Functional Analysis

EXERCISE SHEET 13

Weak Topologies II

Exercise 13.1 (Norm closed ball in $V^*$):
Let $(V, \| \cdot \|)$ be a NVS over $F$. Show that every norm closed ball
\[
\overline{B}(0, \varrho) = \{ x^* \in V^* : \| x^* \| \leq \varrho \}
\]
is weak* closed. Demonstrate that, however, a norm closed and norm bounded set in $V^*$ is not necessarily weak* closed.

[Hint: For the additional statement consider a non-reflexive space $V$ and the canonical embedding of the norm closed ball $\overline{B} = \{ x \in V : \| x \| \leq 1 \}$. Apply Theorem 8.1 and the Theorem of Goldstine to finish the proof.]

Exercise 13.2 (Weak closure of the unit sphere):
Let $(V, \| \cdot \|)$ be an infinite dimensional NVS over $F$. Prove that the weak closure of $S = \{ x \in V : \| x \| = 1 \}$ is $\overline{B}(0, 1) = \{ x \in V : \| x \| \leq 1 \}$.

Exercise 13.3 (Space of continuous functions):
\begin{enumerate}[a)]
\item Let $\varphi : [0, 1] \to \mathbb{R}$ be a continuous function and define $f_n : [0, 1] \to \mathbb{R}, n \in \mathbb{N},$ by $f_n(x) = \varphi(x^n)$. Apply Exercise 12.4 to prove that $(f_n)_{n \in \mathbb{N}} \subset C([0, 1])$ is weak convergent iff $\varphi(0) = \varphi(1)$.
\item Let $X$ be a compact Hausdorff space an let $(f_n)_{n \in \mathbb{N}} \subset C(X), f \in C(X)$, such that $\sup_n \| f_n \| < \infty$ and $f_n(x) \to f(x)$ for all $x \in X$. Prove that there is a sequence $(g_n)_{n \in \mathbb{N}}$ of convex combinations of elements of $(f_n)_{n \in \mathbb{N}}$ such that $g_n \to f$ uniformly.
\item Prove that $C_0(\mathbb{R})$ is a weak* dense subset of $L^\infty(\mathbb{R})$.
\end{enumerate}

Exercise 13.4 (A not weakly sequentially complete space):
Prove that the space $c_0$ is not weakly sequentially complete.

[Hint: Use the fact $c_0^* \cong l^1$. For $n \in \mathbb{N}$ define $y_n = (\eta_n,k)_{k \in \mathbb{N}} \in c_0$ by $\eta_n,k \in \mathbb{C}, \eta_n,k = 0$ if $k > n$ and show that $(y_n)_{n \in \mathbb{N}}$ can not be weakly convergent in $c_0$.]

This sheet will be discussed from Monday, January 30 on.