Example: Differentiation is ill-posed!

Let $f : [0,1] \to \mathbb{R}$ be continuously differentiable. Let $f^\delta(x) = f(x) + n^\delta(x)$ with

$$n^\delta(x) = \sqrt{2}\delta\sin(2\pi kx)$$

for some $k \in \mathbb{N}$. Then

$$\begin{aligned}
\int_0^1 |f^\delta(x) - f(x)|^2 \; dx &= 2\delta^2 \int_0^1 \sin^2(2\pi kx) \; dx, \\
&= \frac{\delta^2}{\pi k} \int_0^{2\pi k} \sin^2(z) \; dz, \\
&= \frac{\delta^2}{\pi k} \left([-\cos(\mathrm{z})\sin(\mathrm{z})]_{z=0}^{2\pi k} + \int_0^{2\pi k} \cos^2(z) \; dz \right), \\
&= \frac{\delta^2}{\pi k} \int_0^{2\pi k} (1 - \sin^2(z)) \; dz.
\end{aligned}$$

It follows $2\int_0^{2\pi k}(1 - \sin^2(z)) \; dz = 2\pi k$ such that

$$\int_0^1 |f^\delta(x) - f(x)|^2 \; dx = \delta^2.$$

Note that the frequency k of the noise can be arbitrarily large. We find

$$\partial_x f^\delta(x) = \partial_x f(x) + \sqrt{2}2\pi k\delta \cos(2\pi kx)$$

such that the squared L^2 error

$$\begin{aligned}
\int_0^1 |\partial_x f^\delta(x) - \partial_x f(x)|^2 \; dx &= 8\pi^2 k^2 \delta^2 \int_0^1 \cos^2(2\pi kx) \; dx \\
&= 4\pi^2\delta^2 k^2,
\end{aligned}$$

or the L^∞ error

$$\sup_{x\in[0,1]} |\partial_x f^\delta(x) - \partial_x f(x)| = \sqrt{2}2\pi\delta k.$$

can be arbitrarily large.

Without regularization and without further information, the error between the exact and noisy solution can be arbitrarily large, even if the noise is arbitrarily small!