Mathematics of Digitalization: Case Study

Massimo Fornasier
Chair of Applied Numerical Analysis
Email: massimo.fornasier@ma.tum.de

Third lecture

Sampling and DFT

Definizione 0.1 *Let* $f \in C(\mathbb{R}^d)$ *and* $\tau = (\tau_1, ..., \tau_d) \in (0, \infty)^d$. *We call the sampling of step* τ *the operator which associates to* f *the function* $f^c \equiv f_{|\tau\mathbb{Z}^d}$. *If* $f \in C_c(\mathbb{R}^d)$, *clearly* f^c *is vanishing out of a compact set of* $\mathbb{Z}^d$. *Moreover, up to a translation, we can always assume that* f^c *is defined on a set* $\tau\mathbb{Z}^d/\tau m\mathbb{Z}^d$ *isomorphic to* $\mathbb{Z}^d_m$. *Hence, the sampling induces an operator acting from* $C_c(\mathbb{R}^d)$ *to* $\ell^2(\mathbb{Z}^d_m)$, *for some* $m = (m_1, ..., m_d) \in \mathbb{N}^d \backslash \{0\}$.

We want to study some of the properties of the operator $\cdot^c$. In particular we wonder which is the relationship between $\mathcal{F}\mathbf{f^c}$ (meant here as DFT!) and $\mathcal{F}f$ (meant as Fourier transform for functions on $\mathbb{R}^d$).

Lemma 0.2 *Let* $\omega = (\omega_1, ..., \omega_d) \in \mathbb{R}^d_+$. *If* $f \in L^1(\mathbb{R}^d)$ *then the series* $\sum_{k \in \mathbb{Z}^d} f(x - \omega k)$ *converges in* $L^1(\omega T^d)$ *to the periodic function* Pf *such that* $\|Pf\|_1 \leq \|f\|_1$. *Moreover for* $k \in \omega\mathbb{Z}^d$ *the Fourier transform for periodic functions on* ωT^d *given by* $\mathcal{F}(Pf)(k)$ *is equal to the Fourier transform for functions on* $\mathbb{R}^d$ *given by* $(\det(\omega))^{-1/2}\mathcal{F}f(k/\omega)$.

Proof of the Lemma

Let $Q = \prod_{i=1}^{d} \omega_i[-\frac{1}{2}, \frac{1}{2})$. We have $\int_Q \sum_{k\in\mathbb{Z}^d} |f(x-\omega k)|dx = \sum_{k\in\mathbb{Z}^d} \int_{Q+\omega k} |f(x)|dx = \int_{\mathbb{R}^d} |f(x)|dx < \infty$. By the Dominated Convergence Theorem we obtain the convergence of the series $\sum_{k\in\mathbb{Z}^d} f(x-\omega k)$ in L^1 to a function Pf such that $\|Pf\|_1 \le \|f\|_1$. By definition of Fourier transform of periodic functions defined on ωT^d we have

$$\mathcal{F}(Pf)(h) = \frac{1}{(\det(\omega))^{1/2}} \int_Q \left(\sum_{k\in\mathbb{Z}^d} f(x-\omega k) \right) e^{-2\pi i (h,x)/\omega} dx =$$

$$= \frac{1}{(\det(\omega))^{1/2}} \sum_{k\in\mathbb{Z}^d} \int_{Q+\omega k} f(x) e^{-2\pi i (h,x+\omega k)/\omega} dx =$$

$$= \frac{1}{(\det(\omega))^{1/2}} \int_{\mathbb{R}^d} f(x) e^{-2\pi i (h,x)/\omega} dx = (\det(\omega))^{-1/2} \mathcal{F}f(h/\omega).$$

Theorem 0.3 (Poisson summation formula) *Let $f \in C(\mathbb{R}^d)$, $|f(x)| \leq C(1+|x|)^{-d-\varepsilon}$ and $|\mathcal{F}f(\hat{x})| \leq C(1+|\hat{x}|)^{-d-\varepsilon}$ for some $C, \varepsilon > 0$. Let $\omega = (\omega_1, ..., \omega_d) \in (0,\infty)^d$. Then the following formula holds*

$$(\det(\omega))^{1/2} \sum_{k \in \mathbb{Z}^d} \mathcal{F}f(\hat{x} - \omega k) = (\det(\frac{1}{\omega}))^{1/2} \sum_{n \in \mathbb{Z}^d} f(n/\omega) e^{-2\pi i(\hat{x}, n/\omega)}.$$

where both sides converge uniformly in ωT^d. In particular for $\hat{x} = 0$ we obtain

$$(\det(\omega))^{1/2} \sum_{k \in \mathbb{Z}^d} \mathcal{F}f(\omega k) = (\det(\frac{1}{\omega}))^{1/2} \sum_{n \in \mathbb{Z}^d} f(n/\omega).$$

Proof of the Theorem

The absolute and uniform convergence of the series follows by the application of the criterion of the integral estimate for the series $\sum_{k\in\mathbb{Z}^d}(1+|k|)^{-n-\varepsilon}$. Then $Pf(x)=\sum_{k\in\mathbb{Z}^d} f(x-\omega k)$ is in $C(\omega T^d)$ and therefore in $L^2(\omega T^d)$. By the Theorem of Fourier and by the previous Lemma we have

$$Pf(x)=\frac{1}{(\det(\omega))^{1/2}}\sum_{h\in\mathbb{Z}^d}\mathcal{F}(Pf)(h)e^{2\pi i(h,x)/\omega}=$$

$$=\frac{1}{\det(\omega)}\sum_{h\in\mathbb{Z}^d}\mathcal{F}f(h/\omega)e^{2\pi i(h,x)/\omega}.$$

Exchanging the roles of f and $\mathcal{F}f$ and observing that $\mathcal{F}\mathcal{F}f=f(-x)$ we obtain the final result.

Corollary 0.4 (Shannon sampling theorem) *Let* $f \in C_c(\mathbb{R}^d)$*,* $|f(x)| \le C(1+|x|)^{-d-\varepsilon}$ *and* $|\mathcal{F}f(\hat{x})| \le C(1+|\hat{x}|)^{-d-\varepsilon}$ *for some* $C, \varepsilon > 0$*. If, for example,* $\operatorname{supp}(f) \subset \prod_{i=1}^{d}[0, a_i]$*, then it holds*

$$\mathcal{F}\mathbf{f^c}(l) = \frac{(\det(m))^{1/2}}{(\det(a))} \left[\sum_{k \in \mathbb{Z}^d} \mathcal{F}f(\hat{x} - \omega k) \right]^c (l/a), \tag{1}$$

where the sampling on the left is made with step $\tau = 1/\omega$ *and the one on the right with step* $1/a$*, so that* $\omega a = m \in \mathbb{N}^d$*.*

Proof of the Corollary

Let us assume $\mathrm{supp}(f) \subset \prod_{i=1}^{d}[0, a_i]$ and denote $a = (a_1, \ldots, a_d)$. We also assume that $a > 0$ is such that $\omega a = m \in \mathbb{N}^d$. But then

$$(\det(\frac{1}{w}))^{1/2} \sum_{n \in \mathbb{Z}^d} f(n/\omega) e^{-2\pi i(\hat{x}, n/\omega)} = \frac{(\det(a))^{1/2}}{(\det(m))^{1/2}} \sum_{n \in \mathbb{Z}_m^d} f(n/\omega) e^{-2\pi i(\hat{x}a, n)/m}.$$

If we assume now that $\hat{x}a = l \in \mathbb{Z}^d$ then by the Poisson summation formula, we get

$$\frac{(\det(m))^{1/2}}{(\det(a))^{1/2}} \sum_{k \in \mathbb{Z}^d} \mathcal{F}f(l/a - \omega k) = \frac{(\det(a))^{1/2}}{(\det(m))^{1/2}} \sum_{n \in \mathbb{Z}_m^d} f(n/\omega) e^{-2\pi i(l, n)/m}.$$

Set $\tau = 1/\omega$, we obtain by definition of the DFT

$$\mathcal{F}\mathbf{f^c}(l) = \frac{(\det(m))^{1/2}}{(\det(a))} \sum_{k\in\mathbb{Z}^d} \mathcal{F}f(l/a - \omega k) = \frac{(\det(m))^{1/2}}{(\det(a))} \left[\sum_{k\in\mathbb{Z}^d} \mathcal{F}f(\hat{x} - \omega k)\right]^c (l/a),$$

where the last sampling is made with step $1/a$.

Sampling theory

Lemma 0.5 *Define* $\operatorname{sinc}(x) \equiv \frac{sin(\pi x)}{\pi x}$*,* $x \in \mathbb{R}$ *and* $Q = \prod_{i=1}^{d} \omega_i[-1/2, 1/2)$*. Then it holds*

$$\frac{1}{\det(\omega)^{1/2}} \int_Q e^{2\pi i(k,\xi)/\omega} e^{2\pi i(\xi,x)} dx = \det(\omega)^{1/2} \prod_{i=1}^{d} \operatorname{sinc}(w_i x_i - k_i), \quad x \in \mathbb{R}^d.$$

Proof. *(Exercise!) Just use Fubini-Tonelli Theorem.*

Theorem 0.6 (of the perturbed sampling in L^2) *Let* $Q=\prod_{i=1}^{d}\omega_i[-1/2,1/2)$ *and* $f\in C(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ *such that* $f_{|\tau\mathbb{Z}^d}\in\ell^2$. *We write* $f=\eta+\epsilon$, *where* $\mathcal{F}\eta=\mathcal{F}f$ *on* Q. *Then it holds*

$$f(x)=\sum_{k\in\mathbb{Z}^n}(f^c(\tau k)-\epsilon^c(\tau k))\prod_{i=1}^{d}\operatorname{sinc}(\tau_i^{-1}x_i-k_i)+\epsilon(x),\quad in\quad L^2(\mathbb{R}^d). \tag{2}$$

PROOF OF THE THEOREM

Let us consider $P[\mathcal{F}\eta](\xi) = \sum_{k\in\mathbb{Z}^d} \mathcal{F}\eta(\xi - \omega k)$. We know already that $P[\mathcal{F}\eta] \in L^2(\omega T^d)$ and thus

$$P[\mathcal{F}\eta](\xi) = \frac{1}{\det(\omega)^{1/2}} \sum_{k\in\mathbb{Z}^d} \mathcal{F}(P[\mathcal{F}\eta])(k) e^{2\pi i(k,\xi)/\omega},$$

where

$$\mathcal{F}(P[\mathcal{F}\eta])(k) = \frac{1}{\det(\omega)^{1/2}} \int_{\omega T^d} P[\mathcal{F}\eta](x) e^{-2\pi i(k,x)/\omega} dx.$$

Clearly the restriction of this function to Q gives $P[\mathcal{F}\eta]_{|Q} = \mathcal{F}\eta$. But then we see immediately that $\mathcal{F}(P[\mathcal{F}\eta])(k) = \frac{1}{\det(\omega)^{1/2}} (f^c(-k/\omega) - \epsilon^c(-k/\omega))$, since $\mathcal{F}\eta$ is not vanishing only on a compact set and thus $\mathcal{F}\eta \in L^1(\mathbb{R}^d)$. But then, substituting $P[\mathcal{F}\eta]$ instead of $\mathcal{F}\eta$ on Q and extending it to 0 out of Q we obtain

$$f(x) = \frac{1}{\det(\omega)^{1/2}} \mathcal{F}^{-1} \left(\sum_{k \in \mathbb{Z}^d} \mathcal{F}(P[\mathcal{F}\eta])(k) e^{2\pi i (k,\xi)/\omega} \chi_Q(\xi) \right)(x) + \epsilon(x) =$$

$$= \sum_{k \in \mathbb{Z}^n} \mathcal{F}(P[\mathcal{F}\eta])(k) \mathcal{F}^{-1} \left(\frac{1}{\det(\omega)^{1/2}} e^{2\pi i (k,\xi)/\omega} \chi_Q(\xi) \right)(x) + \epsilon(x) =$$

$$= \sum_{k \in \mathbb{Z}^n} (f^c(\tau k) - \epsilon^c(\tau k)) \prod_{i=1}^{d} \operatorname{sinc}(\tau_i^{-1} x_i - k_i) + \epsilon(x),$$

where we used the previous Lemma in the last equality and all the equalities hold in L^2.

Definizione 0.7 *For* $Q = \prod_{i=1}^{d} \omega_i[-1/2, 1/2)$*, we define*

$$L^2_Q(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : \mathrm{supp}(\mathcal{F}f) \subset Q\}.$$

We say that $f \in L^2_Q(\mathbb{R}^d)$ *is a* ω*-bandlimited function.*

Corollary 0.8 (Whittacker-Shannon) *If $f \in L^2(\mathbb{R}^d)$ is a ω-bandlimited function, there exists a $\tau_0 > 0$ such that for all $0 < \tau \leq \tau_0$*

$$f(x) = \sum_{k \in \mathbb{Z}^d} f^c(\tau k) \prod_{i=1}^{d} \mathrm{sinc}(\tau_i^{-1} x_i - k_i). \tag{3}$$

Proposition 0.9 (Heisenberg uncertainty principle) *If $f \in C_c(\mathbb{R}^d)$ then its Fourier transform $\mathcal{F}f$ on $\mathbb{R}^d$ cannot be compactly supported.*

Proof. If $f \in C_c(\mathbb{R}^d)$ then $\mathcal{F}f$ is an analitic function (because it has infinitely many bounded derivatives). But a nonzero analitic function has only isolated zeros, hence it cannot be compactly supported (see **http://planetmath.org/zeroesofanalyticfunctionsareisolated** for a proof).