Exercise 14.1 (Characterizing Banach spaces):

Let $(V, \|\cdot\|)$ be a Banach space over $\mathbb{C}$. Prove that there is a compact Hausdorff space $X$ such that $(V, \|\cdot\|)$ is isometrically isomorphic to a closed linear subspace of $(C(X), \|\cdot\|_{\infty})$.

[Hint: Consider $X = B^* = \{x^* \in V^* : \|x^*\| \leq 1\}$ and for $x \in V$ define $f_x(x^*) = x^*(x)$ and convince yourself that $f_x \in C(X)$. Now construct a mapping $T : V \to C(X)$ using $f_x$.]

Exercise 14.2 (Compact operators):

Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be Banach spaces over $\mathbb{F}$ and $B_i = \{x \in V_i : \|x\|_i \leq 1\}$, $i = 1, 2$. A continuous linear operator $T : V_1 \to V_2$ is called compact if $T(B_1) \subset V_2$ is compact. Prove that $T$ is compact if $T^* : V_2^* \to V_1^*$ is weak*-to-norm continuous on weak* compact subsets of $V_2^*$.

[Hint: Make use of the Banach-Alaoglu Theorem and use the fact that a continuous linear operator $T : V_1 \to V_2$ between Banach spaces $V_1$ and $V_2$ is compact iff $T^* : V_2^* \to V_1^*$ is compact (Theorem of Schur).]
**Exercise 14.3 (Relations between weak and norm topology):**

a) Let $(V, \| \cdot \|)$ be a NVS over $\mathbb{F}$. Prove that any weak closed subset $M \subseteq V$ is closed w.r.t. the norm topology.

b) Let $(V, \| \cdot \|)$ be a NVS over $\mathbb{F}$. Show that if $K \subseteq V$ is weakly compact, then $K$ is norm closed and norm bounded.

c) Prove that $B = \{ x \in c_0 : \| x \| \leq 1 \}$ is norm closed and norm bounded but not weakly compact in $c_0$.

**Hint:** b): Note that since $x^* \in V^*$ is weakly continuous, $x^*(K) \subseteq \mathbb{F}$ is compact, and therefore bounded. Consider the set $\{ j(x) : x \in K \}$ and apply the UBP to show that this set is norm bounded. Use a) to complete the proof.

c): Suppose $B$ is weakly compact. Consider the sequence $(s_n)_{n \in \mathbb{N}} \subseteq c_0$, $s_n = e_1 + e_2 + \cdots + e_n$ where $e_k = (\delta_{j,k})_{j \in \mathbb{N}}$. There is a subnet $(s_{F(\iota)})_{\iota \in I}$ s.t. $s_{F(\iota)} \to s$ weakly. Consider $x_n^* \in c_0^*$ defined by $x_n^*(\xi_k)_{k \in \mathbb{N}} = \xi_n$. Now use the definition of a subnet to conclude $x_n^*(s) = 1$ for every $n \in \mathbb{N}$ and demonstrate that this leads to a contradiction.

**Exercise 14.4 (Calvin and Hobbes):**

What do you think how many stripes would the tiger Hobbes have if Calvin was right?

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This sheet will be discussed from **Monday, Februar 6** on.

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