Mathematics of Digitalization: Case Study

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Second lecture

What is the meaning of the Fourier transform? The Fourier transform represents the frequency content of a function/signal. It gives us which are the important oscillating building blocks of a signal and their dinstinctive frequency of oscillation.

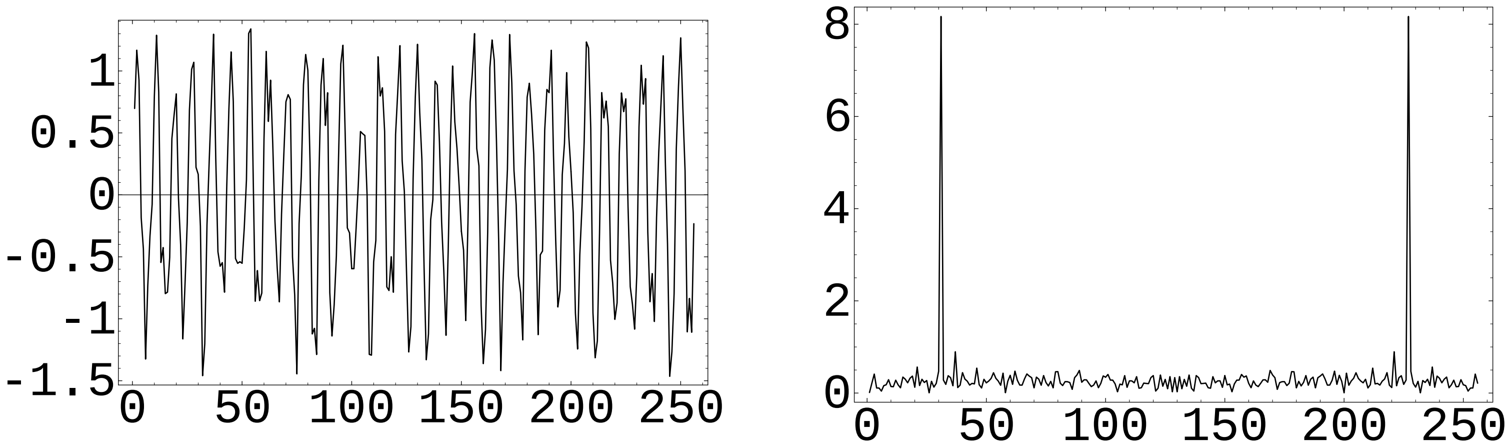

Figure 1: Sinosoidal signal affected by noise: there are two fundamental frequency components

The fortune of Fourier analysis stays essentially in the fact that it is able to describe one of the most frequent behavior in the nature: the wave phanomena, most of which are governed by ovelapping laws of the type

$$y_{\alpha,s_0}(t) = \begin{cases} e^{-\alpha t} e^{2\pi i s_0 t}, & t > 0 \\ 0, & t < 0. \end{cases}$$

$y_{\alpha,s_0}(t)$ is called *Lorenzian function*.

When, for example, some molecules are excited by electomagnetic radiation, damped oscillations are induced and described by the law $y_{\alpha,s_0}(t)$.

Every building block of the molecule has its own unique special oscillations. The Lorenzians are called in this case the *molecular spectrum*.

From here comes the idea which “costed” the Nobel prize to Richard Ernst for chemistry (1991) for the development of a powerful instrument to determine the structure of large complex organic molecules.

We want now to discuss how in practice we can use the tools of the Fourier analysis.

The execises we showed on simple functions show us that it is not so trivial to compute Fourier series or transforms.

For relatively simple functions such as $f(x) = \frac{1}{a^2+x^2}$ the computation of the Fourier transform is far from being trivial. In this case we have $\mathcal{F}f(w) = \frac{\pi}{a}e^{-|w|a}$.

A holomorphic complex function $f(z)$ (i.e., differentiable) on a domain Ω except for a point a, called singularity, can be expanded in a power series, called the Laurent series, $f(z) = \sum_{n\in\mathbb{Z}} c_n(z-a)^n$. The specific coefficient c_{-1} is called *residual* and we write $Res(f,a) = c_{-1}$. One can show that if f is holomorphic on a domain Ω which contains the halfplane $\{\mathrm{Im}(z) \geq 0\}$ except for a finite number of non real singularities and if the limit $\lim_{|z|\to\infty, \mathrm{Im}(z)\geq 0} f(z) = 0$ then we have for every $\alpha > 0$,

$$\lim_{r\to\infty} \int_{-r}^{r} f(x)e^{i\alpha x}dx = 2\pi i \sum \{Res(f(\cdot)e^{i\alpha \cdot}, a), \mathrm{Im}(a) > 0\}.$$

Analogous result holds for $\alpha < 0$ and for $\{\mathrm{Im}(z) < 0\}$, but changing the sign of the integral.

It is certainly more reasonable to search for an approximation on a discrete domain and a proper definition of discrete Fourier transform, which, in some sense, can approximate well the behavior of the continuous Fourier transform. Moreover, we need to make sure that the computation of the discrete Fourier transform (DFT) can be executed in reasonable time: otherwise it does not pay off.

Let's proceed as usual in the language of the Hilbert spaces. We consider now the cyclic group $\mathbb{Z}_n = \frac{\mathbb{Z}}{n\mathbb{Z}}$, $n \in \mathbb{N}\backslash\{0\}$ of the rest classes modulo n. In practice we can write

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\},$$

reminding us that such elements represent the rest of the integer number with respect to the division by n. For example, in $\mathbb{Z}_3$ the element $h = 4$ corresponds to 1 as $3 = 3 \cdot 1 + \mathbf{1}$. Hence, in particular, $2 + 2 = 1$ in $\mathbb{Z}_3$. Moreover, $\ell^2(\mathbb{Z}_n) = \{(c_0, \ldots, c_{n-1}) | \sum_{k=0}^{n-1} |c_i|^2 < \infty\} = \mathbb{C}^n$.

One can show by induction (exercise!)

$$1 + z + z^2 + \cdots + z^{n-1} = \begin{cases} n, & z = 1 \\ (z^n - 1)/(z - 1), & \text{otherwise.} \end{cases} \tag{1}$$

But then it is not difficult to show (exercise!) that for $z = e^{2\pi i(k-l)/n}$ we have

$$\langle \frac{1}{\sqrt{n}} (e^{2\pi i k l/n})_{l \in \mathbb{Z}_n}, \frac{1}{\sqrt{n}} (e^{2\pi i l m/n})_{m \in \mathbb{Z}_n} \rangle = \sum_{m=0}^{n-1} e^{2\pi i m(k-l)/n} = \delta_{k,l}.$$

Hence $\{\frac{1}{\sqrt{n}} (e^{2\pi i k l/n})_{l \in \mathbb{Z}_n}\}_{k \in \mathbb{Z}_n}$ is an orthonormal basis for the finite dimensional (Hilbert) space $\ell^2(\mathbb{Z}_n)$.

Every discrete signal $\mathbf{f}$ of length n can be written as:

$$\mathbf{f} = \frac{1}{n} \sum_{k=0}^{n-1} \langle \mathbf{f}, (e^{2\pi i k l / n})_{l \in \mathbb{Z}_n} \rangle (e^{2\pi i k l / n})_{l \in \mathbb{Z}_n}. \tag{2}$$

Moreover we have that

$$\mathcal{F}\mathbf{f}(k) = \frac{1}{\sqrt{n}} \langle \mathbf{f}, (e^{2\pi i k l / n})_{l \in \mathbb{Z}_n} \rangle = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \mathbf{f}(l) e^{-2\pi i k l / n}, \tag{3}$$

is again a signal of length n. The operator $\mathcal{F} : \ell^2(\mathbb{Z}_n) \to \ell^2(\mathbb{Z}_n)$ is called discrete Fourier transform or DFT and it realizes all the properties of the Fourier transforms already defined before for periodic functions and functions on the real line. For example, prove by exercise that

$$\|\mathbf{f}\|_{\ell^2} = \|\mathcal{F}\mathbf{f}\|_{\ell^2}.$$

If we assume that one operation is either a sum or a multiplication, then the computational cost of a DFT is of $2n$ sums and multiplications times n, i.e., $2n^2$. As one complex operation costs twice a single real, the total costs is $C(DFT)(n) = 4n^2$. If we had a PC able to compute 10^7 operations per second and we are able to compute a DFT of a signal of length $n = 1000$ in

$$4 \cdot 1000^2 \cdot \frac{1}{10^7} = 0.4\,\mathrm{sec}\,.$$

But already for a signal of length $n = 16384 = 2^{14}$ the cost is of $107\,\mathrm{sec}$. A simple audio signal of few seconds, sufficiently densely sampled, can easily reach length $n = 2^{14}$!.

Given a signal $\mathbf{f}$ of length n, we define the translation operator the one given by

$$T_m\mathbf{f}(k) = \mathbf{f}(k-m), \quad m \in \mathbb{Z}_n. \tag{4}$$

and the modulation operator by

$$M_m\mathbf{f}(k) = e^{2\pi imk/n}\mathbf{f}(k), \quad m \in \mathbb{Z}_n. \tag{5}$$

Moreover, we define the operator of *upsampling* and *doubling* as

$$U\mathbf{f}(h) = \begin{cases} \mathbf{f}(h/2), & \mathrm{mod}(h,2) = 0 \\ 0, & \text{otherwise,} \end{cases} \tag{6}$$

$$D\mathbf{f} = \frac{1}{2}(\mathbf{f}, \mathbf{f}), \tag{7}$$

for $h \in \mathbb{Z}_{2n}$.

The action of the DFT with respect to these operators is given by (exercise!)

$$\mathcal{F} \cdot T_m \mathbf{f}(k) = M_{-m} \cdot \mathcal{F}\mathbf{f}(k), \quad \mathcal{F} \cdot M_m \mathbf{f}(k) = T_m \cdot \mathcal{F}\mathbf{f}(k). \tag{8}$$

$$\mathcal{F} \cdot U\mathbf{f}(h) = D \cdot \mathcal{F}\mathbf{f}(h). \tag{9}$$

Let's now consider a vector of length $n = 2^2 = 4$ given by $\mathbf{f} = (f_0, f_1, f_2, f_3)$. Let us show how is it possible to compute $\mathbf{f}$ from the single f_i by simple operations:

$$
\begin{array}{cccc}
f_0 & f_2 & f_1 & f_3 \\
\downarrow U & \downarrow U & \downarrow U & \downarrow U \\
(f_0, 0) & (f_2, 0) & (f_1, 0) & (f_3, 0) \\
\downarrow I & \downarrow T_{-1} & \downarrow I & \downarrow T_{-1} \\
(f_0, 0) & (0, f_2) & (f_1, 0) & (0, f_3) \\
\searrow & \downarrow & \downarrow & \swarrow \\
 & (f_0, f_2) & (f_1, f_3) & \\
 & \downarrow U & \downarrow U & \\
 & (f_0, 0, f_2, 0) & (f_1, 0, f_3, 0) & \\
 & \downarrow I & \downarrow T_{-1} & \\
 & (f_0, 0, f_2, 0) & (0, f_1, 0, f_3) & \\
 & \searrow & \swarrow & \\
 & \multicolumn{2}{c}{\mathbf{f}} &
\end{array}
$$

Observing now that $\mathcal{F} f_i = f_i$ for all $i = 0, ... f_{n-1}$, by applying the DFT on the previous diagram and substituting to U, T_{-1} resp. D, M_1 as given in the commutation rules, we generate another diagram, actually a recursive algorithm to compute the DFT $\mathcal{F}\mathbf{f}$.

$$
\begin{array}{cccc}
f_0 & f_2 & f_1 & f_3 \\
\downarrow D & \downarrow D & \downarrow D & \downarrow D \\
(*,*) & (*,*) & (*,*) & (*,*) \\
\downarrow I & \downarrow M_1 & \downarrow I & \downarrow M_1 \\
(*,*) & (*,*) & (*,*) & (*,*) \\
\searrow & \downarrow & \downarrow & \swarrow \\
 & (*,*) & (*,*) & \\
 & \downarrow D & \downarrow D & \\
 & (*,*,*,*) & (*,*,*,*) & \\
 & \downarrow I & \downarrow M_1 & \\
 & (*,*,*,*) & (*,*,*,*) & \\
 & \searrow & \swarrow & \\
\end{array}
$$

$$\mathcal{F}\mathbf{f}$$

Let us assume that the computation of I and D are negligible (as it is in practice!). We need to compute only the cost of a single M_1 which, applied to a vector of length l costs $l-1$. If we assume now that $n = 2^m$, starting from the bottom of the diagram we do only one M_1 on the length of the original vector and then a cost given by $2^0(\frac{n}{2^0} - 1)$. This cost has to summed to the one of the level above, where one has to execute $2(\frac{n}{2} - 1)$ operations corresponding to twice M_1 computed on vector of half the original length. And so on, obtaining that the total cost is given by

$$C(FFT)(n) = \sum_{k=0}^{m-1} 2^k(\frac{n}{2^k} - 1) =$$

$$= \sum_{k=0}^{m-1} \left(2^m - 2^k\right) = m2^m - 2^m + 1 = n\log_2(n) - n + 1. \quad (10)$$

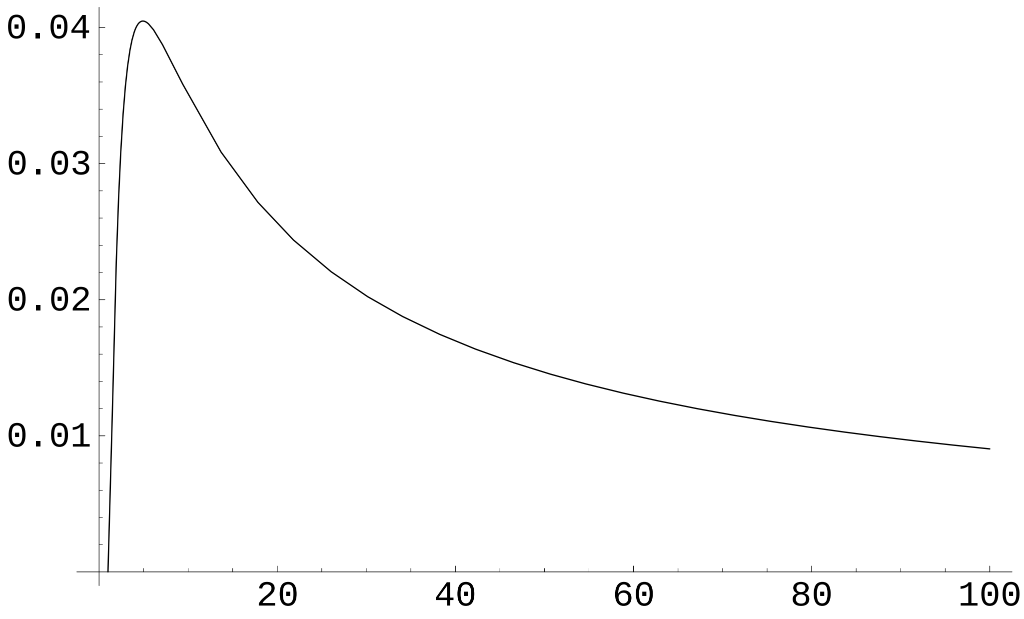

Figure 2: Ratio $\frac{C(FFT)(n)}{C(DFT)(n)}$ for growing n

Hence a PC is able to produce the FFT of a signal of length $n = 16384$ in

$$(2^{14}14 - 2^{14} + 1) \cdot \frac{1}{10^7} = 0.021\,\mathrm{sec},$$

against the $107\,\mathrm{sec}$ which one may need by means of the direct DFT!

Relationship between discrete and continuous transforms

In the following the dimension is fixed $d \geq 1$. For $\tau = (\tau_1, ..., \tau_d) \in \mathbb{R}^d_+$ we define $\det(\tau) = \tau_1 \cdot \ldots \tau_d$ and $\omega = \frac{1}{\tau}$. We set $T^d = (0,1)^d$. We denote (v, w) the scalar product on $\mathbb{R}^d$. If f is a periodic function over $\tau T^d = \prod_{i=1}^d (0, \tau_i)$ we define its Fourier transform as follows:

$$\mathcal{F}f(h) = \frac{1}{(\det(\tau))^{1/2}} \int_{\tau T^d} f(x) e^{-2\pi i (h,x)/\tau} dx, \quad h \in \mathbb{Z}^d.$$

and for a function f defined on the Euclidean space $\mathbb{R}^d$ we define the Fourier transform

$$\mathcal{F}f(\hat{x}) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i (x,\hat{x})} dx, \quad \hat{x} \in \mathbb{R}^d.$$

If $\mathbf{f}$ is a function in $\ell_2(\mathbb{Z}^d_m)$, for $m \in \mathbb{Z}^d_+$, then we denote

$$\mathcal{F}\mathbf{f}(k) = \frac{1}{\det(m)^{1/2}} \sum_{h \in \mathbb{Z}^d_m} f(h) e^{-2\pi i (h,k)/m}, \quad k \in \mathbb{Z}^d_m.$$

We use the same symbol $\mathcal{F}$ differently depending on the context.

Sampling and DFT

Definizione 0.1 *Let $f \in C(\mathbb{R}^d)$ and $\tau = (\tau_1, ..., \tau_d) \in (0, \infty)^d$. We call the sampling of step τ the operator which associates to f the function $f^c \equiv f_{|\tau\mathbb{Z}^d}$. If $f \in C_c(\mathbb{R}^d)$, clearly f^c is vanishing out of a compact set of $\mathbb{Z}^d$. Moreover, up to a translation, we can always assume that f^c is defined on a set $\tau\mathbb{Z}^d/\tau m\mathbb{Z}^d$ isomorphic to $\mathbb{Z}^d_m$. Hence, the sampling induces an operator acting from $C_c(\mathbb{R}^d)$ to $\ell^2(\mathbb{Z}^d_m)$, for some $m = (m_1, ..., m_d) \in \mathbb{N}^d \backslash \{0\}$.*

We want to study some of the properties of the operator $\cdot^c$. In particular we wonder which is relationship between $\mathcal{F}\mathbf{f^c}$ (meant here as DFT!) and $\mathcal{F}f$ (meant as Fourier transform for functions on $\mathbb{R}^d$).

Lemma 0.2 *Let* $\omega = (\omega_1, ..., \omega_d) \in \mathbb{R}^d_+$*. If* $f \in L^1(\mathbb{R}^d)$ *then the series* $\sum_{k\in\mathbb{Z}^d} f(x - \omega k)$ *converges in* $L^1(\omega T^d)$ *to the periodic function* Pf *such that* $\|Pf\|_1 \leq \|f\|_1$*. Moreover for* $k \in \omega\mathbb{Z}^d$ *the Fourier transform for periodic functions on* $`(\omega T^d$ *the function* $\mathcal{F}(Pf)(k)$ *is equal to the Fourier transform for functions on* $\mathbb{R}^d$ *given by* $(\det(\omega))^{-1/2}\mathcal{F}f(k/\omega)$*.*

Proof of the Lemma

Let $Q = \prod_{i=1}^{d} \omega_i[-\frac{1}{2}, \frac{1}{2})$. We have $\int_Q \sum_{k\in\mathbb{Z}^d} |f(x-\omega k)|dx = \sum_{k\in\mathbb{Z}^d} \int_{Q+\omega k} |f(x)|dx = \int_{\mathbb{R}^d} |f(x)|dx < \infty$. By the Dominated Convergence Theorem we obtain the convergence of the series $\sum_{k\in\mathbb{Z}^d} f(x-\omega k)$ in L^1 to a function Pf such that $\|Pf\|_1 \leq \|f\|_1$. By definition of Fourier transform of periodic functions defined on ωT^d we have

$$\mathcal{F}(Pf)(h) = \frac{1}{(\det(\omega))^{1/2}} \int_Q \left(\sum_{k\in\mathbb{Z}^d} f(x-\omega k) \right) e^{-2\pi i(h,x)/\omega} dx =$$

$$= \frac{1}{(\det(\omega))^{1/2}} \sum_{k\in\mathbb{Z}^d} \int_{Q+\omega k} f(x) e^{-2\pi i(h,x+\omega k)/\omega} dx =$$

$$= \frac{1}{(\det(\omega))^{1/2}} \int_{\mathbb{R}^d} f(x) e^{-2\pi i(h,x)/\omega} dx = (\det(\omega))^{-1/2} \mathcal{F}f(h/\omega).$$

Theorem 0.3 (Formula della somma di Poisson) *Let $f \in C(\mathbb{R}^d)$, $|f(x)| \leq C(1+|x|)^{-d-\varepsilon}$ and $|\mathcal{F}f(\hat{x})| \leq C(1+|\hat{x}|)^{-d-\varepsilon}$ for some $C, \varepsilon > 0$. Let $\omega = (\omega_1, ..., \omega_d) \in (0,\infty)^d$. Then the following formula holds*

$$(\det(\omega))^{1/2} \sum_{k\in\mathbb{Z}^d} \mathcal{F}f(\hat{x} - \omega k) = (\det(\frac{1}{\omega}))^{1/2} \sum_{n\in\mathbb{Z}^d} f(n/\omega) e^{-2\pi i(\hat{x}, n/\omega)}.$$

where both sides converge uniformly in ωT^d. In particular for $\hat{x} = 0$ we obtain

$$(\det(\omega))^{1/2} \sum_{k\in\mathbb{Z}^d} \mathcal{F}f(\omega k) = (\det(\frac{1}{\omega}))^{1/2} \sum_{n\in\mathbb{Z}^d} f(n/\omega).$$

Proof of the Theorem

The absolute and uniform convergence of the series follows by the application of the criterion of the integral estimate for the series $\sum_{k\in\mathbb{Z}^d}(1+|k|)^{-n-\varepsilon}$. Then $Pf(x)=\sum_{k\in\mathbb{Z}^d} f(x-\omega k)$ is in $C(\omega T^d)$ and therefore in $L^2(\omega T^d)$. By the Theorem of Fourier and by the previous Lemma we have

$$Pf(x) = \frac{1}{(\det(\omega))^{1/2}} \sum_{h\in\mathbb{Z}^d} \mathcal{F}(Pf)(h) e^{2\pi i(h,x)/\omega} =$$

$$= \frac{1}{\det(\omega)} \sum_{h\in\mathbb{Z}^d} \mathcal{F}f(h/\omega) e^{2\pi i(h,x)/\omega}.$$

Exchanging the roles of f and $\mathcal{F}f$ and observing that $\mathcal{F}\mathcal{F}f = f(-x)$ we obtain the final result.

Corollary 0.4 (Shannon sampling theorem) *Let* $f \in C_c(\mathbb{R}^d)$, $|f(x)| \leq C(1+|x|)^{-d-\varepsilon}$ *and* $|\mathcal{F}f(\hat{x})| \leq C(1+|\hat{x}|)^{-d-\varepsilon}$ *for some* $C, \varepsilon > 0$. *If, for example,* $\mathrm{supp}(f) \subset \prod_{i=1}^{d}[0, a_i]$, *then it holds*

$$\mathcal{F}\mathbf{f^c}(l) = \frac{(\det(m))^{1/2}}{(\det(a))} \left[\sum_{k \in \mathbb{Z}^d} \mathcal{F}f(\hat{x} - \omega k) \right]^c (l/a), \tag{11}$$

where the sampling on the left is made with step $\tau = 1/\omega$ *and the one on the right with step* $1/a$, *so that* $\omega a = m \in \mathbb{N}^d$.

Proof of the Corollary

Let us assument $\mathrm{supp}(f) \subset \prod_{i=1}^{d}[0, a_i]$ and denote $a = (a_1, \ldots, a_d)$. We also assume that $a > 0$ is such that $\omega a = m \in \mathbb{N}^d$. But then

$$(\det(\frac{1}{w}))^{1/2} \sum_{n \in \mathbb{Z}^d} f(n/\omega) e^{-2\pi i (\hat{x}, n/\omega)} = \frac{(\det(a))^{1/2}}{(\det(m))^{1/2}} \sum_{n \in \mathbb{Z}_m^d} f(n/\omega) e^{-2\pi i (\hat{x} a, n)/m}.$$

If we assume now that $\hat{x} a = l \in \mathbb{Z}^d$ then by the Poisson summation formula, we get

$$\frac{(\det(m))^{1/2}}{(\det(a))^{1/2}} \sum_{k \in \mathbb{Z}^d} \mathcal{F} f(l/a - \omega k) = \frac{(\det(a))^{1/2}}{(\det(m))^{1/2}} \sum_{n \in \mathbb{Z}_m^d} f(n/\omega) e^{-2\pi i (l, n)/m}.$$

Set $\tau = 1/\omega$, we obtain by definition of the DFT

$$\mathcal{F}\mathbf{f}^{\mathbf{c}}(l) = \frac{(\det(m))^{1/2}}{(\det(a))} \sum_{k \in \mathbb{Z}^d} \mathcal{F}f(l/a - \omega k) = \frac{(\det(m))^{1/2}}{(\det(a))} \left[\sum_{k \in \mathbb{Z}^d} \mathcal{F}f(\hat{x} - \omega k) \right]^c (l/a),$$

where the last sampling is made with step $1/a$.