Chapter 2
Linear Ill-Posed Problems

Ill-Posed Problems in Image and Signal Processing
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What we have seen so far...

- **Differentiation**: Finding \( u(x) \) for given
  \[
  \int_0^x u(y) \, dy
  \]
is **ill-posed**.

- **Inverse heat equation**: Finding \( u(x,0) \) for given
  \[
  u(x,T) = \int_0^{\pi} k(x,y,T)f(y) \, dy,
  \]
  \[
  k(x,y,T) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 T} \sin(nx) \sin(ny).
  \]
is **ill-posed**.

- **Deconvolution**: Finding \( u(x) \) for given
  \[
  \int_{\Omega} k(x-y)u(y) \, dy
  \]
  with smoothing kernel \( k \) is **ill-posed**.

**Question**

Is the inversion of integral operators ill-posed in general?
Why are problems ill-posed?

We want to study and understand our introductory examples in more detail.
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**Observation:** Our introductory problems can be written as

\[ f = Au \]

for linear operators \( A : X \rightarrow Y \) between Hilbert spaces \( X, Y \).
Why are problems ill-posed?

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**Observation:** Our introductory problems can be written as

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**Strategy:** Understand finite dimensional case first!
Finite dimensional linear operators

Linear operators between two finite dimensional Hilbert spaces: Matrices.
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Making our life easier: Consider a linear operator from a finite dimensional Hilbert space into itself: \( A \in \mathbb{R}^{n \times n} \).
Finite dimensional linear operators

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Corresponding finite dimensional linear inverse problem: Find $u$ from given

$$f = Au$$
Finite dimensional linear operators

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Corresponding finite dimensional linear inverse problem: Find $u$ from given

$$f = Au$$

Making our life even easier: $A$ is symmetric and positive definite.
Finite dimensional linear operators

Symmetric positive definite $A \in \mathbb{R}^{n \times n}$:

$$A = VSV^T,$$

with

- diagonal matrix $S$, $S_{i,i} = \lambda_i$ eigenvalues,
- $\lambda_1 \geq ... \geq \lambda_n > 0$,
- $V$ orthonormal matrix of eigenvectors.

Assume scaling: $\lambda_1 = 1$. Condition number $\kappa = \frac{1}{\lambda_n}$. 
Finite dimensional linear operators

Assume \( f = Au, \ f^\delta = Au^\delta \), with \( \|f^\delta - f\| \leq \delta \):
Finite dimensional linear operators

Assume \( f = Au, \ f^\delta = Au^\delta \), with \( \| f^\delta - f \| \leq \delta \):

\[
\begin{align*}
u^\delta - u &= VS^{-1} V^T (f^\delta - f) \\
\Rightarrow \| u^\delta - u \| &= \| VS^{-1} V^T (f^\delta - f) \| \\
&= \| S^{-1} V^T (f^\delta - f) \| \\
&\leq \frac{1}{\lambda_n} \| V^T (f^\delta - f) \| \\
&= \frac{\delta}{\lambda_n} = \kappa \delta
\end{align*}
\]
Finite dimensional linear operators

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\end{align*}
\]

→ Noise amplification: reciprocal of smallest eigenvalue!

→ Continuous dependence on the data!

→ Well-posed, but for small \( \lambda_n \) ill-conditioned!

→ In infinite dimensions: infinitely many \( \lambda_n \to 0 \)!
Question

What can we do against the instability?

Idea:

• Approximate $A$ by $A_\alpha = A + \alpha I$ with $\alpha > 0$.

• The smallest eigenvalue is $\lambda_n + \alpha > \alpha$.

• Approximate the solution to $Au = f$ for given noisy data $f_\delta$ by $u_\alpha = (A - I)^{-1} f_\delta$.
Finite dimensional linear operators

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- The smallest eigenvalue is $\lambda_n + \alpha > \alpha$.
- Approximate the solution to $Au = f$ for given noisy data $f^\delta$ by $u_\alpha = A_\alpha^{-1} f^\delta$.

**Computation on the board.**
Linear inverse problems in infinite dimensions.
Some basics...

Definition: Banach space
A normed vector space $X$ which is complete is called a Banach space. Being complete means that every Cauchy sequence converges in $X$.

Definition: Hilbert space
A vector space $X$ equipped with a scalar product $\langle \cdot, \cdot \rangle$ which is complete with respect to the induced norm $\|x\| = \sqrt{\langle x, x \rangle}$ is called a Hilbert space.

Convention
Unless stated otherwise, $X$ and $Y$ are real Hilbert spaces.
Some basics...

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Proposition: Closed subspaces

A nonempty subspace $M \subset X$ is closed if and only if $(x_n) \subset M$, \(\lim_{n \to \infty} x_n = x\) implies $x \in M$. 
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Definition: Closure

The closure $\overline{M}$ of $M \subset X$ is defined as

$$\overline{M} = M \cup \left\{ x \mid \exists (x_n) \in M \text{ with } \lim_{n \to \infty} x_n = x \right\}$$
Definition: Orthogonal complement

The *orthogonal complement* of the set $M \subset X$ is

$$M^\perp = \{ x \in X \mid \langle x, m \rangle = 0 \ \forall \ m \in M \}.$$
Banach and Hilbert spaces

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**Theorem: Direct sum**

Let $M \subset X$ be any closed subspace of $X$. Then

$$X = M + M^\perp.$$
Linear operators

**Definition: Linear operators**

A mapping $A : \mathcal{D}(A) \subset X \rightarrow Y$ is called a *linear operator*, if the domain $\mathcal{D}(A)$ is a subspace of $X$ and for all $x_1, x_2 \in \mathcal{D}(A)$, and all $\alpha \in \mathbb{R}$

\[
A(x_1 + x_2) = Ax_1 + Ax_2 \\
A(\alpha x_1) = \alpha Ax_1
\]
Definition: Boundedness

We say that a linear operator $A : \mathcal{D}(A) \subset X \to Y$ is bounded if there exists a $c \in \mathbb{R}$ such that for all $x \in \mathcal{D}(A)$

$$\|Ax\|_Y \leq c\|x\|_X.$$
## Linear operators

### Definition: Boundedness

We say that a linear operator $A : D(A) \subset X \rightarrow Y$ is bounded if there exists a $c \in \mathbb{R}$ such that for all $x \in D(A)$

$$\|Ax\|_Y \leq c\|x\|_X.$$ 

### Examples

- If $A : X \rightarrow Y$ is a linear operator with $\dim(X) < \infty$, then $A$ is bounded.
- First lecture: The derivative operator $\partial_x : C^1([0, 1]) \subset L^2([0, 1]) \rightarrow L^2([0, 1])$ is unbounded. (Example with $\sin(2\pi kx)$)
Theorem: Boundedness and Continuity

Let $A : \mathcal{D}(A) \subset X \to Y$ be a linear operator. Then the following three statements are equivalent:

- $A$ is continuous.
- $A$ is bounded.
- $A$ is continuous at $x = 0$.

Proof: Exercises
**Theorem: Boundedness and Continuity**

Let \( A : \mathcal{D}(A) \subset X \to Y \) be a linear operator. Then the following three statements are equivalent:

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**Proof:** Exercises

**Notation**

- \( \mathcal{L}(X, Y) \) set of all continuous linear operators from \( X \) to \( Y \).
- \( \mathcal{N}(A) := \{ x \in X \mid Ax = 0 \} \) *nullspace* of \( A \)
- \( \mathcal{R}(A) := \{ y \in Y \mid \exists x \in X \text{ with } Ax = y \} \) *range* of \( A \)
Definition: Open

An operator $A : X \to Y$ is called open, if for every open set $M \subset X$ in $X$ the set $A(M) \subset Y$ is open in $Y$.

\footnote{See D. Werner, Funktionalanalysis. Springer 2005.}
**Definition: Open**

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**Theorem: Open mapping theorem**

If $A \in \mathcal{L}(X, Y)$ is surjective, then $A$ is open. \(^1\)

---

\(^1\)See D. Werner, Funktionalanalysis. Springer 2005.
Linear operator equations

The previous analysis was done for symmetric positive definite matrices $A \in \mathbb{R}^{n \times n}$. What can we do for a general $A \in \mathcal{L}(X,Y)$?

What could happen?

• If $A$ is not surjective, $Au = f$ might not have a solution.

Definition

We call $u$ a least-squares solution of $Au = f$ if $\|Au - f\| = \inf\{\|Av - f\| | v \in X\}$.

• If $A$ is not injective, the least-squares solution might not be unique.

Definition

We call $u$ a minimal-norm solution of $Au = f$ if $\|u\| = \inf\{\|v\| | v \text{ is least-squares solution of } Au = f\}$. 
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Linear operator equations

**Question**

Can we define a linear operator that computes minimal norm solutions?
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Definition (Moore-Penrose inverse)

Let $A \in \mathcal{L}(X, Y)$ and let $\tilde{A} : \mathcal{N}(A) \to \mathcal{R}(A)$ denote its restriction. Then the Moore-Penrose generalized inverse $A^\dagger$ is defined as the unique linear extension of $\tilde{A}^{-1}$ to

$$\mathcal{D}(A^\dagger) := \mathcal{R}(A) \oplus \mathcal{R}(A) \perp$$

with $\mathcal{N}(A^\dagger) = \mathcal{R}(A) \perp$. 
The Moore-Penrose inverse is well-defined.
The Moore-Penrose inverse is well-defined.

**Theorem: Moore-Penrose equations**

The Moore-Penrose generalized inverse $A^\dagger$ meets the following four Moore-Penrose equations

1. $AA^\dagger A = A$
2. $A^\dagger AA^\dagger = A^\dagger$
3. $A^\dagger A = I - P$
4. $AA^\dagger = Q_{|\mathcal{D}(A^\dagger)}$

where $P : X \to \mathcal{N}(A)$ and $Q : Y \to \overline{\mathcal{R}(A)}$ are the orthogonal projectors onto the nullspace of $A$, $\mathcal{N}(A)$, and onto the closure of the range of $A$, $\overline{\mathcal{R}(A)}$, respectively.
Linear operator equations

**Theorem: Minimal-norm solutions**

For a given $f \in \mathcal{D}(A^\dagger)$, the equation $Ax = f$ has a unique minimal-norm solution given by

$$x^\dagger := A^\dagger f.$$  

The set of all least-squares solutions is given by $\{x^\dagger\} + \mathcal{N}(A)$.

**Proof: Board.**
Gaussian normal equation

A word about adjoint operators...
Gaussian normal equation

A word about adjoint operators...

Theorem: Gaussian normal equation

For a given $f \in \mathcal{D}(A^\dagger)$, $x \in X$ is a least-squares solution of $Ax = f$ if and only if $x$ satisfies the Gaussian normal equation

$$A^* Ax = A^* f.$$

Proof: Board.
Gaussian normal equations

Observations:

- $x^\dagger = A^\dagger y$ is the minimal-norm solution, i.e. the least-squares solution with minimal norm.
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$$A^* Ax = A^* y \quad (1)$$
Gaussian normal equations

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- $\mathbf{x}^\dagger = \mathbf{A}^\dagger \mathbf{y}$ is the minimal-norm solution, i.e. the least-squares solution with minimal norm.
- All least-squares solutions meet
  \[
  \mathbf{A}^* \mathbf{A} \mathbf{x} = \mathbf{A}^* \mathbf{y}
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  (1)
- $\mathbf{A}^\dagger \mathbf{y} = (\mathbf{A}^* \mathbf{A})^\dagger \mathbf{A}^* \mathbf{y}$.
Gaussian normal equations

Observations:

- $x^\dagger = A^\dagger y$ is the minimal-norm solution, i.e. the least-squares solution with minimal norm.
- All least-squares solutions meet
  \[ A^* A x = A^* y \] (1)
- \[ A^\dagger y = (A^* A)^\dagger A^* y. \]
- Possible to approximate $A^* A$ instead of $A$.

(cf. Landweber iteration!)
Linear operator equations

For any linear operator equation $f = Au, \ A \in \mathcal{L}(X, Y)$, we now have a (possibly naive) way of finding a solution via $u = A^\dagger f$.

When is this approach naive?
Linear operator equations

For any linear operator equation $f = Au$, $A \in \mathcal{L}(X, Y)$, we now have a (possibly naive) way of finding a solution via $u = A^\dagger f$.

When is this approach naive?

**Proposition: Discontinuity of $A^\dagger$**

$A^\dagger$ is continuous if and only if $\mathcal{R}(A)$ is closed.

**Proof: Board.**
Compact linear operators

Definition: Compact linear operator

$A \in \mathcal{L}(X, Y)$ is said to be compact if for every bounded sequence $\{x_n\} \subset X$, $\{Ax_n\}$ has a convergent subsequence.
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Remark: Be careful with the dimensions of your space!

Example: Identity operator on X for finite and infinite dimensional X.
Theorem: Ill-posedness of compact linear operators

Let $A \in \mathcal{L}(X, Y)$ be compact, and let the dimension of $\mathcal{R}(A)$ be infinite. Then $A^\dagger$ is discontinuous.

Proof: Board.
Compact linear operators

What kind of operators are compact?
Compact linear operators

What kind of operators are compact?

Theorem: Operators with Hilbert-Schmidt kernel are compact

Let

\[ Au(x) = \int_\Omega k(x, y)u(y) \, dy \]

with kernel \( k \in L^2(\Omega \times \Omega) \). Then \( A \in \mathcal{L}(L^2(\Omega), L^2(\Omega)) \) is compact.

Proof: Board.

A kernel \( k \in L^2(\Omega \times \Omega) \) is called a Hilbert-Schmidt kernel from \( \Omega \times \Omega \to \mathbb{R} \).
Examples for compact linear operators

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  with smoothing kernel $k$ is ill-posed.
## Compact linear operators

### What kind of operators are compact?

#### Facts about compact operators

- Let $A : X \to Y$ be a compact linear operator. Then $A$ is bounded, i.e. $A \in \mathcal{L}(X, Y)$.

- Let $A \in \mathcal{L}(X, Y)$ be compact and $B \in \mathcal{L}(Z, X)$. Then $AB$ is compact.

- Let $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Z, X)$ be compact. Then $AB$ is compact.

- Let $A \in \mathcal{L}(X, Y)$ be compact. Then $A^*$ is compact.

$X, Y, Z$ Hilbert spaces.
A little summary

What did we learn so far?

- No solution exists
A little summary

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- $A^\dagger$ continuous $\iff \mathcal{R}(A)$ closed.
- $A$ compact, $\mathcal{R}(A)$ infinite dimensional
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  - \( A^\dagger \) continuous \( \iff \mathcal{R}(A) \) closed.
  - A compact, \( \mathcal{R}(A) \) infinite dimensional \( \Rightarrow A^\dagger \) not continuous.
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- Third criterion for well posedness? $\rightarrow A^\dagger$ continuous.
- $A^\dagger$ continuous $\iff \mathcal{R}(A)$ closed.
- $A$ compact, $\mathcal{R}(A)$ infinite dimensional $\Rightarrow A^\dagger$ not continuous.
- Integral equation with H.S. kernel $\Rightarrow A$ compact
Study compact lin. operators $A$!
How does $A^\dagger$ look like?
Theorem: Eigendecomposition

Let $A \in \mathcal{L}(X, X)$ be self-adjoint and compact. Then there exist at most countably many nonzero eigenvalues $\{\lambda_n\}, \ n \in I$, of $A$. All eigenvalues are real and for a set of orthonormal eigenvectors $\{u_n\}$ with $\|u_n\| = 1$ one has

$$Ax = \sum_{n \in I} \lambda_i \langle x, u_n \rangle u_n$$

---

Eigendecomposition

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Ideas: If $A \in \mathcal{L}(X, Y)$ is compact ...

• ... then $B := A^* A$ is compact and self-adjoint.

\[ Bx = \sum_{n \in I} \sigma_n^2 u_n \langle x, u_n \rangle \quad \forall x \in X \]
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- ... then $B := A^* A$ is compact and self-adjoint.
  
  $$Bx = \sum_{n \in I} \sigma_n^2 u_n \langle x, u_n \rangle \quad \forall x \in X$$

- ... then $C := AA^*$ is compact and self-adjoint.
  
  $$Cy = \sum_{n \in \tilde{I}} \tilde{\sigma}_n^2 v_n \langle y, v_n \rangle \quad \forall y \in Y$$

Further computations on the board.
Singular value decomposition

Singular value decomposition (SVD)

Any compact linear operator $A \in \mathcal{L}(X, Y)$ has a representation

$$Ax = \sum_{n \in I} \sigma_n \langle x, u_n \rangle v_n$$

$$A^* y = \sum_{n \in I} \sigma_n \langle y, v_n \rangle u_n$$

with the orthonormal singular vectors $u_n$ and $v_n$, and singular values $\sigma_n > 0$. 
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Singular value decomposition

**Singular value decomposition (SVD)**

Any compact linear operator $A \in \mathcal{L}(X, Y)$ has a representation

$$Ax = \sum_{n \in I} \sigma_n \langle x, u_n \rangle v_n$$

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**Sanity check:**

$$\left\| \sum_{n=1}^{N} \sigma_n \langle x, u_n \rangle v_n \right\|^2 = \sum_{n=1}^{N} \sigma_n^2 \langle x, u_n \rangle^2 \leq \sigma_1 \sum_{n=1}^{N} \langle x, u_n \rangle^2 \leq \sigma_1 \|x\|^2$$
Singular value decomposition

Can we use the SVD for the Moore-Penrose inverse?

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**Theorem: Singular values of compact operators**

Let $A \in \mathcal{L}(X, Y)$ be compact. The zero is the only possible accumulation point for the singular values $\sigma_n$.

Proof: Exercise.
Singular value decomposition

From

\[ A^\dagger y = \sum_{n \in I} \frac{1}{\sigma_n} \langle y, v_n \rangle u_n \]

we can see

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**Singular value decomposition**

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