_{n=1}^{3} c_n \tilde{g}_n = (\begin{pmatrix} -2 \\ \frac{4}{3} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix}) = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

If I choose another dual, for instance

$$\{h_n\}_{n\in\{1,2,3\}} = \left\{ \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

we have again the recovery of f

$$f = \sum_{n=1}^{3} c_n h_n = (\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}) = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$