Weekly Homework 7

Ill-Posed Problems in Image and Signal Processing

Due on Monday Nov. 24th, 2014

Exercise 1 (4 points). Let \( A \in \mathcal{L}(X,Y) \) have a non-closed range and let \( \{ R_\alpha \} \) be a linear regularization for \( A^\dagger \). Let \( \alpha = \alpha(\delta,y^\delta) \) be a parameter choice rule such that \((R_\alpha,\alpha)\) is a convergent regularization method for all \( y \in \mathcal{D}(A^\dagger) \). Prove that there can be no function \( r : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( r(\delta) \to 0 \) for \( \delta \to 0 \), such that
\[
\| A^\dagger y - R_{\alpha(\delta,y^\delta)} y^\delta \| \leq r(\delta)
\]
holds for all \( y \in \mathcal{D}(A^\dagger) \) with \( \| y \| \leq 1 \), all \( \delta > 0 \) and \( y^\delta \in Y \) with \( \| y - y^\delta \| \leq \delta \). In other words, show that in general the convergence of a regularization method is arbitrarily slow.

\textbf{Hint: Assume that there exists such a function. Consider that}
\[
\sup \{ \| R_{\alpha(\delta,x^\delta)} x^\delta - x \| \mid x \in \mathcal{N}(A) \cap \mathcal{N}(B_1(0)), \ y^\delta \in Y, \ Ax - y^\delta \| \leq \delta \} \leq r(\delta)
\]
where \( B_1(0) = \{ y \in Y \mid \| y \| \leq 1 \} \). Use a particular choice of \( y^\delta \). Finally construct a contradiction to the fact that \( A^\dagger \) cannot be continuous.

Exercise 2 (4 points). Let \( A \in \mathbb{R}^{n \times m}, \ B \in \mathbb{R}^{k \times m} \). Show that any \( \hat{u} \in \mathbb{R}^m \) that meets
\[
(A^T A + \alpha B^T B) \hat{u} = A^T f
\]
is a minimizer of
\[
E(u) = \frac{1}{2} \| Au - f \|^2 + \frac{\alpha}{2} \| Bu \|^2.
\]
Under which conditions on \( A \) and \( B \) can we show uniqueness of the minimizer \( \hat{u} \)?

Exercise 3 (4 points). A function \( E : X \to \mathbb{R} \) (with \( X \) being a normed vector space) is called \textit{convex} if for all \( u, v \in X \) and all \( \beta \in [0,1] \) it holds that
\[
E(\beta u + (1 - \beta)v) \leq \beta E(u) + (1 - \beta) E(v),
\]
and called \textit{strictly convex} if the strict inequality holds for all \( u \neq v, \ \beta \in ]0,1[ \) in the above.

\begin{itemize}
  \item Prove that that the function \( E(u) \) in (1) is convex.
  \item Prove that any local minimum of a convex function is a global minimum. If the function is strictly convex, this minimum is unique.
\end{itemize}

Note: We could have considered \( E : C \to \mathbb{R}, \ C \subset X \) being a convex subset.

Please turn!
Exercise 4 (4 points). Download the data for this exercise from the courses website. You find an image of some parrots that has been corrupted since someone wrote some text on top of the image. Additionally, you find the text in a separate image. The goal of this exercise is to restore an image in which the text is removed. The text image is a binary image \( r \) where \( r_{i,j} = 1 \) if there is no text at pixel \((i, j)\) and \( r_{i,j} = 0 \) if there is text at pixel \((i, j)\).

- Create a sparse matrix \( P_r \), such that \( \text{reshape}(P_r \tilde{u}, \text{size}(u))(i, j, c) = r(i, j) \cdot u(i, j, c) \), where \( \tilde{u} \) denotes writing the whole RGB image as one long vector (\( u(:) \) in Matlab).

- Create a sparse matrix \( K \) such that
  \[
  K \tilde{u} = \begin{pmatrix}
  \nabla \tilde{u}_1 \\
  \nabla \tilde{u}_2 \\
  \nabla \tilde{u}_3
  \end{pmatrix},
  \]
  where \( u_1, u_2 \) and \( u_3 \) denote the red, green and blue channel of an RGB image and \( \nabla \) denotes a discretization of the gradient. (Hint: remeber \texttt{kron, sparse, spdiags}.)

- Solve
  \[
  (P_r^T P_r + \alpha K^T K) \tilde{u} = P_r^T \tilde{f}
  \]
  for \( \tilde{u} \), where \( \tilde{f} \) denotes the vectorized version of the corrupted parrot image. Reshape and display \( u \) for different values of \( \alpha \).