Minimal-norm solution by Moore-Penrose inverse

**Theorem 1.** For a given $f \in \mathcal{D}(A^\dagger)$, the equation $Ax = f$ has a unique minimal-norm solution given by

$$x^\dagger := A^\dagger f.$$

The set of all least-squares solutions is given by $\{x^\dagger\} + \mathcal{N}(A)$.

**Proof.** Define

$$S = \{z \in X \mid Az = Qf\}.$$

Since $f \in \mathcal{D}(A^\dagger) = \mathcal{R}(A) + \mathcal{R}(A)^\perp$, it follows that $Ax^\dagger = AA^\dagger f = Qf$ and hence $x^\dagger \in S$. For all $z \in S$ we have

$$\|Az - f\| = \|Qf - f\| \leq \|Ax - f\|,$$

for all $x$, which means that

$$S = \{z \in X \mid z \text{ is least-squares solution of } Az = f\} \neq \emptyset.$$

Let $\hat{z} \in S$ be the element with minimal norm. Such an element exists since the inverse image $A^{-1}(\{Qf\})$ is closed due to the continuity of $A$. Obviously $S = \hat{z} + \mathcal{N}(A)$. Furthermore

$$\hat{z} = (I - P)\hat{z} = A^\dagger A\hat{z} = A^\dagger Qf = A^\dagger AA^\dagger f = A^\dagger f = x^\dagger$$

$\square$