Anisotropic mesh adaptation for the Ambrosio-Tortorelli model: application to quasi-static crack propagation

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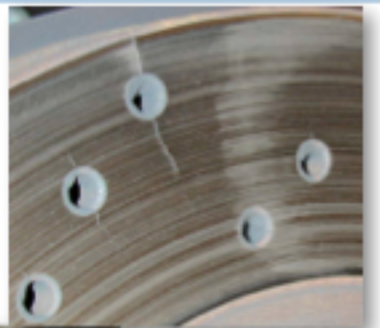
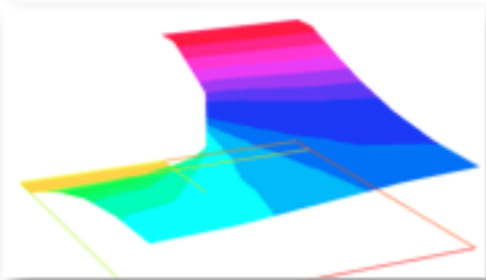
Francfort and Marigo formulated a model which requires the minimization of a functional based on the Griffith principle of energy balance between elastic energy and a fictitious crack energy:

$$\hat{u}(t) \in \underset{\substack{u \in SBV(\Omega),\\ u|_{\Omega_D} = g(t)|_{\Omega_D}}}{\arg\min} \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx + \kappa \mathcal{H}^{N-1}(\Gamma)$$

where Ω_D is the part of the domain Ω to which we apply a force and κ is the elasticity constant of the material.





Problems

- Nonconvex and nonsmooth minimization involving unknown functions and sets
- Any discretization is an additional bias towards a proper fracture propagation

Approximation with a smoother functional by Ambrosio and Tortorelli

$$J_\varepsilon(u,v) = \int_\Omega (v^2+\eta)|\nabla u|^2 dx + \kappa \int_\Omega \left[\frac{1}{4\varepsilon}(1-v)^2 + \varepsilon|\nabla v|^2\right] dx$$

Introducing a time discretization $0 = t_0 < t_1 < \dots < t_N = T$, the minimization process is:

$$(u_\varepsilon(t_k), v_\varepsilon(t_k)) \in \underset{\substack{u \in SBV(\Omega),\ u|_{\Omega_D} = g(t_k)|_{\Omega_D},\\ v \in H^1(\Omega;[0,1]),\ v \le v_\varepsilon(t_{k-1})}}{\arg\min} J_{\varepsilon,k}(u,v)$$

The constraint $v \le v_\varepsilon(t_{k-1})$ avoid the welding of the crack in the future time steps.

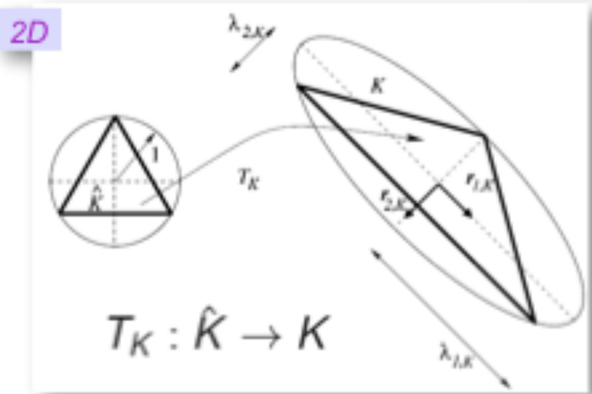
How to satisfy the constraints?

We enforce them by introducing penalization terms:

$$J^m_{\varepsilon,k}(u,v) = \int_\Omega (v^2+\eta)|\nabla u|^2 dx + \kappa \int_\Omega \left[\frac{1}{4\varepsilon}(1-v)^2 + \varepsilon|\nabla v|^2\right] dx + \frac{1}{\gamma_A}\int_{\Omega_D} (g(t_k)-u)^2 dx + \frac{1}{\gamma_B}\int_{CR_{k-1}} v^2 dx.$$

Forcing term applied on the set Ω_D | Constraint over v to avoid the crack welding

Numerical approximation exploits the properties of the anisotropic mesh



$x = T_K(\hat{x}) = M_K \hat{x} + t_K$
$M_K = B_K Z_K$
Z_K is orthogonal
$B_K = R_K^t \Lambda_K R_K$ is s.p.d.
$R_K = [r_{1,K}\ \ r_{2,K}]$ is orthonormal
$\Lambda_K = \begin{bmatrix} \lambda_{1,K} & 0 \\ 0 & \lambda_{2,K} \end{bmatrix}$

Approximation of functions u and v by piecewise linear functions. Residual estimate

Proposition

Let $u_h, v_h \in X_h$ be such that $J'^m_{h,k}(u_h, v_h, \varphi_h, \psi_h) = 0$ for all $\varphi_h, \psi_h \in X_h$. Then

$$|J'^m_{\varepsilon,k}(u_h, v_h; \varphi, \psi)| \lesssim \rho_h(\varphi) + \sigma_h(\psi) \qquad \forall \varphi, \psi \in H^1(\Omega)$$

where $\rho_h, \sigma_h : H^1(\Omega) \to \mathbb{R}$ are:

$$\rho_h(\varphi) = \sum_{K \in \mathcal{T}_h} \rho_K(u_h, v_h) \left[\sum_{i=1}^2 \lambda_{i,K}^2 (r_{i,K}^T G_{\Delta_K}(\varphi) r_{i,K})\right]^{\frac{1}{2}}$$

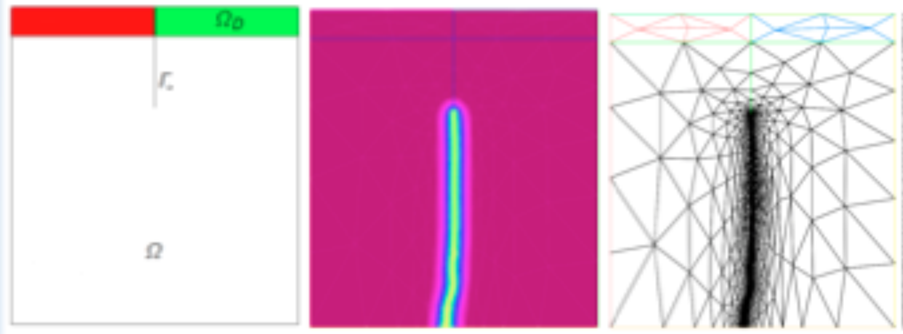
$$\sigma_h(\psi) = \sum_{K \in \mathcal{T}_h} \sigma_K(u_h, v_h) \left[\sum_{i=1}^2 \lambda_{i,K}^2 (r_{i,K}^T G_{\Delta_K}(\psi) r_{i,K})\right]^{\frac{1}{2}}$$

$$G_{\Delta_K}(\chi) := \sum_{K \in \Delta_K} \begin{bmatrix} \int_K \left(\frac{\partial \chi}{\partial x_1}\right)^2 dx & \int_K \frac{\partial \chi}{\partial x_1}\frac{\partial \chi}{\partial x_2} dx \\ \int_K \frac{\partial \chi}{\partial x_1}\frac{\partial \chi}{\partial x_2} dx & \int_K \left(\frac{\partial \chi}{\partial x_2}\right)^2 dx \end{bmatrix}$$

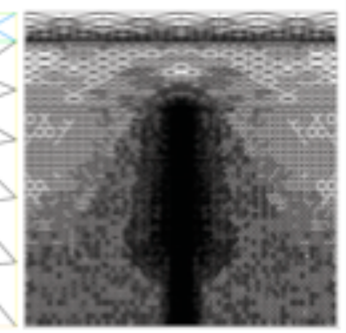
NUMERICAL RESULTS

Adaptive evolution minimization algorithm

Set $v^1 = 1$ if $k = 0$, $v^1 = v(t_{k-1})$ else, and initial mesh $\mathcal{T}^1 = \mathcal{T}_{k-1}$
while $|\#\mathcal{T}^n - \#\mathcal{T}^{n-1}| >$ ADAPTOL **do**
  while $\|v^n_{i-1} - v^n_i\|_{L^\infty(\Omega)} \geq$ VTOL **do**
    $u^n_i = \arg\min_{z \in X_h} J^m_{h,k}(z, v^n_i)$;
    $v^n_{i+1} = \arg\min_{z \in X_h} J^m_{h,k}(u^n_i, z)$;
  end while
  Generate the new metric $\mathcal{M}_{n+1}$ using $(u^n_i, v^n_{i+1}, \mathrm{REFTOL})$;
  Create the new mesh $\mathcal{T}^{n+1}$;
  Set $u^n = u^n_i$ and $v^n = v^n_{i+1}$;
end while
$u_h(t_k) = u^n$, $v_h(t_k) = v^n$, and $\mathcal{T}_k = \mathcal{T}^{n+1}$.



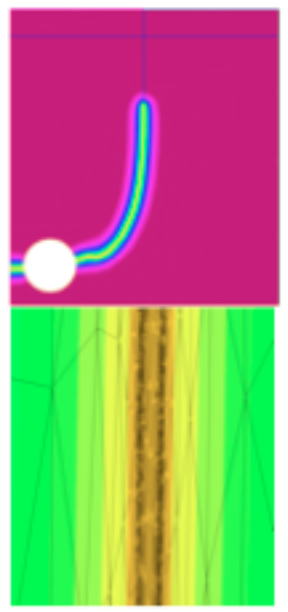
The domain, the crack path, and the mesh obtained with the anisotropic adaptive evolution minimization algorithm



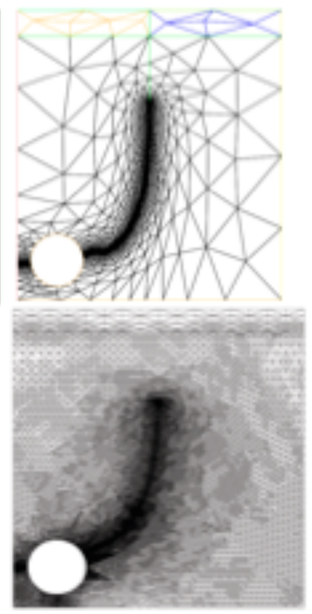
Adaptive grid used by E. Süli and al.[1]

Test 1. The crack behaves like expected and goes straight down. We used a mesh with very few elements
N_{el}=21304

Test 2. We add a hole in the domain. We expect that the fracture "feels" that the hole is a weak part of the domain and goes into it.



Crack path, mesh, and a particular of Test 2



Adaptive grid used by E. Süli and al.[1]

The mesh follows very closely the crack path, without influencing the evolution of the crack itself.
The number of triangles of the mesh is
N_{el} = 48558

[1]Burke, Siobhan; Ortner, Christoph; Süli, Endre. *An adaptive finite element approximation of a variational model of brittle fracture.* SIAM J. Numer. Anal. 48 (2010), no. 3, 980-1012.