Some considerations on the definition of a regularization method

We have two conditions. The first one is
\[
\limsup_{\delta \to 0} \{ \|R_{\alpha(\delta,y^{\delta})}y^{\delta} - A^{\dagger}y\| \mid y^{\delta} \in Y, \|y - y^{\delta}\| \leq \delta \} = 0. \tag{1}
\]

Let us consider what that means. Let \( y^{\delta} \in Y \) be arbitrary with \( \|y - y^{\delta}\| \leq \delta \). Consider our usual estimate
\[
\|A^{-1}y - R_{\alpha}y^{\delta}\| \leq \left( \|A^{\dagger}y - R_{\alpha}y\| + \|R_{\alpha}y - R_{\alpha}y^{\delta}\| \right).
\]

The approximation error converges to zero if \( R_{\alpha}y \to A^{\dagger}y \) as \( \alpha \) goes to zero. After extending \( \rho \) goes to zero. Hence it makes sense to require any parameter choice rule to approach zero as \( \delta \) decreases, which is exactly the second criterion for a convergent regularization method, i.e.
\[
\limsup_{\delta \to 0} \{ \alpha(\delta,y^{\delta}) \mid y^{\delta} \in Y, \|y - y^{\delta}\| \leq \delta \} = 0. \tag{2}
\]

It seems like the above definition of regularization is rather complicated, and indeed is given here just to state the most general definition of a regularization method. In practice one most often finds a family of operators for which \( R_{\alpha}y \to A^{\dagger}y \) for all \( y \in D(A^{\dagger}) \), which then immediately allows to define a regularization method as the next theorem tells us:

**Theorem 1.** Let \( A \in \mathcal{L}(X,Y) \) be compact and \( R_{\alpha} \in \mathcal{L}(Y,X) \) be a family of operators with \( \alpha \in \mathbb{R}^{+} \). If
\[
R_{\alpha}y \to A^{\dagger}y
\]
for all \( y \in D(A^{\dagger}) \) as \( \alpha \to 0 \) (pointwise convergence), then \( \{ R_{\alpha} \} \) is a regularization of \( A^{\dagger} \). In particular, there exists an a-priori parameter choice rule \( \alpha \) such that \( (R_{\alpha},\alpha) \) is a convergent regularization method.

**Proof.** Let \( y \in D(A^{\dagger}) \) be arbitrary but fixed. Due to the pointwise convergence, we can find a monotone function \( \sigma : \mathbb{R}^{+} \to \mathbb{R}^{+} \) such that for every \( \epsilon > 0 \)
\[
\|R_{\sigma(\epsilon)}y - A^{\dagger}y\| \leq \frac{\epsilon}{2}.
\]

The operator \( R_{\sigma(\epsilon)} \) is continuous for a fixed \( \epsilon \) and hence there exists a \( \rho(\epsilon) \) such that
\[
\|R_{\sigma(\epsilon)}y - R_{\sigma(\epsilon)}z\| \leq \frac{\epsilon}{2} \quad \text{for all } z \text{ with } \|y - z\| \leq \rho(\epsilon).
\]

Without restriction of generality, we can assume that \( \rho \) is continuous, strictly monotonically increasing and
\[
\lim_{\epsilon \to 0} \rho(\epsilon) = 0.
\]

Due to the monotonicity \( \rho \) is invertible, the inverse is continuous, and also goes to zero as its argument goes to zero. After extending \( \rho^{-1} \) to \( \mathbb{R}^{+} \) we define the parameter choice rule
\[
\alpha : \mathbb{R}^{+} \to \mathbb{R}^{+}
\]
\[
\delta \mapsto \sigma(\rho^{-1}(\delta))
\]

For \( \delta = \rho(\epsilon) \) we have \( \alpha(\delta) = \sigma(\rho^{-1}(\rho(\epsilon))) = \sigma(\epsilon) \), and hence
\[
\|A^{\dagger}y - R_{\alpha}y^{\delta}\| \leq \|A^{\dagger}y - R_{\alpha}y\| + \|R_{\alpha}y - R_{\alpha}y^{\delta}\| \leq \epsilon
\]
for all \( y^{\delta} \) with \( \|y - y^{\delta}\| \leq \delta \). Hence, \( (R_{\alpha},\alpha) \) is a convergent regularization method. \( \square \)

In words the above theorem tells us that every continuous pointwise convergent family of operators \( R_{\alpha} \) allows to define a regularization strategy. Regarding the reverse claim, by definition we know that for all \( y \in D(A) \) we have
\[
\lim_{\delta \to 0} R_{\alpha(\delta,y)}y - A^{\dagger}y = 0.
\]

Now if \( \alpha \) is continuous, then we can conclude the pointwise convergence of \( R_{\beta} \) for \( \beta \to 0 \). If \( \alpha \) is not continuous, we just have the pointwise convergence in the range of \( \alpha \).