Mathematics of Digitalization: Case Study

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Fourth lecture

Summary of the results obtained so far

So far we showed that the sampling followed by a DFT

$$\begin{array}{ccccc} & \overset{\cdot^c}{} & & \overset{\mathcal{F}}{} & \\ C_c & \rightarrow & \ell^2(\mathbb{Z}_m^d) & \rightarrow & \ell^2(\mathbb{Z}_m^d) \\ f & \rightarrow & \mathbf{f^c} & \rightarrow & \mathcal{F}\mathbf{f^c} \end{array}$$

is equivalent to apply a continuous Fourier transform, one periodization and a sampling again

$$\begin{array}{ccccccc} & \overset{\mathcal{F}}{} & & \overset{P}{} & & \overset{\cdot^c}{} & \\ C_c & \rightarrow & L^2(\mathbb{R}^d) & \rightarrow & L^2(\omega T^d) & \rightarrow & \ell^2(\mathbb{Z}^d) \\ f & \rightarrow & \mathcal{F}f & \rightarrow & P[\mathcal{F}f] & \rightarrow & (P[\mathcal{F}f])^c \end{array}$$

Moreover **IF** $\mathcal{F}f$ is a function decaying very rapidly we have

$$\mathcal{F}f(w) \approx P[\mathcal{F}f](w), \quad \forall w \in Q$$

and we can therefore assume the following good approximations

$$
\begin{aligned}
\mathcal{F}f(w) &\approx \sum_{l\in\mathbb{Z}^d, l/a\in Q} (P[\mathcal{F}f])^c(l/a) \prod_{i=1}^{d} \operatorname{sinc}(a_i w_i - l_i) \\
&= \frac{\det(a))}{(\det(m))^{1/2}} \sum_{l\in\mathbb{Z}^d, l/a\in Q} \mathcal{F}\mathbf{f}^{\mathbf{c}}(l) \prod_{i=1}^{d} \operatorname{sinc}(a_i w_i - l_i).
\end{aligned}
$$

THEREFORE if f is a function of compact support, whose Fourier transform decay fast (and this is ensured as soon as a function is sufficiently smooth!) then the Fourier transform can be well-approximated by applying a DFT on the samples of f and the formula given by the Wittacker-Shannon sampling theorem. Hence, for functions, where it is difficult to apply the techniques of the residuals, we can attempt the approximation by means of discrete Fourier transform.

In the following we assume that $\ell^2(\tau\mathbb{Z}^d)$ is endowed with the scalar product $\langle f, g\rangle_{\ell^2} := \det(\tau)\sum_{k\in\mathcal{Z}^d} f(\tau k)\overline{g(\tau k)}$.

Theorem 0.1 (Perturbed scalar product) *Let us assume that* $f, g \in C(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ *tali che* $f_{|\tau\mathbb{Z}^d}, g_{|\tau\mathbb{Z}^d} \in \ell^2$. *Then the following approximation on the scalar products of* f *and* g *holds:*

$$|\langle f, g\rangle_{L^2} - \langle f_{|\tau Z^d}, g_{|\tau Z^d}\rangle_{\ell^2}| \leq$$

$$\leq \|\varepsilon_f\|_{\ell^2}\|g\|_{\ell^2} + \|\varepsilon_g\|_{\ell^2}\|f\|_{\ell^2} + \|\varepsilon_f\|_{\ell^2}\|\varepsilon_g\|_{\ell^2} + \|\varepsilon_f\|_{L^2}\|\varepsilon_f\|_{L^2}. \quad (1)$$

Proof of the Theorem

By the Theorem of the Perturbed Sampling in L^2 we obtain by considering the scalar products:

$$\begin{aligned}\langle f,g\rangle_{L^2} &= \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx \\ &= \int_{\mathbb{R}^d}\left(\sum_{k\in\mathbb{Z}^d}(f(\tau k)-\varepsilon_f(\tau k))\prod_{i=1}^{d}\operatorname{sinc}(\tau_i^{-1}x_i-k_i)+\varepsilon_f(x)\right)\times \\ &\qquad \times\overline{\left(\sum_{h\in\mathbb{Z}^d}(g(\tau h)-\varepsilon_g(\tau h))\prod_{i=1}^{d}\operatorname{sinc}(\tau_i^{-1}x_i-h_i)+\varepsilon_g(x)\right)}\,dx.\end{aligned}$$

As the functions of the type η_* are orthogonal in L^2 to functions of the type ε_*, we have:

$$\langle f,g\rangle_{L^2} = \sum_{h,k\in\mathbb{Z}^d}(f(\tau k)-\varepsilon_f(\tau k))\overline{(g(\tau h)-\varepsilon_g(\tau h))}\times$$

$$\times \left(\int_{\mathbb{R}^d} \prod_{i=1}^{d} \operatorname{sinc}(\tau_i^{-1} x_i - k_i) \operatorname{sinc}(\tau_i^{-1} x_i - h_i) dx \right) + \langle \varepsilon_f, \varepsilon_g \rangle_{L^2} =$$

and by the orthogonality of the "sinc" these identities continue as follows

$$\begin{aligned} &= \det(\tau) \cdot \sum_{k \in \mathbb{Z}^d} (f(\tau k) - \varepsilon_f(\tau k)) \overline{(g(\tau k) - \varepsilon_g(\tau k))} + \langle \varepsilon_f, \varepsilon_g \rangle_{L^2} \\ &= \langle f_{|\tau \mathbb{Z}^d}, g_{|\tau \mathbb{Z}^d} \rangle_{\ell^2} + \langle f_{|\tau \mathbb{Z}^d}, (\varepsilon_g)_{|\tau \mathbb{Z}^d} \rangle_{\ell^2} + \langle (\varepsilon_f)_{|\tau \mathbb{Z}^d}, g_{|\tau \mathbb{Z}^d} \rangle_{\ell^2} \\ &\qquad + \langle (\varepsilon_f)_{|\tau \mathbb{Z}^d}, (\varepsilon_g)_{|\tau \mathbb{Z}^d} \rangle_{\ell^2} + \langle \varepsilon_f, \varepsilon_g \rangle_{L^2} \end{aligned}$$

Hence, by using the Cauchy-Schwarz inequalty, we obtain the following estimate:

$$|\langle f, g \rangle_{L^2} - \langle f_{|\tau \mathbb{Z}^d}, g_{|\tau \mathbb{Z}^d} \rangle_{\ell^2}| \leq$$

$$\leq \|\varepsilon_f\|_{\ell^2} \|g\|_{\ell^2} + \|\varepsilon_g\|_{\ell^2} \|f\|_{\ell^2} + \|\varepsilon_f\|_{\ell^2} \|\varepsilon_g\|_{\ell^2} + \|\varepsilon_f\|_{L^2} \|\varepsilon_f\|_{L^2}.$$

Remark 0.2 *If f, g are two ω-bandlimited functions we have the identity*

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}dx \ \ dx = \det(\tau) \sum_{k \in \mathbb{Z}^n} f^c(\tau k)\overline{g^c(\tau k)}.$$

Digitalization and recovery: redundancy

We considered so far a system of digitalization by sampling: for every $f \in L^2_Q$ the coefficient map $C : L^2_Q \to \ell^2$ is given by $C(f) = \{f(\tau k)\}_{k \in \mathbb{Z}^d}$ and it can be inverted by the recovery map $R : \ell^2 \to L^2_Q$ defined by $R(\overrightarrow{c}) = \sum_{k \in \mathbb{Z}^d} c_k \prod_{i=1}^d \mathrm{sinc}(\tau_i^{-1} x_i - k_i)$. Indeed we have $I = R \circ C$. Moreover, by the previous remark, we know that this system of digitalization is stable, since we have an equivalence of norms. Unfortunately the system is NOT redundant as C is the unique coefficient map which is the right-inverse R. In fact the $\mathrm{sinc}(\tau_i^{-1} x_i - k_i)$ constitute an orthonormal basis! Can we construct redundant systems?

Translation and Modulation

As for discrete signals $\mathbb{Z}_n^d$, also for functions on $\mathbb{R}^d$ we can define the operators of translation and modulation:

$$T_x f(t) = f(t-x), \quad M_w f(t) = e^{2\pi i w t} f(t). \tag{2}$$

and it holds (exercise!)

$$\mathcal{F} T_x = M_{-x} \mathcal{F}, \quad \mathcal{F} M_w = T_w \mathcal{F}. \tag{3}$$

Moreover if $f, g \in L^2(\mathbb{R}^d)$ we define the $L^2(\mathbb{R}^d)$ function given by the convolution of f and g by

$$f \star g(x) = \int_{\mathbb{R}^d} f(x-y) g(y) dy. \tag{4}$$

and we have that (exercise!)

$$\mathcal{F}(f \star g) = \mathcal{F}(f) \cdot \mathcal{F}(g). \tag{5}$$

Proposition 0.3 *The translation operator is given by* $T_x g(t) = g(t-x)$ *for any function* g *on* $\mathbb{R}^d$. *Given band-limited function* $g \in L^1(\mathbb{R}^d)$ *such that* $\mathcal{F}g \neq 0$ *on a compact set* $\Omega \subset \mathbb{R}^d$, *then there exists* $\tau > 0$ *such that for all* $f \in L^2_\Omega(\mathbb{R}^d)$

$$f = \sum_{k \in \mathbb{Z}^d} c_k(f) T_{\tau k} g, \tag{6}$$

for suitable coefficients $(c_k(f))_{k \in \mathbb{Z}^d} \in \ell^2$ *depending continuously on* f. *Such coefficients are NOT in general unique.*

Proof of the Proposition

We known that $\{\tau^{d/2}e^{2\pi i\tau kx}\}_{k\in\mathbb{Z}^d}$ is and orthonormal basis for $L^2([-1/(2\tau),1/(2\tau)]^d)$. For $\tau>0$ sufficiently small $\Omega\subset[-1/(2\tau),1/(2\tau)]^d$. There exists $g_1\in L^1$ a bandlimited function, such that $\mathcal{F}g_1\cdot\mathcal{F}g\equiv 1$ on Ω. Hence for $f\in L^2_\Omega(\mathbb{R}^d)$ and for $\tau>0$ sufficiently small

$$\mathcal{F}f(w)=[(\mathcal{F}f\mathcal{F}g_1)\mathcal{F}g](w)=\sum_{k\in\mathbb{Z}^d}\tau^d\langle\mathcal{F}f\mathcal{F}g_1,e^{2\pi i\tau kx}\rangle e^{2\pi i\tau kw}\mathcal{F}g(w). \tag{7}$$

By applying the inverse Fourier transform we obtain

$$f=\sum_{k\in\mathbb{Z}^d}\tau^d(f\star g_1)(\tau k)T_{\tau k}g, \tag{8}$$

where $\star$ is the convolution symbol

$$f \star g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy.$$

In particular we can prove

$$\sum_{k\in\mathbb{Z}^d} \tau^d |(f \star g_1)(\tau k)|^2 = \|f \star g_1\|_2^2 \leq \|f\|_2^2 \|g_1\|_1^2. \tag{9}$$

Hence, if we set $c_k(f) = \tau^d (f \star g_1)(\tau k)$ we have

$$\sum_{k\in\mathbb{Z}} |c_k(f)|^2 \leq \left(\tau^d \|g_1\|_1^2\right) \|f\|_2^2. \tag{10}$$

In particular,
$\sum_{k\in\mathbb{Z}} |\langle f, T_{\tau k} g\rangle|^2 = \sum_{k\in\mathbb{Z}} |\langle \mathcal{F}f, e^{2\pi i\tau k w}\mathcal{F}g\rangle|^2 \leq \tau^{-d} \|f\|_2^2 \|\mathcal{F}g\|_\infty^2.$

Let us now consider $Q = \prod_{i=1}^{d} \omega_i[-1/2, 1/2)$. Surely we have that $\{\omega j + Q\}_{j\in\mathbb{Z}^d}$ define a disjoint covering of $\mathbb{R}^d$, that means $\mathbb{R}^d = \bigcup_{j\in\mathbb{Z}^d}(\omega j + Q)$. Every function $f \in L^2(\mathbb{R}^d)$ can be written as the following decomposition:

$$f = \mathcal{F}^{-1}\left(\sum_{j\in\mathbb{Z}^d} \mathcal{F}f \cdot \chi_{\omega j+Q}\right) = \sum_{j\in\mathbb{Z}^d} \mathcal{F}^{-1}\left(\mathcal{F}f \cdot \chi_{\omega j+Q}\right). \quad (11)$$

Each function $\mathcal{F}^{-1}\left(\mathcal{F}f \cdot \chi_{\omega j+Q}\right)$ is bandlimited with $\mathrm{supp}((\mathcal{F}f \cdot \chi_{\omega j+Q})) \subset \omega j + Q \Leftrightarrow \mathrm{supp}(T_{\omega j}\left(\mathcal{F}f \cdot \chi_{\omega j+Q}\right)) \subset Q$.

Hence, we have

$$M_{-\omega j}\mathcal{F}^{-1}\left(\mathcal{F}f\cdot\chi_{\omega j+Q}\right)=M_{-\omega j}f_j,$$

is a ω-bandlimited function, where we set $f_j=\mathcal{F}^{-1}\left(\mathcal{F}f\cdot\chi_{\omega j+Q}\right)$. From the Wittacker-Shannon sampling Theorem we have immediately that

$$M_{-\omega j}f_j=\sum_{k\in\mathbb{Z}^d}f_j(\tau k)\prod_{i=1}^{d}\operatorname{sinc}(\tau_i^{-1}x_i-k_i),$$

and

$$f_j=\sum_{k\in\mathbb{Z}^d}f_j(\tau k)M_{\omega j}\prod_{i=1}^{d}\operatorname{sinc}(\tau_i^{-1}x_i-k_i). \tag{12}$$

Combining the previous expressions we get

$$f = \sum_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} f_j(\tau k) M_{\omega j} T_{\tau k} \left(\prod_{i=1}^{d} \operatorname{sinc}(\tau_i^{-1} x_i) \right). \qquad (13)$$

The Gabor Transform

Let us define

$$V_g(f)(b,w) = \langle f, M_w T_b g\rangle = \int_{\mathbb{R}^d} f(t) e^{-2\pi i w\cdot t}\overline{g(t-b)}dt \qquad (14)$$

For $\tau > 0$ fixed small and $\omega = 1/\tau$ we have

$$\begin{aligned} f &= \sum_{j\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d} f_j(\tau k) M_{\omega j} T_{\tau k} \underbrace{\left(\prod_{i=1}^{d} \operatorname{sinc}(\tau_i^{-1} x_i)\right)}_{:=g} \\ &= \sum_{j,k\in\mathbb{Z}^d} \underbrace{\frac{1}{\det(\tau)} V_g(f)(\tau k, \omega j)}_{:=f_j(\tau k)} M_{\omega j} T_{\tau k} g \end{aligned}$$

A signal f is mapped to the time-frequancy plane by means of the transform $V_g(f)(b,w)$, realized by considering the scalar product in L^2 of f with respec to an analyzing function g modified by an application of a translation and a modulation.

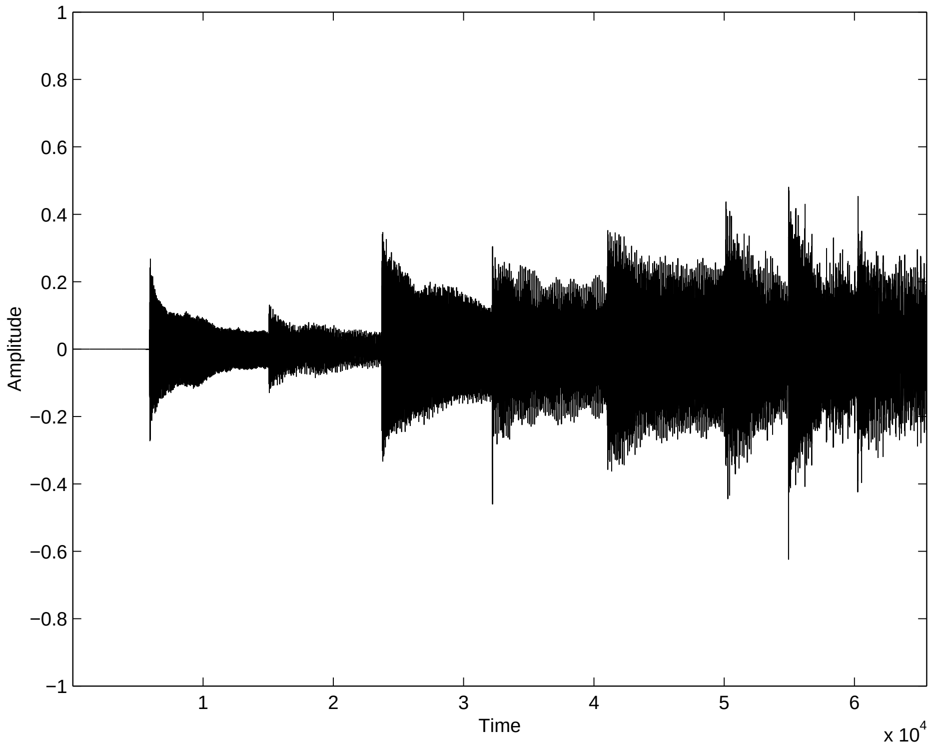

Figure 1: A test signal

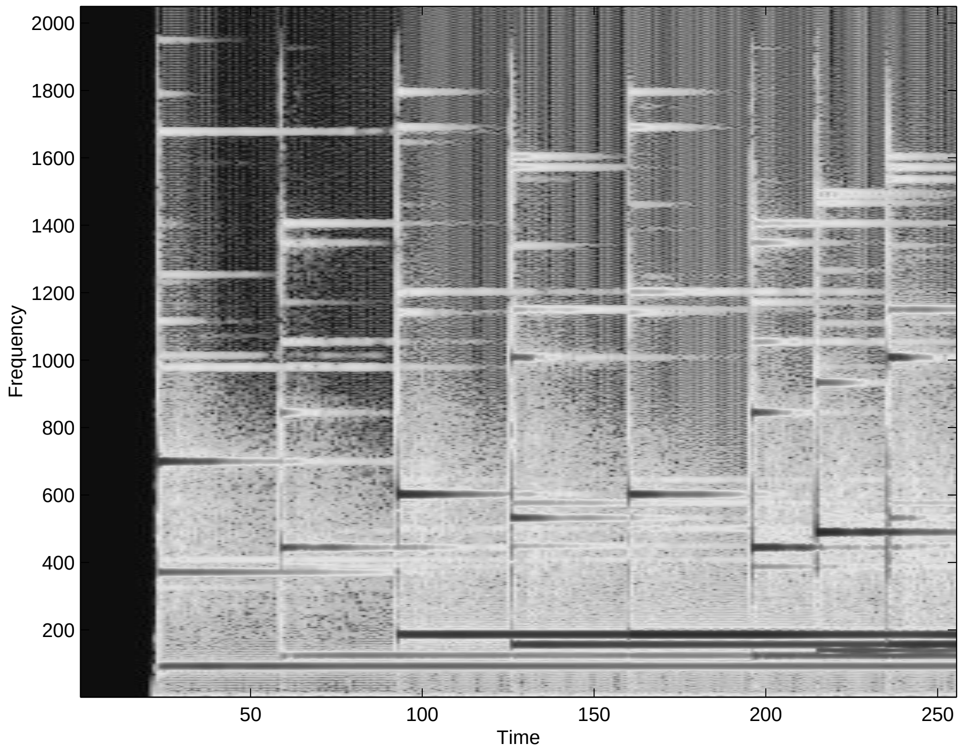

Figure 2: Spectrogram of $|V_g f(b,w)|^2$ of the test signal

Uncertainty principle

Theorem 0.4 (Heisnberg uncertainty principle) *Let $g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$ two arbitrary scalars. Then*

$$\left(\int (x-a)^2 |g(x)|^2 dx\right)^{1/2} \left(\int (\omega-b)^2 |\mathcal{F}g(\omega)|^2 d\omega\right)^{1/2} \geq \frac{\|g\|_2^2}{4\pi}.$$

If $g(x) = e^{-\pi|x|^2}$ is a Gaussian, then it holds

$$\left(\int (x-a)^2 |g(x)|^2 dx\right)^{1/2} \left(\int (\omega-b)^2 |\mathcal{F}g(\omega)|^2 d\omega\right)^{1/2} = \frac{\|g\|_2^2}{4\pi}.$$

A window (analyzing function) cannot be arbitrarily concentrated in time and frequency. The Gaussian is the function which is equally concentrated in time and frequency.

Hence for an arbitrary window function g, the Gabor transform has more resolution either in time or in frequency. In particular the function $g(x) = \mathrm{sinc}(x)$ is badly localized in time. We wonder how can we recover f by means of superpositions of functions of the type $M_w T_x g$ with g different from $\mathrm{sinc}(x)$

$$f = \sum_{j,k \in \mathbb{Z}^d} c_{j,k} M_{\omega j} T_{\tau k} g.$$