Developing the singular value decomposition

Let \( v_n \) be an eigenvector to \( C = A A^* \) with eigenvalues \( \sigma_n^2 \). We have

\[
\sigma_n^2 A^* v_n = A^* C v_n = A^* A A^* v_n = B A^* v_n,
\]
and see that \( \sigma_n^2 \) is an eigenvalue of \( B \) to eigenvector \( A^* v_n \)! Similarly, we compute

\[
\sigma_n^2 A u_n = A B u_u = A A^* A u_n = C A u_n,
\]
and conclude that \( A u_n \) is an eigenvector of \( C \) with eigenvalue \( \sigma_n^2 \). Thus, without restriction of generality, we say

\[
\tilde{\sigma}_n = \sigma_n, \quad v_n = \frac{A u_n}{\|A u_n\|},
\]

The norm \( \|A u_n\| \) can further be computed as

\[
\|A u_n\|^2 = \langle A^* A u_n, u_n \rangle = \sigma_n^2 \|u_n\|^2 = \sigma_n^2.
\]

The latter implies

\[
A^* v_n = A^* \frac{A u_n}{\|A u_n\|} = \frac{\sigma_n^2}{\|A u_n\|} u_n = \sigma_n u_n,
\]

and one similarly shows \( A u_n = \sigma_n v_n \).

Now

\[
A A^* y = \sum_{n=1}^{\infty} \sigma_n^2 \langle v_n, y \rangle v_n,
\]

\[
= \sum_{n=1}^{\infty} \sigma_n \langle A u_n, y \rangle v_n,
\]

\[
= \sum_{n=1}^{\infty} \sigma_n \langle u_n, A^* y \rangle v_n,
\]

such that we have shown that

\[
A x = \sum_{n=1}^{\infty} \sigma_n \langle u_n, x \rangle v_n \tag{1}
\]

for all \( x \in \mathcal{R}(A^*) \). By considering a convergent sequence, one readily shows that the representation holds for all \( x \in \overline{\mathcal{R}(A^*)} \). Furthermore, \( \mathcal{R}(A^*)^\perp = \mathcal{N}(A) \) which shows that (1) holds for all \( x \in X \). Similarly, one can show the representation

\[
A^* y = \sum_{n=1}^{\infty} \sigma_n \langle v_n, y \rangle u_n \tag{2}
\]

for all \( y \in Y \).

The representations (1) and (2) are called the singular value decomposition of \( A \) and \( A^* \).