Examples of Regularizations - Pros and Cons

We have seen the following examples of regularization methods:

• Truncated SVD
  
  – **Pro:** We need only finitely many singular values and vectors for this regularization.
  
  – **Con:** As $\alpha$ decreases, the number of singular values and vectors can increase very quickly. Interestingly, the slow $\sigma_n \to 0$ the more singular values we need.

• Lavrientiev
  
  – **Pro:** If $A$ is self-adjoint and positive definite (particularly the $\{u_n\}$ form a basis of $X = Y$), we have that for
    $$x_\alpha := R_\alpha y = \sum_{n \in I} \frac{1}{\sigma_n + \alpha} \langle y, v_n \rangle u_n,$$
    one obtains that
    $$(A + \alpha I)x_\alpha = \sum_{n \in I} (\sigma_n + \alpha) \langle x_\alpha, u_n \rangle u_n$$
    $$= \sum_{n \in I} \langle y, u_n \rangle u_n = y.$$  

    Thus, although the number of singular values and singular vectors we need to compute the Lavrientiev regularization using the $g_\alpha$ is infinite, one does not need to compute the SVD at all. Instead one determines $x_\alpha$ as the solution to the linear system
    $$(A + \alpha I)x = y.$$  

    – **Con:** The restriction of being positive definite is very restrictive. From the fact that the $u_n$ form a basis one can see that such an operator can only exist in separable Hilbert spaces.

• Tikhonov regularization
  
  – **Pro:** For any compact linear operator $A$ we have that for
    $$x_\alpha := R_\alpha y = \sum_{n \in I} \frac{\sigma_n}{\sigma_n^2 + \alpha} \langle y, v_n \rangle u_n,$$
    one obtains that
    $$(A^* A + \alpha I)x_\alpha = \sum_{n \in I} \frac{\sigma_n^2}{\sigma_n^2 + \alpha} \langle x_\alpha, u_n \rangle u_n + \sum_{n \in I} \frac{\sigma_n}{\sigma_n^2 + \alpha} \langle y, v_n \rangle u_n$$
    $$= \sum_{n \in I} \frac{\sigma_n^2}{\sigma_n^2 + \alpha} \langle y, \sigma_n v_n \rangle u_n + \sum_{n \in I} \frac{\sigma_n}{\sigma_n^2 + \alpha} \langle y, \sigma_n v_n \rangle u_n$$
    $$= \sum_{n \in I} \langle y, \sigma_n v_n \rangle u_n$$
    $$= \sum_{n \in I} \langle y, A u_n \rangle u_n$$
    $$= \sum_{n \in I} \langle A^* y, u_n \rangle u_n = A^* y,$$

    where we used that the $u_n$ are a basis for $\mathcal{N}(A)^\perp = \mathcal{R}(A^*)$. Hence, instead of computing the singular value decomposition, one can compute the solution to the linear system
    $$(A^* A + \alpha I)x = A^* y.$$  

    Note that Tikhonov regularization is basically Lavrientiev regularization applied to the Gaussian normal equation.
Con: Nothing we can see so far. Thus, we will discuss Tikhonov regularization in more detail in the next chapter.

- Landweber iteration

Con: There is no easy formula for the upper bound on $x^*_\delta$, such that it seems difficult to give a parameter choice rule for convergence.

Pro: The above drawback can easily be fixed and shows a nice property of Landweber iteration.

$$\|x^k - x^{k+1}\|^2 = \|x^k - (x^k + \tau A^* (y^\delta - Ax^k))\|^2$$
$$= \|x^k - x^k\|^2 + \tau^2 \|A^* (y^\delta - Ax^k)\|^2 - 2\tau \langle x^k - x^k, A^* (y^\delta - Ax^k)\rangle$$
$$= \|x^k - x^k\|^2 + \tau^2 \|A^* (y^\delta - Ax^k)\|^2 - 2\tau \langle y - y^\delta, y^\delta - Ax^k\rangle$$
$$= \|x^k - x^k\|^2 + \tau^2 \|A^* (y^\delta - Ax^k)\|^2 - 2\tau \langle y - y^\delta, y^\delta - Ax^k\rangle$$
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$$= \|x^k - x^k\|^2 + \tau^2 \|A^* (y^\delta - Ax^k)\|^2 - 2\tau \langle y - y^\delta, y^\delta - Ax^k\rangle$$

This gives rise to the so-called discrepancy-principle. We pick a number $\gamma > 1$ (e.g. $\gamma = \frac{2}{2 - \tau \|A\|^2}$) and use the a-posteriori stopping rule

$$k^*(\delta, y^\delta) := \inf \{ k \mid \|y^\delta - Ax^k\| > \gamma \delta \}.$$

With a little extra afford, one can show that the above rule indeed makes the Landweber iteration a convergent regularization method (of optimal order, see Theorem 6.5, p.159, in Engl, Hanke, Neubauer, *Regularization of Inverse Problems*).

In practice another advantage of iterative regularization is that one can 'look how the solution evolves' over the iteration, and stop the iteration by visual inspection. Choosing a different $\alpha$ in Tikhonov regularization means recomputing the linear system.