Continuity of $A^\dagger$

**Theorem 1.** $A^\dagger$ is continuous if and only if $\mathcal{R}(A)$ is closed.

**Proof.** Since $\mathcal{R}(A)$ and $\mathcal{N}(A)^\perp$ are closed, $(\mathcal{R}(A), \langle \cdot, \cdot \rangle_Y)$ and $(\mathcal{N}(A)^\perp, \langle \cdot, \cdot \rangle_X)$ are Hilbert spaces again. Hence, by the open mapping theorem, the restriction $\tilde{A} : \mathcal{N}(A)^\perp \rightarrow \mathcal{R}(A)$ is open. Therefore, the inverse image of $\tilde{A}^{-1}$ of any open set is open which shows that $\tilde{A}^{-1}$ is continuous, i.e. bounded.

Let $y \in Y = \mathcal{R}(A) + \mathcal{R}(A)^\perp = \mathcal{D}(A^\dagger)$ be arbitrary with $y = y_1 + y_2$, $y_1 \in \mathcal{R}(A)$, $y_2 \in \mathcal{R}(A)^\perp$. Then

$$\|A^\dagger y\| = \|\tilde{A}^{-1} y_1\| \leq \|\tilde{A}^{-1}\| \|y_1\| \leq \|\tilde{A}^{-1}\| \|y\|,$$

which shows the continuity of $A^\dagger$.

Now let $A^\dagger$ be continuous. Then $A^\dagger$ has a unique continuous extension $\overline{A}^\dagger$ to $Y$. We can conclude that $A\overline{A}^\dagger = Q$ is the projection onto $\mathcal{R}(A)$. Hence for any $y \in \mathcal{R}(A)$ it holds that

$$y = Qy = A\overline{A}^\dagger Q \in \mathcal{R}(A),$$

which means $\mathcal{R}(\overline{A}) \subset \mathcal{R}(A)$ and concludes the proof. \qed