Integral operators with Hilbert-Schmidt kernel are compact

**Theorem 1.** Let

\[ Au(x) = \int_{\Omega} k(x, y)u(y) \, dy \]

with kernel \( k \in L^2(\Omega \times \Omega) \). Then \( A \in \mathcal{L}(L^2(\Omega), L^2(\Omega)) \) is compact.

**Proof.** The space \( L^2(\Omega) \) has a countable orthonormal basis (ONB). (A Hilbert space has a countable orthonormal basis if and only if it is separable.) Note that if \( \phi_i \) is an ONB of \( L^2(\Omega) \) then \( \phi_i(x) \phi_j(y) \) is an ONB of \( L^2(\Omega \times \Omega) \). We write

\[ k(x, y) = \sum_{i, j=1}^{\infty} k_{i,j} \phi_i(x) \phi_j(y) , \]

with

\[ k_{i,j} = \int_{\Omega} \int_{\Omega} k(x, y) \phi_i(x) \phi_j(y) \, dx \, dy. \]

We now define

\[ k_n(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{\infty} k_{i,j} \phi_i(x) \phi_j(y) , \]

and

\[ A_n u(x) = \int_{\Omega} k_n(x, y)u(y) \, dy. \]

Obviously, \( A_n \) maps from \( L^2(\Omega) \) into a finite-dimensional subspace of \( L^2(\Omega) \). The range of \( A_n \) is finite-dimensional and hence \( A_n \) is compact. We find

\[ \|(A - A_n)u\|^2 = \left\| \int_{\Omega} (k(x, y) - k_n(x, y))u(y) \, dy \right\|^2 \]

\[ = \int_{\Omega} \left( \int_{\Omega} (k(x, y) - k_n(x, y))u(y) \, dy \right)^2 \, dx \]

\[ \leq \int_{\Omega} \left( \int_{\Omega} (k(x, y) - k_n(x, y))^2 \, dy \right) \left( \int_{\Omega} u^2(y) \, dy \right) \, dx \]

\[ = \left( \int_{\Omega} \int_{\Omega} (k(x, y) - k_n(x, y))^2 \, dy \, dx \right) \left( \int_{\Omega} u^2(y) \, dy \right) \]

\[ = \left( \sum_{i=n+1}^{\infty} \sum_{j=1}^{\infty} |k_{i,j}|^2 \right) \|u\|^2. \]

Since \( \|k\|^2_{L^2(\Omega \times \Omega)} = \sum_{i, j=1}^{\infty} |k_{i,j}|^2 < \infty \), the above sum has to go to zero as \( n \to \infty \). We conclude that \( A_n \to A \) in the operator norm which yields the assertion with the following Theorem:

**Theorem 2.** Let \( A : X \to Y \) be a linear operator and \( A_n \in \mathcal{L}(X, Y) \) a sequence of compact operators. If \( A_n \xrightarrow{n \to \infty} A \) in the operator norm, then \( A \) is compact.

The proof of this theorem can be found on page 408, theorem 8.1-5, of *Introductory Functional Analysis with Applications* by Erwin Kreyszig, Wiley 1989.

**Proof.** Let \( \{x_n\} \) be a bounded sequence in \( X \). Since \( A_1 \) is compact, there exists a subsequence \( x_{n}^k \) such that \( A_1 x_{n}^k \) converges. Iteratively/inductively, we now construct a subsequence as follows: If \( x_{n}^k \) is a subsequence of \( x_n \) such that \( A_k x_{n}^k \) converges, \( x_{n}^{k+1} \) is still bounded and we can find a subsequence \( x_{n}^{k+1} \) of \( x_n \) such that \( A_{k+1} x_{n}^{k+1} \) converges.
Consider the sequence \((x_n^n)_{n \in \mathbb{N}}\). There exist natural numbers \(r_n\) such that \(x_{r_n} = x_n^n\). Clearly, for all \(k \in \mathbb{N}\), \(A_k x_{r_n}\) converges. Now

\[
\|Ax_{r_n} - Ax_{r_m}\| = \|Ax_{r_n} - A_k x_{r_n} + A_k x_{r_n} - A_k x_{r_m} + A_k x_{r_m} - Ax_{r_m}\|
\leq \|Ax_{r_n} - A_k x_{r_n}\| + \|A_k x_{r_n} - A_k x_{r_m}\| + \|A_k x_{r_m} - Ax_{r_m}\|
\leq \|A - A_k\|\|x_{r_n}\| + \|A_k x_{r_n} - A_k x_{r_m}\| + \|A_k - A\|\|x_{r_m}\|
\]

Now the first and third term can become arbitrary small since \(x_n\) is bounded and \(A_k\) converges to \(A\) and the middle term can become arbitrary small by the construction of the subsequence. Thus, we found a subsequence of \(x_n\) for which \(Ax_{r_n}\) is a Cauchy sequence and therefore convergent.