**What the TV-semnorn means**

Consider the BV-seminorm

\[ |u|_{BV} := \sup_{\phi \in C^\infty_c(\Omega, \mathbb{R}^n), |\phi(x)|\leq 1} \int_\Omega u \nabla \cdot \phi \, dx. \]

Let us look at some special cases.

- **Let** \( u \in C^1 \). Then
  \[
  \int_\Omega u \nabla \cdot \phi \, dx = \int_\Omega \nabla u \cdot \phi \, dx.
  \]
  For non-zero derivatives the supremum over the \( \phi \) would be attained for
  \[
  \phi = \frac{\nabla u}{|\nabla u|}
  \]
  and we can use the density of \( C^\infty_c \) in \( C_c \) to show that
  \[
  |u|_{BV} = \int \sqrt{(\partial_x u)^2 + (\partial_y u)^2} \, dx =: \|\nabla u\|_{2,1}.
  \]
  For a \( u \) with zero derivatives one can extend the above \( \phi \) from the set of \( \nabla u \neq 0 \) to the whole domain \( \Omega \) and still approximate it arbitrarily closely with \( C^\infty_c \) functions.

- **For** \( u \in W^{1,1} \) we can repeat the same arguments as in the previous case (along with the density of \( C^\infty_c(\Omega, \mathbb{R}^n) \) in \( L^2 \)) and also obtain \( |u|_{BV} = \|\nabla u\|_{2,1} \).

- **Let us consider**
  \[
  u(x) = \begin{cases} 
  1 & \text{for } x \in D \\
  0 & \text{else.}
  \end{cases}
  \]
  for a piecewise smooth \( \partial D \), i.e. the example we previously used to show that \( W^{1,1} \) is too small to contain all reasonable images. We obtain
  \[
  |u|_{BV} = \sup_{\phi \in C^\infty_c(\Omega, \mathbb{R}^n), |\phi(x)|\leq 1} \int_\Omega u \nabla \cdot \phi \, dx
  = \sup_{\phi \in C^\infty_c(\Omega, \mathbb{R}^n), |\phi(x)|\leq 1} \int_D \nabla \cdot \phi \, dx
  = \sup_{\phi \in C^\infty_c(\Omega, \mathbb{R}^n), |\phi(x)|\leq 1} \int_{\partial D} \phi \cdot n \, d\sigma
  = \int_{\partial D} d\sigma,
  \]
  such that the total variation is finite as long as \( \partial D \) has a finite \( n-1 \) dimensional Hausdorff-measure. We can see that in this case the total variation is nothing but the length of the curve \( \partial D \).

- **Another intuition of what the total variation means** can be obtained geometrically for \( \Omega = [a,b] \). If you like hiking and climbing, you are very familiar with the total variation, since it is nothing but the sum of total altitude difference.