Example of a noncompact operator with discontinuous pseudoinverse.

Let us take the bounded sequence \( \{u_n\} \) for \( n \neq k^2 \). Then \( A \) acts as the identity on \( u_n \) and \( \{Au_n\} \) does not have a convergent subsequence since \( \|u_n - u_m\| = \sqrt{2} \) for all \( n \neq m \). This shows that \( A \) is not compact.

Now we consider \( \{u_{n^2}\}_{n \in \mathbb{N}} \). We have that \( Au_{n^2} = \frac{1}{n^2} u_{n^2} \), which can be arbitrarily close to zero. However, as \( A \) is injective (\( u_{n^2} \in \mathcal{N}(A)^\perp \)) we have that

\[
A^\dagger \frac{1}{n^2} u_{n^2} = \hat{A}^{-1} \frac{1}{n^2} u_{n^2} = u_{n^2},
\]

which has norm one. This shows

\[
Au_{n^2} \to 0, \quad \text{but} \quad A^\dagger Au_{n^2} \not\to 0,
\]

which shows that \( A^\dagger \) is discontinuous.